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Dirichlet Problem with Measurable Data in Rectifiable Domains

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Abstract. The study of the Dirichlet problem with arbitrary measurable data for harmonic functions in the unit disk $\mathbb{D}$ is due to the dissertation of Luzin.

The paper [11] was devoted to the Dirichlet problem with continuous boundary data for quasilinear Poisson equations in smooth (C^1) domains.

The present paper is devoted to the Dirichlet problem with arbitrary measurable (over natural parameter) boundary data for the quasilinear Poisson equations in any Jordan domains with rectifiable boundaries.

For this purpose, it is constructed completely continuous operators generating nonclassical solutions of the Dirichlet boundary-value problem with arbitrary measurable data for the Poisson equations $\triangle U = G$ with the sources $G \in L^p$, $p > 1$.

The latter makes it possible to apply the Leray-Schauder approach to the proof of theorems on the existence of regular nonclassical solutions of the measurable Dirichlet problem for quasilinear Poisson equations of the form $\triangle U(z) = H(z) \cdot Q(U(z))$ for multipliers $H \in L^p$ with $p > 1$ and continuous functions $Q : \mathbb{R} \to \mathbb{R}$ with $Q(t)/t \to 0$ as $t \to \infty$.

Here the boundary values are interpreted in the sense of angular (along nontangential paths) limits that are a traditional tool of the geometric function theory in comparison with variational interpretations in PDE.

As consequences, we give applications to some concrete semi-linear equations of mathematical physics, arising under modelling various physical processes such as diffusion with absorption, plasma states, stationary burning etc.

1. Introduction

The research of boundary-value problems with arbitrary measurable data is due to the famous dissertation of Luzin where he has studied the corresponding Dirichlet problem for harmonic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

In this connection, recall that the following deep result of Luzin was one of the main theorems of his dissertation, see e.g. his paper [14], dissertation [15], p. 35, and its reprint [16], p. 78, adopted to the segment $[0, 2\pi]$.

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Theorem A. *For any measurable function* $\varphi : [0, 2\pi] \to \mathbb{R}$*, there is a continuous function* $\Phi : [0, 2\pi] \to \mathbb{R}$ *such that* $\Phi' = \varphi$ *a.e. on* $[0, 2\pi]$.

Just on the basis of Theorem A, Luzin proved the next significant result of his dissertation, see e.g. [16], p. 80, formulated in terms of angular (along nontangential paths) limits that then became a traditional tool in the geometric function theory, see e.g. monographs [6], [12], [19] and [20].

Theorem B. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a* $2\pi-$*periodic measurable function. Then there is a harmonic function* u *in* $\mathbb{D}$ *such that* $u(z) \to \varphi(\vartheta)$ *for a.e.* $\vartheta \in \mathbb{R}$ *as* $z \to e^{i\vartheta}$ *along any nontangential path.*

Note that the Luzin dissertation was later on published only in Russian as book [16] with comments of his pupils Bari and Men'shov already after his death. A part of its results was also printed in Italian [17]. However, Theorem A was published in English in the Saks book [24] as Theorem VII(2.3). Hence Frederick Gehring in [7] has rediscovered Theorem B and his proof on the basis of Theorem A in fact coincided with the original proof of Luzin.

Corollary 5.1 in [21] has strengthened Theorem B as the next, see also [22].

Theorem C. *For each measurable function* $\varphi : \partial\mathbb{D} \to \mathbb{R}$*, the space of all harmonic functions* $u : \mathbb{D} \to \mathbb{R}$ *with the angular (along nontangential paths) limits* $\varphi(\zeta)$ *for a.e.* $\zeta \in \partial\mathbb{D}$ *has the infinite dimension.*

2. Definitions and preliminary remarks

First of all, recall that a **completely continuous** mapping from a metric space M_1 into a metric space M_2 is defined as a continuous mapping on M_1 which takes bounded subsets of M_1 into relatively compact ones of M_2, i.e. with compact closures in M_2. When a continuous mapping takes M_1 into a relatively compact subset of M_1, it is nowadays said to be **compact** on M_1.

The notion of completely continuous (compact) operators is due essentially, in the simplest partial cases, to Hilbert and Riesz F., see the corresponding comments of Section VI.12 in [4], and to Leray and Schauder in the general case. Recall more some definitions and the fundamental result of the celebrated paper [13].

