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Weyl Type Theorems for a Class of Operator Matrices

Weihong Zhang[a,b], Guojun Hai[a], Alatancang Chen[b]

[a] *School of Mathematical Sciences, Inner Mongolia University, Hohhot, 010021, China*
[b] *School of Mathematical Sciences, Inner Mongolia Normal University, Hohhot, 010022, China*

Abstract. It is well known that not all operators satisfy Weyl type theorems simultaneously. In this paper, we denote an operator matrix and consider how eight Weyl type theorems hold for it when certain entry operators satisfy Weyl type theorems. Moreover, we characterize its spectral structure. Finally, the relevant conclusions are promoted to infinite dimensional Hamilton operator.

1. Introduction

Throughout this paper, let X be a separable Hilbert space and let $\mathcal{B}(X)$ denote the set of all bounded linear operators from X to X. If $T \in \mathcal{B}(X)$, we write T^*, $\mathcal{N}(T)$, $\mathcal{R}(T)$, $p(T)$ and $q(T)$ for the adjoint operator, the null space, the range space, ascent and descent, respectively. Let $\alpha(T) = \dim\mathcal{N}(T)$, $\beta(T) = \operatorname{codim}\mathcal{R}(T)$. If $T \in \mathcal{B}(X)$ is such that $\mathcal{R}(T)$ is closed and $\alpha(T) < \infty$, then T is called upper semi-Fredholm operator. If $\beta(T) < \infty$, then T is a lower semi-Fredholm operator. Let the set of all upper (lower) semi-Fredholm operators is written as $F_+(X)$ ($F_-(X)$). Let $F(X) := F_+(X) \cap F_-(X)$ ($F_\pm(X) := F_+(X) \cup F_-(X)$) denote the set of all Fredholm (semi-Fredholm) operators. The index of Fredholm operator $T \in \mathcal{B}(X)$, denoted $\operatorname{ind}(T)$, is given by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$. The class of all Weyl operators is defined by $W(X) = \{T \in F(X) : \operatorname{ind}(T) = 0\}$, and the set of all upper semi-Weyl operators is given by $W_+(X) := \{T \in F_+(X) : \operatorname{ind}(T) \leq 0\}$. The set of all Browder operators is denoted by $B(X) = \{T \in F(X) : p(T) = q(T) < \infty\}$, the set of all upper semi-Browder operators and lower semi-Browder operators are defined by $B_+(X) = \{T \in F_+(X) : p(T) < \infty\}$ and $B_-(X) = \{T \in F_-(X) : q(T) < \infty\}$ respectively.

The essential spectrum $\sigma_e(T)$, the Weyl spectrum $\sigma_w(T)$, the Browder spectrum $\sigma_b(T)$, the essential approximate point spectrum $\sigma_{ea}(T)$ and the Browder essential approximate point spectrum $\sigma_{ab}(T)$ are defined

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Email addresses: zhangwh@mail.imu.edu.cn (Weihong Zhang), 3695946@163.com (Guojun Hai), alatanca@imu.edu.cn (Alatancang Chen)

by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin F(X)\},$$
$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W(X)\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin B(X)\},$$
$$\sigma_{ea}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin W_+(X)\},$$
$$\sigma_{ab}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \notin B_+(X)\}.$$

We define the following subset of the spectrum:

$$\pi_{00}(T) = \{\lambda \in \mathrm{iso}\sigma(T) : 0 < \alpha(T - \lambda I) < \infty\},$$
$$\pi_{00}^a(T) = \{\lambda \in \mathrm{iso}\sigma_a(T) : 0 < \alpha(T - \lambda I) < \infty\},$$
$$p_{00}(T) = \sigma(T) \setminus \sigma_b(T),$$
$$p_{00}^a(T) = \sigma_a(T) \setminus \sigma_{ab}(T),$$

where iso$\triangle$ is the set of all isolated points of $\triangle$.

In 1909, H. Weyl [13] shown that the complement in the spectrum of the "Weyl spectrum"with hermitian operators coincides with the isolated points of spectrum which are eigenvalues of finite multiplicity, and this findings was called Weyl's theorem. Whereafter, Weyl's theorem has been extended and transformed by many authors. In this article, we focus our attention on Weyl type theorems which includes Weyl's theorem, a-Weyl's theorem, Browder's theorem, a-Browder's theorem, property (w), property (aw), property (b) and property (ab). Specifically defined as follows:

Definition 1.1. *Let $T \in \mathcal{B}(X)$. We say that*
(1) *Weyl's theorem holds for T if $\sigma(T)\backslash\sigma_w(T) = \pi_{00}(T)$.*
(2) *a-Weyl's theorem holds for T if $\sigma_a(T)\backslash\sigma_{ea}(T) = \pi_{00}^a(T)$.*
(3) *Browder's theorem holds for T if $\sigma(T)\backslash\sigma_w(T) = p_{00}(T)$.*
(4) *a-Browder's theorem holds for T if $\sigma_a(T)\backslash\sigma_{ea}(T) = p_{00}^a(T)$.*
(5) *T has the property (w) if $\sigma_a(T)\backslash\sigma_{ea}(T) = \pi_{00}(T)$.*
(6) *T has the property (aw) if $\sigma(T)\backslash\sigma_w(T) = \pi_{00}^a(T)$.*
(7) *T has the property (b) if $\sigma_a(T)\backslash\sigma_{ea}(T) = p_{00}(T)$.*
(8) *T has the property (ab) if $\sigma(T)\backslash\sigma_w(T) = p_{00}^a(T)$.*

The following diagram show the relationship between various Weyl type theorems

$$\begin{array}{ccccccccc}
a\text{-}\mathtt{W} & \Rightarrow & \mathtt{W} & \Leftarrow & (w) & \Rightarrow & (b) & \Rightarrow & a\text{-}\mathtt{B} \\
\Downarrow & & \Downarrow & & & & \Downarrow & & \Downarrow \\
a\text{-}\mathtt{B} & \Rightarrow & \mathtt{B} & \Leftarrow & (aw) & \Rightarrow & (ab) & \Rightarrow & \mathtt{B},
\end{array}$$

where the abbreviations $\mathtt{W}$, a-$\mathtt{W}$, $\mathtt{B}$, a-$\mathtt{B}$, (w), (aw), (b) and (ab) to signify that an operator $T \in \mathcal{B}(X)$ obeys Weyl's theorem, a-Weyl's theorem, Browder's theorem, a-Browder's theorem, property (w), property (aw), property (b) and property (ab). We refer the reader to [1, 3, 5, 10] for more details about Weyl type theorems.

It is well known that, not all operators satisfy these theorems and properties simultaneously.

Example 1.2 ([14]). *Let*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & -T_1^* \end{bmatrix},$$

where $T_1 : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, for any $x = (x_1, x_2, x_3, \cdots) \in l^2(\mathbb{N})$, we denote

$$T_1 x := (0, \frac{x_1}{2}, \frac{x_2}{3}, \cdots),$$

then

$$T_1^*x = (\frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \cdots).$$

According to a simple calculation shows that

$$\sigma(T) = \sigma_a(T) = \sigma_w(T) = \sigma_b(T) = \{0\},$$

and

$$0 \in \sigma_{ab}(T), \pi_{00}^a(T) = \{0\}.$$

Therefore, T satisfy Browder's theorem and property (ab), but property (aw) fails for T.

In general, Weyl type theorems may or may not hold for an operator matrix of the form $\left[\begin{smallmatrix} T_1 & T_2 \\ 0 & T_3 \end{smallmatrix}\right]$ for which Weyl type theorems hold for entry operators T_1 and T_3, (see [10]).

Example 1.3 ([10]). *Let*

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_3 \end{bmatrix},$$

the operators T_1, T_2 and T_3 on $l^2(\mathbb{N})$ are defined by

$$\begin{aligned}
T_1(x_1, x_2, x_3, \cdots) &= (0, x_1, 0, \frac{1}{2}x_2, 0, \frac{1}{3}x_3, \cdots), \\
T_2(x_1, x_2, x_3, \cdots) &= (0, 0, x_2, 0, x_3, 0, x_4, \cdots), \\
T_3(x_1, x_2, x_3, \cdots) &= (0, x_2, 0, x_4, \cdots).
\end{aligned}$$