Leray and Schauder extend as follows the Brouwer degree to compact perturbations of the identity I in a Banach space B, i.e. a complete normed linear space. Namely, given an open bounded set $\Omega \subset B$, a compact mapping $F : B \to B$ and $z \notin \Phi(\partial\Omega)$, $\Phi := I - F$, the **(LeraySchauder) topological degree** $\deg[\Phi, \Omega, z]$ of Φ in Ω over z is constructed from the Brouwer degree by approximating the mapping F over Ω by mappings F_ε with range in a finite-dimensional subspace B_ε (containing z) of B. It is showing that the Brouwer degrees $\deg[\Phi_\varepsilon, \Omega_\varepsilon, z]$ of $\Phi_\varepsilon := I_\varepsilon - F_\varepsilon$, $I_\varepsilon := I|_{B_\varepsilon}$, in $\Omega_\varepsilon := \Omega \cap B_\varepsilon$ over z stabilize for sufficiently small positive ε to a common value defining $\deg[\Phi, \Omega, z]$ of Φ in Ω over z.

This topological degree algebraically counts the number of fixed points of $F(\cdot) - z$ in Ω and conserves the basic properties of the Brouwer degree as additivity and homotopy invariance. Now, let a be an isolated fixed point of F. Then the **local (LeraySchauder) index** of a is defined by $\operatorname{ind}[\Phi, a] := \deg[\Phi, B(a, r), 0]$ for small enough $r > 0$. $\operatorname{ind}[\Phi, 0]$ is called by **index** of F. In particular, if $F \equiv 0$, correspondingly, $\Phi \equiv I$, then the index of F is equal to 1.

The fundamental Theorem 1 in [13] can be formulated in the following way:

Proposition 1. *Let* B *be a Banach space, and let* $F(\cdot, \tau) : B \to B$ *be a family of operators with* $\tau \in [0, 1]$*. Suppose that the following hypotheses hold:*

(H1) $F(\cdot, \tau)$ *is completely continuous on* B *for each* $\tau \in [0, 1]$ *and uniformly continuous with respect to the parameter* $\tau \in [0, 1]$ *on each bounded set in* B*;*

(H2) *the operator* $F := F(\cdot, 0)$ *has finite collection of fixed points whose total index is not equal to zero;*

(H3) *the collection of all fixed points of the operators* $F(\cdot, \tau)$, $\tau \in [0, 1]$*, is bounded in* B.

Then the collection of all fixed points of the family of operators $F(\cdot, \tau)$ *contains a continuum along which* τ *takes all values in* $[0, 1]$.

Let us go back to the discussion of the results of Luzin in Introduction.

Remark 1. Applying the Cantor ladder type functions, namely, continuous nondecreasing functions $C : [0, 2\pi] \to \mathbb{R}$ with $C(0) = 0$, $C(2\pi) = 1$ and $C'(t) = 0$ for a.e. $t \in [0, 2\pi]$, see e.g. Section 8.15 in [8], we may assume in Theorem A that $\Phi(0) = 0 = \Phi(2\pi)$. On the same base, using uniform continuity of the function Φ on $[0, 2\pi]$ and applying sequentially fragmentations of the segment to arbitrarily small parts, we may assume in Theorem A that $|\Phi(t)| < \varepsilon$ for every prescribed $\varepsilon > 0$ and, in particular, that $|\Phi(t)| < 1$ for all $t \in [0, 2\pi]$. Thus, in view of arbitrariness of $\varepsilon > 0$, there is the infinite collection of such Φ for each φ. Furthermore, applying series of pair of (nondecreasing and nonincreasing) functions of the Cantor ladder type on the segments $[2^{-(k+1)}\pi, 2^{-k}\pi]$, $k = 1, 2, \ldots$ it is easy to see that the space of such functions Φ has the infinite dimension.

By the proof of Theorem B, see [15], [16] or [7], $u(z) = \frac{\partial}{\partial \vartheta} U(z)$, where

$$U(re^{i\vartheta}) \;=\; \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta - t)+r^2}\, \Phi(e^{it})\, dt\,, \tag{1}$$

i.e., for a function Φ from Theorem A, u can be calculated in the explicit form

$$u(re^{i\vartheta}) \;=\; -\,\frac{r}{\pi} \int\limits_0^{2\pi} \frac{(1-r^2)\sin(\vartheta - t)}{(1-2r\cos(\vartheta - t)+r^2)^2}\, \Phi(e^{it})\, dt\,. \tag{2}$$

Remark 2. Later on, it was shown by Theorems 3 in [23] that the Luzin harmonic functions $u(z)$ can be represented as the **Poisson–Stieltjes integrals**

$$\mathbb{U}_\Phi(z) \;=\; \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} P_r(\vartheta - t)\, d\,\Phi(e^{it}) \quad \forall\, z = re^{i\vartheta},\ r \in (0,1)\,,\ \vartheta \in [-\pi, \pi]\,, \tag{3}$$

where $P_r(\Theta) = (1-r^2)/(1-2r\cos\Theta + r^2)$, $r < 1$, $\Theta \in \mathbb{R}$, is the **Poisson kernel**.