Then

$$\sigma(T_1) = \sigma_w(T_1) = \{0\}, \sigma(T_3) = \sigma_w(T_3) = \{0, 1\},$$

and

$$\pi_{00}(T_1) = \pi_{00}(T_3) = \emptyset, p_{00}(T_1) = p_{00}(T_3) = \emptyset,$$

which says that Weyl's theorem and Browder's theorem hold for T_1 and T_3. Also a straighforward calculation shows that

$$\sigma(T) = \sigma_w(T) = \{0, 1\}, \pi_{00}(T) = \{0\}, \text{ and } p_{00}(T) = \emptyset,$$

which implies that Weyl's theorem fails for T, while Browder's theorem holds for T.

For this reason, many authors have been studied the Weyl type theorems for upper triangular operator matrices(see[1, 5, 10]), and it is necessary to study that these Weyl type theorems are equivalent to each other when the operator satisfy certain conditions. While, the study of operator matrices arises naturally from the following fact: if T is a bounded linear operator on a Hilbert space and M is an invariant subspace for T, then T has a 2×2 oparetor matrix representation of the form

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} : M \oplus M^\perp \longrightarrow M \oplus M^\perp,$$

and one way to study operators is to see them as entries of simpler operators (see [11, p.1059]). This is a working theory which is based on the problem that studied a class of operator matrices.

Recently, in[12], they provide several types of Hamilton type operator matrices including $\left[\begin{smallmatrix} T_1 & T_2 \\ 0 & JT_1^*J \end{smallmatrix}\right]$. In this paper, we focus on the operator matrix $\left[\begin{smallmatrix} T_1 & T_2 \\ 0 & JT_1^*J \end{smallmatrix}\right]$ where J is an unitary operator with $J^2 = -I$. To simplify our notation, we will henceforth identify $M_{(T_1,T_2)} = \{\left[\begin{smallmatrix} T_1 & T_2 \\ 0 & JT_1^*J \end{smallmatrix}\right] \in \mathcal{B}(X \oplus X) : J$ is an unitary operator with $J^2 = -I\}$. In this case, our aim is to use imformation about the entries T_1 and T_2 to investigate various Weyl type theorems of operator matrix T.

2. Preliminaries

The Hamilton system is an important branch in dynamical systems, all real physical processes with negligible dissipations, no matter whether they are classical, quantum, or relativistic, and of finite or infinite degress of freedom, can always be cast in the suitable Hamiltonian form. While infinite dimensional Hamilton operators come from the infinite dimensional Hamilton systems, and have deep mechanical background[4, 6, 7].

Definition 2.1 ([4, 12]). *Let $T : D(T) \subseteq X \times X \to X \times X$ be a densely defined closed operator. If T satisfies $(JT)^* = JT$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, then T is called an infinite dimensional Hamilton operator.*

Now we introduce a new class of operators related to Hamilton operator.

Definition 2.2 ([12]). *An operator $T \in \mathcal{B}(X)$ is called a Hamilton type operator if there exists an unitary operator J on X such that $J^2 = -I$ and $(JT)^* = JT$. In this case, we say that T is a Hamilton type operator with unitary operator J.*

Clearly, infinite dimensional Hamilton operator is Hamilton type operator.

Definition 2.3 ([8]). *Let $T \in \mathcal{B}(X)$. An operator T has the single-valued extension property, abbreviated SVEP, if, for every open set $G \subseteq \mathbb{C}$, the only analytic solution $f : G \to X$ of the equation $(T - \lambda I)f(\lambda) = 0$ for all $\lambda \in G$ is the zero function on G.*

Lemma 2.4. *Let $T \in \mathcal{B}(X)$ be a Hamilton type operator. We have:*

(a) $\sigma(T) = -\sigma(T^*)$, $\sigma_a(T) = -\sigma_a(T^*)$.
(b) $\sigma_w(T) = -\sigma_w(T^*)$, $\sigma_b(T) = -\sigma_b(T^*)$.
(c) $\sigma_e(T) = -\sigma_e(T^*)$, $\sigma_{ea}(T) = -\sigma_{ea}(T^*)$.