The corresponding analytic functions in $\mathbb{D}$ with the real parts $u(z)$ can be represented as the **Schwartz–Stieltjes integrals**

$$\mathbb{S}_\Phi(z) \;=\; \frac{1}{2\pi} \int\limits_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z}\, d\,\Phi(\zeta)\,, \quad z \in \mathbb{D}\,, \tag{4}$$

because of the Poisson kernel is the real part of the (analytic in the variable z) **Schwartz kernel** $(\zeta + z)/(\zeta - z)$. Integrating (4) by parts, see Lemma 1 and Remark 1 in [23], we obtain also the more convenient form of the representation

$$\mathbb{S}_\Phi(z) \;=\; \frac{z}{\pi} \int\limits_{\partial\mathbb{D}} \frac{\Phi(\zeta)}{(\zeta - z)^2}\, d\,\zeta\,, \quad z \in \mathbb{D}\,. \tag{5}$$

3. On completely continuous Dirichlet operators

Here we essentially apply the **logarithmic (Newtonian) potential $\mathcal{N}_G$ of sources** $G \in L^p(\mathbb{C})$, $p > 1$, with compact supports given by the formula:

$$\mathcal{N}_G(z) \;:=\; \frac{1}{2\pi} \int\limits_{\mathbb{C}} \ln|z - w|\, G(w)\, d\,m(w)\,. \tag{6}$$

Recall that by Theorem 2 in [11], $\mathcal{N}_G \in W^{2,p}_{\rm loc}(\mathbb{C})$ and $\triangle\mathcal{N}_G \ = \ G$ a.e., moreover, $N_G \in W^{1,q}_{\rm loc}(\mathbb{C})$ for $q > 2$ and, consequently, N_G is locally Hölder continuous. Furthermore, $N_G \in C^{1,\alpha}_{\rm loc}(\mathbb{C})$ with $\alpha = (p-2)/p$ if $p > 2$ and with any $\alpha \in (0,1)$ if $p = \infty$.

In addition, the collection $\{N_G\}$ is equicontinuous if the collection $\{G\}$ is bounded by the norm in $L^p(\mathbb{C})$ and with supports in $\mathbb{D}$ and, in addition,

$$\|N_G\|_C \ \le \ M \cdot \|G\|_p \tag{7}$$

on each compact set S in $\mathbb{C}$, where M is a constant depending only on S and, in particular, the restriction of N_G to $\overline{\mathbb{D}}$ is a completely continuous bounded linear operator, see e.g. Theorem 1 in [11].

By Theorem C in Introduction, there is a space of harmonic functions u in the unit disk $\mathbb{D}$ of the infinite dimension with the angular limits a.e. on $\partial\mathbb{D}$

$$\lim_{z\to\zeta} u(z) \ = \ \psi(\zeta) \ := \ \varphi(\zeta) \ - \ \varphi_G(\zeta)\,, \qquad \varphi_G(\zeta) := N_G(\zeta)\,. \tag{8}$$

Note that $U := u + N_G|_{\mathbb{D}}$ with such u are Hölder continuous solutions of the Poisson equation $\triangle U = G$ a.e. in the class $W^{2,p}_{\rm loc}(\mathbb{D}) \cap W^{1,q}_{\rm loc}(\mathbb{D})$, $q > 2$, with the angular limits

$$\lim_{z\to\zeta} U(z) \ = \ \varphi(\zeta) \quad \text{a.e. on } \partial\mathbb{D}\,. \tag{9}$$

By Remarks 1 and 2 such a harmonic function $u : \mathbb{D} \to \mathbb{R}$ can be obtained in the form of the real part of the analytic function

$$\mathbb{S}_\Psi(z) \ := \ \frac{z}{\pi} \int\limits_{\partial\mathbb{D}} \frac{\Psi(\zeta)}{(\zeta - z)^2}\, d\,\zeta\,, \quad z \in \mathbb{D}\,, \tag{10}$$

where Ψ is an antiderivative of the function ψ from Theorem A in Introduction.

Consequently, such a harmonic function u can be represented in the form

$$u(z) \ = \ u_0(z) \ - \ u_G(z)\,, \quad u_0(z) \ := \ \mathrm{Re}\,\mathbb{S}_\Phi(z)\,, \quad u_G(z) \ := \ \mathrm{Re}\,\mathbb{S}_{\Phi_G}(z)\,, \tag{11}$$

where Φ and Φ_G are antiderivatives of φ and φ_G in Theorem A, respectively. Note that the harmonic function u_0 does not depend on the sources G at all.

By Remark 1 the Dirichlet problem always has many solutions for each boundary date φ and source G in the sense of angular limits a.e. on $\partial\mathbb{D}$. Of course, axiom of choice by Zermelo makes it possible to choose one of such correspondence named further as a Dirichlet operator but the latter with such a random choice can be completely discontinuous.

Let us choose such a function Φ_G to guarantee that the correspondence $G \mapsto U = u + N_G|_{\mathbb{D}}$ is a Dirichlet operator $\mathcal{D}_G$ that is completely continuous on compact sets in $\mathbb{D}$ and generates solutions of the Poisson equation $\triangle U = G$ a.e. in the class $W^{2,p}_{\rm loc}(\mathbb{D}) \cap W^{1,q}_{\rm loc}(\mathbb{D})$, $q > 2$, with Dirichlet boundary condition (9).

Namely, the following function Φ_G is an antiderivative for the function φ_G:

$$\Phi_G(\zeta) \ := \ \int\limits_0^{\vartheta} N_G(e^{i\theta})\, d\,\theta \ - \ S(\vartheta)\,, \quad \zeta = e^{i\vartheta}, \quad \theta\,, \vartheta \in [0, 2\pi]\,, \tag{12}$$

where $S : [0, 2\pi] \to \mathbb{C}$ is either zero or a singular function of the form

$$S(\vartheta) \ := \ C(\vartheta) \int\limits_0^{2\pi} N_G(e^{i\theta})\, d\,\theta\,, \quad \zeta = e^{i\vartheta}, \quad \theta\,, \vartheta \in [0, 2\pi]\,, \tag{13}$$

with a singular function $C : [0, 2\pi] \to [0, 1]$ of the Cantor ladder type, i.e., C is continuous, nondecreasing, $C(0) = 0$, $C(2\pi) = 1$ and $C' = 0$ a.e., see e.g. Section 8.15 in [8].

Setting $u_G = \mathrm{Re}\, \mathbb{S}_{\Phi_G}$, it is easy to see by (7) that

$$|\Phi_G(\zeta)| \;\le\; 4\pi M \cdot \|G\|_p \qquad \forall\, \zeta \in \partial\mathbb{D} \tag{14}$$

and by (5) that, for constants C_r and C_r^* depending only on $r \in (0, 1)$,

$$|u_G(z)| \;\le\; |\mathbb{S}_{\Phi_G}(z)| \;\le\; C_r \cdot \|G\|_p\,, \quad \forall\, z \in \mathbb{D}_r\,, \tag{15}$$

$$|u_G(z_1) - u_G(z_2)| \le |\mathbb{S}_{\Phi_G}(z_1) - \mathbb{S}_{\Phi_G}(z_2)| \le C_r^* \|G\|_p |z_1 - z_2|, z_1, z_2 \in \mathbb{D}_r. \tag{16}$$

Consequently, the operator $u_G := \mathrm{Re}\, \mathbb{S}_{\Phi_G}$ is completely continuous on compact sets in $\mathbb{D}$ by the Arzela-Ascoli theorem, see e.g. Theorem IV.6.7 in [4]. Thus:

Lemma 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be measurable. Then there is a Dirichlet operator $\mathcal{D}_G$ over sources $G : \mathbb{D} \to \mathbb{C}$ in $L^p(\mathbb{D})$, $p > 1$, generating locally Hölder continuous solutions $U : \mathbb{D} \to \mathbb{R}$ of the Poisson equation $\triangle U = G$ in the class $W^{2,p}_{\mathrm{loc}} \cap W^{1,q}_{\mathrm{loc}}(\mathbb{D})$, $q > 2$, with the Dirichlet boundary condition (9) in the sense of angular limits a.e. on $\partial\mathbb{D}$, that is completely continuous over $\mathbb{D}_r$ for each $r \in (0, 1)$. Moreover, these solutions U belong to the class $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$ if $p > 2$ and with any $\alpha \in (0, 1)$ if $p = \infty$.*