Proof. The proof is similar to Theorem 2.2.3 in [14]. □

Lemma 2.5. *Let J be an unitary operator with $J^2 = -I$. Then the following assertions hold:*

(a) T has the SVEP if and only if JTJ has the SVEP.
(b) If T is a Hamilton type operator with J, then
$\sigma_a(JTJ) = \sigma_a(T^*)$, $\sigma_e(JTJ) = \sigma_e(T^*)$.

Proof. (a) Since J is an unitary operator with $J^2 = -I$, then it is easily seen that $JTJ - \lambda I = J(T + \lambda I)J$ for every $\lambda \in \mathbb{C}$. Hence the assertion is immediate from Definition 2.3.

(b) To establish the assertion, it remains, by Lemma 2.4, to be seen that

$$\sigma_a(T) = -\sigma_a(JTJ),\ \sigma_e(T) = -\sigma_e(JTJ).$$

Because of the method is similar, we shall only prove the identity $\sigma_a(T) = -\sigma_a(JTJ)$. Consider an arbitrary $\lambda \in \sigma_a(JTJ)$, then there exists $\{x_n\} \subseteq X$ with $\| x_n \| = 1$ such that

$$\lim_{n\to\infty} \| (JTJ - \lambda I)x_n \| = 0$$

for every $n \in \mathbb{N}$. Moerover, if T is Hamilton type operator, then it is clear that

$$\lim_{n\to\infty} \| (T + \lambda I)Jx_n \| = 0$$

and $\|Jx_n\| = 1$, and therefore $-\lambda \in \sigma_a(T)$. Hence $\sigma_a(JTJ) \subseteq -\sigma_a(T)$.
On the other hand, we can obtain that

$$\sigma_a(T) = \sigma_a(JJTJJ) = \sigma_a(J(JTJ)J) \subseteq -\sigma_a(JTJ).$$

This completes the proof of (b). □

3. Main results

Theorem 3.1. *Let $T \in M_{(T_1,T_2)}$. Suppose that T_1 is Hamilton type operator and has the SVEP. Then the following statements are equivalent:*

(a) T_1 satisfies Weyl's theorem.
(b) T_1 satisfies a-Weyl's theorem.
(c) T_1 has property (w).
(d) T_1 has property (aw).
(e) T satisfies Weyl's theorem.
(f) T satisfies a-Weyl's theorem.
(g) T has property (w).
(h) T has property (aw).

Proof. If T_1 has the SVEP, then T_1^* has the SVEP by Corollary 3.1 of [12]. Hence $\sigma(T_1) = \sigma_a(T_1)$. Moreover, we obtain that $\pi_{00}(T_1) = \pi_{00}^a(T_1)$. Now observe that, we have $\sigma_w(T_1) = \sigma_{ea}(T_1)$. Indeed, $\sigma_{ab}(T_1) = \sigma_{ea}(T_1) \cup \mathrm{acc}\sigma_a(T_1)$ holds for every $T_1 \in \mathcal{B}(X)$, and therefore we need only to prove $\sigma_w(T_1) \subseteq \sigma_{ab}(T_1)$. If $\lambda \notin \sigma_{ab}(T_1)$, then $T_1 - \lambda I \in F_+(X)$, and $p(T_1 - \lambda I) < \infty$. Since T_1^* has the SVEP, it follows that $q(T_1 - \lambda I) < \infty$. Therefore, we conclude that $\alpha(T_1 - \lambda I) = \beta(T_1 - \lambda I)$, hence $\lambda \notin \sigma_w(T_1)$.

Assume that T_1 satisfies Weyl's theorem, then

$$\sigma(T_1)\backslash\sigma_w(T_1) = \pi_{00}(T_1).$$

It is immediate that

$$\sigma_a(T_1)\backslash\sigma_{ea}(T_1) = \pi_{00}^a(T_1),$$
$$\sigma_a(T_1)\backslash\sigma_{ea}(T_1) = \pi_{00}(T_1),$$

and

$$\sigma(T_1)\backslash\sigma_w(T_1) = \pi_{00}^a(T_1).$$

This completes the proof of the implication, (a) ⇔ (b) ⇔ (c) ⇔ (d).

Since T_1 has the SVEP, we obtain from part (a) of Lemma 2.5, JT_1^*J has the SVEP. Hence T has the SVEP. It is simple to show that, T^* has the SVEP in terms of $T^* = \begin{bmatrix} T_1^* & 0 \\ T_2^* & JT_1J \end{bmatrix}$. Hence (e) ⇔ (f) ⇔ (g) ⇔ (h).

We next claim that (a) ⇔ (e). The following equations are ture by assumption and Lemma 2.5

$$\begin{aligned} \sigma(T) = \sigma_a(T) &= \sigma_a(T) \cup S(T_1^*) \\ &= \sigma_a(T_1) \cup \sigma_a(JT_1^*J) \\ &= \sigma_a(T_1) \cup \sigma_a(T_1) \\ &= \sigma(T_1), \end{aligned}$$

$$\begin{aligned} \sigma_w(T) = \sigma_e(T) &= \sigma_e(T) \cup [S(T_1^*) \cap S(JT_1^*J)] \\ &= \sigma_e(T_1) \cup \sigma_e(JT_1^*J) \\ &= \sigma_e(T_1) \cup \sigma_e(T_1) \\ &= \sigma_w(T_1), \end{aligned}$$

where $S(T) := \{\lambda \in \mathbb{C} : T \textit{ has no SVEP at } \lambda\}$. On the other hand, we observe that $\pi_{00}(T) = \pi_{00}(T_1)$. Indeed, since $\sigma(T) = \sigma(T_1)$, it suffices to show that

$$0 < \alpha(T - \lambda I) < \infty \Leftrightarrow 0 < \alpha(T_1 - \lambda I) < \infty$$

for every $\lambda \in \mathrm{iso}\sigma(T)$.

Given an arbitrary $\lambda \in \mathrm{iso}\sigma(T)$, we note that

$$\mathcal{N}(T_1 - \lambda I) \oplus \{0\} \subseteq \mathcal{N}(T - \lambda I).$$

Hence, we conclude that

$$0 < \alpha(T - \lambda I) < \infty \Rightarrow 0 < \alpha(T_1 - \lambda I) < \infty.$$

Conversely, let $0 < \alpha(T_1 - \lambda I) < \infty$ for $\lambda \in \mathrm{iso}\sigma(T_1)$. Since T_1 is Hamilton type operator, we obtain that

$$0 < \alpha(T_1^* + \lambda I) < \infty.$$

Let $\alpha(T_1^* + \lambda I) = k < \infty$, and consider a linearly independent set $\{e_1, e_2, \cdots, e_k\} \subseteq \mathcal{N}(T_1^* + \lambda I)$. If $\sum\limits_{i=1}^{k} a_i J e_i = 0$ for $a_i \in \mathbb{C},\ i = 1, 2, \cdots, k$, then

$$0 = J \sum_{i=1}^{k} a_i J e_i = -\sum_{i=1}^{k} a_i e_i,$$

and therefore $a_i = 0$ for all $i = 1, 2, \cdots, k$. Thus $\{Je_1, Je_2, \cdots, Je_k\}$ are linearly independent set in $J\mathcal{N}(T_1^* + \lambda I)$. Hence

$$\alpha(JT_1^*J - \lambda I) = k < \infty.$$

Moreover, if $JT_1^*J - \lambda I$ is injective, then it is easy to deduce that $T_1 - \lambda I$ is also injective. So $\alpha(JT_1^*J - \lambda I) > 0$, that is, $0 < \alpha(T - \lambda I) < \infty$. Hence

$$0 < \alpha(T_1 - \lambda I) < \infty \Rightarrow 0 < \alpha(T - \lambda I) < \infty.$$

Therefore (a) $\Leftrightarrow$ (e). □

Remark 3.2. *In the above Theorem, T is Hamilton type operator with some unitary operators not necessarily J.*

Corollary 3.3. *It is true that any of T_1 or T in the statements of Theorem 3.1 can be changed to T_1^* or T^*.*

Proof. Hamilton type operator $T \in \mathcal{B}(X)$ satisfies Weyl's theorem if and only if T^* also satisfies. So we assert that it is found by assumption. Indeed, it is simple to prove that $\pi_{00}(T) = -\pi_{00}(T)^*$ by a similar method in [14]. Moreover if T satisfies Weyl's theorem, then, from Lemma 2.4, we have

$$\sigma(T^*) \backslash \sigma_w(T^*) = -\pi_{00}(T) = \pi_{00}(T^*).$$

Hence T^* satisfies Weyl's theorem. Therefore this completes the proof. □

The following simple consequence of the preceding Theorem 3.1 will be useful in the study of the spectral theory with operator matrix.