Remark 3. Note that the nonlinear operator $\mathcal{D}_G$ constructed above is not bounded except the trivial case $\Phi \equiv 0$ because then $\mathcal{D}_0 = \mathbb{S}_\Phi \neq 0$. However, the restriction of the operator $\mathcal{D}_G$ to $\mathbb{D}_r$ under each $r \in (0, 1)$ is bounded at infinity in the sense that $\max\limits_{z \in \mathbb{D}_r} |\mathcal{D}_G(z)| \le M \cdot \|G\|_p$ for some $M > 0$ and all G with large enough $\|G\|_p$. Note also that by Remark 1 we are able always to choose Φ for any φ, including $\varphi \equiv 0$, which is not identically 0 in the unit disk $\mathbb{D}$.

4. On the Dirichlet problem in the unit disk

In this section we study the solvability of the Dirichlet problem for semi-linear Poisson equations of the form $\triangle U(z) = H(z) \cdot Q(U(z))$ in the unit disk $\mathbb{D}$. The Leray–Schauder approach, see Proposition 1 in Section 2, Lemma 1 and Remark 3 from the last section allow us to reduce the problem to the study of the Dirichlet problem for the linear Poisson equation.

Note that hypothesis (H2) in Section 2 will be automatically satisfied in the proof of the next theorem because the initial operator $F(\cdot) := F(\cdot, 0) \equiv 0$ and hence F has the only one fixed point (at the origin) and its index is equal to 1.

Theorem 1. *Let $\varphi : \partial\mathbb{D} \to \mathbb{R}$ be a measurable function. Suppose that $H : \mathbb{D} \to \mathbb{R}$ is a function in the class $L^p(\mathbb{D})$, $p > 1$, with compact support in $\mathbb{D}$ and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with*

$$\lim_{t \to \infty} \frac{Q(t)}{t} \;=\; 0\,. \tag{17}$$

Then there is a locally Hölder continuous solution $U : \mathbb{D} \to \mathbb{R}$ in the class $W^{2,p}_{\mathrm{loc}} \cap W^{1,q}_{\mathrm{loc}}(\mathbb{D})$ with some $q > 2$ of the semi-linear Poisson equation

$$\triangle U(z) \;=\; H(z) \cdot Q(U(z)) \qquad \text{a.e. in } \mathbb{D}\,, \tag{18}$$

with the angular limits

$$\lim_{z \to \zeta} U(z) \;=\; \varphi(\zeta) \qquad \text{a.e. on } \partial\mathbb{D}\,. \tag{19}$$

Furthermore, this solution U belongs to the class $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{D})$ with $\alpha = (p-2)/p$ if $p > 2$ and with any $\alpha \in (0, 1)$ if $p = \infty$.

Proof. If $\|H\|_p = 0$ or $\|Q\|_C = 0$, then any harmonic function in Theorem B gives the desired solution of (18). Thus, we may assume that $\|H\|_p \neq 0$ and $\|Q\|_C \neq 0$. Set $Q_*(t) = \max\limits_{|\tau|\leq t} |Q(\tau)|$, $t \in \mathbb{R}^+ := [0, \infty)$. Then the function $Q_* : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing and, moreover, by (17)

$$\lim_{t\to\infty} \frac{Q_*(t)}{t} \ = \ 0\,. \tag{20}$$

By Lemma 1 and Remark 3 we obtain the family of operators $F(G;\tau) : L^p_H(\mathbb{D}) \to L^p_H(\mathbb{D})$, where $L^p_H(\mathbb{D})$ consists of functions $G \in L^p(\mathbb{D})$ with supports in the support of H,

$$F(G;\tau) \ := \ \tau H \cdot Q(\mathcal{D}^*_G) \qquad \forall\ \tau \in [0,1] \tag{21}$$

which satisfies hypothesis H1-H3 of Theorem 1 in [13], see Proposition 1. Indeed:

H1). First of all, by Lemma 1 the function $F(G;\tau) \in L^p_H(\mathbb{D})$ for all $\tau \in [0,1]$ and $G \in L^p_H(\mathbb{C})$ because the function $Q(\mathcal{D}^*_G)$ is continuous and, furthermore, the operators $F(\cdot\,;\tau)$ are completely continuous for each $\tau \in [0,1]$ and even uniformly continuous with respect to the parameter $\tau \in [0,1]$.

H2). The index of the operator $F(\cdot\,;0)$ is obviously equal to 1.