Corollary 3.4. *Let $T \in M_{(T_1,T_2)}$. If T_1^* has the SVEP, then*

(a) $\sigma(T) = \sigma(T_1)$.
(b) $\sigma_a(T) = \sigma_a(T_1)$.
(c) $\sigma_w(T) = \sigma_w(T_1)$.
(d) $\sigma_e(T) = \sigma_e(T_1)$.

Corollary 3.5. *Let T_1 be a Hamilton type operator with J and has the SVEP. Then, Weyl's theorem holds for T_1 if and only if a-Weyl's theorem holds for $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_1 \end{bmatrix}$.*

Proof. Since T_1 is Hamilton type operator with J, then $T_1 = JT_1^*J$. Hence the proof follows from Theorem3.1. □

Theorem 3.6. *Let $T \in M_{(T_1,T_2)}$. Suppose that T_2 is Hamilton type operator with J and T_1 is Hamilton type opeator with the SVEP. Then the following statements hold:*

(a) *T_1 satisfies Browder's theorem.*
(b) *T_1 satisfies a-Browder's theorem.*
(c) *T_1 has property (b).*
(d) *T_1 has property (ab).*
(e) *T satisfies Browder's theorem.*
(f) *T satisfies a-Browder's theorem.*
(g) *T has property (b).*
(h) *T has property (ab).*

Proof. Since T_1 is Hamilton type operator and has the SVEP, we obtain that

$$\sigma_w(T_1) = \sigma_b(T_1) = \sigma_{ea}(T_1) = \sigma_{ab}(T_1),$$

it is means that T_1 satisfies Browder's theorem and a-Browder's theorem. In the same way, we have

$$\sigma_a(T_1)\backslash\sigma_{ea}(T_1) = \sigma(T_1)\backslash\sigma_b(T_1) = p_{00}(T_1)$$

and

$$\sigma(T_1)\backslash\sigma_w(T_1) = \sigma_a(T_1)\backslash\sigma_{ab}(T_1) = p^a_{00}(T_1).$$

Hence T_1 have property (b) and peoperty (ab).

In terms of the operator

$$T = \begin{bmatrix} T_1 & T_2 \\ 0 & JT_1^*J \end{bmatrix},$$

it is easy to obtain that both T and T^* have the SVEP. Indeed, T is Hamilton type operator with $J' = \left[\begin{smallmatrix} 0 & J \\ J & 0 \end{smallmatrix}\right]$. Therefore, we have

$$\sigma_w(T) = \sigma_b(T) = \sigma_{ea}(T) = \sigma_{ab}(T)$$

and

$$\sigma(T) = \sigma_a(T), p_{00}(T) = p^a_{00}(T).$$

Hence T satisfies Browder's theorem, a-Browder's theorem, property (b) and property (ab), respectively. □

Corollary 3.7. *Let $T \in M_{(T_1,T_2)}$ where T_1 and T_1^* have the SVEP. Then a-Browder's theorem hold for T.*

Proof. The proof is derived directly from Theorem3.6. □

It comes naturally that the above-mentioned statements described in Theorem 3.1 and 3.6 be equivalent under what conditions holds for T.

We now determine a few concepts applied in following statement. Let $\sigma_T(x) = \mathbb{C}\backslash\rho_T(x)$ be the local spectrum of T at $x \in X$, and define the local spectral subspace of T, $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ for each subset F of $\mathbb{C}$. Assume that $\lambda \in \text{iso}\sigma(T)$, let $P_{\{\lambda\}}$ denote the spectral projector determined by the set $\{\lambda\}$ using the usual holomorphic functional calculus. For more details see [8].

Theorem 3.8. *Let $T \in \mathcal{B}(X)$. If $X_T(\{\lambda\}) = \mathcal{N}(T - \lambda I)$ for each $\lambda \in iso\sigma(T)$. Then T satisfies Weyl's theorem if and only if T satisfies Browder's theorem.*

Proof. Assume that T satisfies Browder's theorem, then $\sigma_w(T) \supseteq \sigma(T) \setminus \pi_{00}(T)$. We will show that

$$\sigma_w(T) \subseteq \sigma(T) \setminus \pi_{00}(T).$$

Let $\lambda \in \pi_{00}(T)$. Since $X_T(\{\lambda\}) = \mathcal{N}(T - \lambda I)$, it follows that $X_T(\{\lambda\})$ is finite dimensional for each $\lambda \in \text{iso}\sigma(T)$. Since T has the SVEP at $\lambda \in \text{iso}\sigma(T)$, then $P_{\{\lambda\}}(X) = X_T(\{\lambda\})$. Therefore $T - \lambda I + P_{\{\lambda\}}$ is invertible, moreover, $T - \lambda I \in W(X)$ by [2, Proposition 2]. Hence T satisfies Weyl's theorem. On the other hand, if $T \in \mathcal{B}(X)$ obeys Weyl's theorem, then T satisfies Browder's theorem. □

Corollary 3.9. *Let $T \in M_{(T_1,T_2)}$. Assume that T_2 is Hamilton type operator with J and T_1 is Hamilton type operator with the SVEP, $X_{T_1}(\{\lambda\}) = \mathcal{N}(T_1 - \lambda I)$. Then all of the statements described in Theorem 3.1 and Theorem 3.6 hold for T and T_1.*

Remark 3.10. *(a) The statements (a) to (d) of Theorem 3.1, the Hamilton type operator can be replaced by infinite dimensional Hamilton operator.*

(b) In Theorem 3.1, 3.6 and Corollary 3.4, 3.9, the conclusions still hold when the condition SVEP is changed to $T_\sigma - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$, where $T_\sigma \in \{T, T_1\}$.

(c) Theorems 3.1, 3.6, 3.8 remain true while corresponding conditions replace by the following statement:let $T = \begin{bmatrix} T_1 & T_2 \\ 0 & -T_1^* \end{bmatrix} \in \mathcal{B}(X \oplus X)$, where T_1 is an infinite dimensional Hamilton operator with $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ which has the SVEP, and T_2 is self-adjoint operator.

Recall that an operator $T \in \mathcal{B}(X)$ is called isoloid if every $\lambda \in \mathrm{iso}\sigma(T)$ is an eigenvalue of T.

Theorem 3.11. *Let $T \in M_{(T_1,T_2)}$ where T_1 and T_1^* have the SVEP.*

(a) If T_1 is isoloid and Weyl's theorem hold for T_1 and T_1^*, then Weyl's theorem holds for T.

(b) The following statements are equivalent:

(i) Weyl's theorem holds for T.

(ii) a-Weyl's theorem holds for T.

(iii) Property (w) holds for T.

Proof. (a) Since T_1 is an isoloid operator with the SVEP, it follows from Corollary 2.5 by [5] that

$$\sigma_w(\begin{bmatrix} T_1 & 0 \\ 0 & JT_1^*J \end{bmatrix}) = \sigma_w(T_1) \cup \sigma_w(JT_1^*J).$$

Therefore, $\begin{bmatrix} T_1 & 0 \\ 0 & JT_1^*J \end{bmatrix}$ satisfies Weyl's theorem from Lemma 10 by [9]. Then T satisfies Weyl's theorem from Corollary 3.9 by [5].

(b) Assume that T satisfies Weyl's theorem. Since T_1 and T_1^* have the SVEP, it follows from Corollary 3.3 and the proof of Theorem 3.1 that

$$\sigma_a(T)\backslash\sigma_{ea}(T) = \sigma(T)\backslash\sigma_w(T) = \pi_{00}(T) = \pi_{00}^a(T).$$

Hence a-Weyl's theorem and property (w) hold for T, and so we have (i) $\Rightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii). However, it is obvious that (ii) $\Rightarrow$ (i). $\square$

Corollary 3.12. *Let T_1 be Hamilton type operator with J and has the SVEP. If T_1 is isoloid and Weyl's theorem holds for T_1, then a-Weyl's theorem and property (w) hold for $T = \begin{bmatrix} T_1 & T_2 \\ 0 & T_1 \end{bmatrix}$.*

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