H3). Let us assume that solutions of the equations $G = F(G;\tau)$ is not bounded in $L^p_H(\mathbb{D})$, i.e., there is a sequence of functions $G_n \in L^p_H(\mathbb{D})$ with $\|G_n\|_p \to \infty$ as $n \to \infty$ such that $G_n = F(G_n;\tau_n)$ for some $\tau_n \in [0,1]$, $n = 1, 2, \ldots$. However, then by Remark 3 we have that, for some constant $M > 0$,

$$\|G_n\|_p \ \leq \ \|H\|_p\, Q_*\left(M\,\|G_n\|_p\right)$$

and, consequently,

$$\frac{Q_*(\,M\,\|G_n\|_p)}{M\,\|G_n\|_p} \ \geq \ \frac{1}{M\,\|H\|_p} \ > \ 0 \tag{22}$$

for all large enough n. The latter is impossible by condition (20). The obtained contradiction disproves the above assumption.

Thus, by Theorem 1 in [13] there is a function $G \in L^p_H(D)$ with $F(G;1) = G$, and by Lemma 1 the function $U := \mathcal{D}^*_G$ gives the desired solution of (18). □

Remark 4. Moreover, by the proof of Theorem 1, $U = \mathcal{D}^*_G$, where $\mathcal{D}^*_G$ is the Dirichlet operator described in the last section, Lemma 1, and the support of G is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$ and on the function Q.

In addition, the source $G : \mathbb{D} \to \mathbb{C}$ is a fixed point of the nonlinear operator $\Omega_G := h \cdot Q(\mathcal{D}^*_G) : L^p_H(\mathbb{D}) \to L^p_H(\mathbb{D})$, where $L^p_H(\mathbb{D})$ consists of functions G in $L^p(\mathbb{D})$ with supports in the support of H.

5. The case of rectifiable domains

In this section we extend the result from the last section to arbitrary domains with rectifiable boundaries.

Theorem 2. *Let D be a Jordan domain with a rectifiable boundary and let a function $\varphi : \partial D \to \mathbb{R}$ be measurable over the natural parameter.*

Suppose that $H : D \to \mathbb{R}$ is in $L^p(D)$ for $p > 1$ with compact support in D and $Q : \mathbb{R} \to \mathbb{R}$ is a continuous function with

$$\lim_{t\to\infty} \frac{Q(t)}{t} \ = \ 0\,. \tag{23}$$

Then there is a locally Hölder continuous solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc} \cap W^{1,q}_{\rm loc}(D)$ with some $q > 2$ of the semi-linear Poisson equation

$$\triangle U(\xi) \ = \ H(\xi) \cdot Q(U(\xi)) \qquad \textit{a.e. in } D\,, \tag{24}$$

with the angular limits

$$\lim_{\xi\to\omega} U(\xi) = \varphi(\omega) \qquad a.e.\ on\ \partial D\,. \tag{25}$$

Furthermore, this solution U belongs to the class $C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ if $p > 2$ and with any $\alpha \in (0,1)$ if $p = \infty$.

Proof. Let c be a conformal mapping of D onto $\mathbb{D}$ that exists by the Riemann mapping theorem, see e.g. Theorem II.2.1 in [9]. Now, by the Caratheodory theorem, see e.g. Theorem II.3.4 in [9], c is extended to a homeomorphism $\tilde{c}$ of $\overline{D}$ onto $\overline{\mathbb{D}}$. Set $c_* = \tilde{c}|_{\partial D}$. If ∂D is rectifiable, then by the theorem of F. and M. Riesz length $c_*^{-1}(E) = 0$ whenever $E \subset \partial\mathbb{D}$ with $|E| = 0$, see e.g. Theorem II.C.1 and Theorems II.D.2 in [12]. Conversely, by the Lavrentiev theorem $|c_*(\mathcal{E})| = 0$ whenever $\mathcal{E} \subset \partial D$ and length $\mathcal{E} = 0$, see [16], see also the point III.1.5 in [20].

Hence c_* and c_*^{-1} transform measurable sets into measurable sets. Indeed, every measurable set is the union of a sigma-compact set and a set of measure zero, see e.g. Theorem III(6.6) in [24], and continuous mappings transform compact sets into compact sets. Thus, function $\varphi : \partial D \to \mathbb{R}$ is measurable with respect to the natural parameter on ∂D if and only if the function $\tilde{\varphi} = \varphi \circ c_*^{-1} : \partial\mathbb{D} \to \mathbb{R}$ is so.

Now, set $\tilde{H} = |C'|^2 \cdot H \circ C$, where C is the inverse conformal mapping $C := c^{-1} : \mathbb{D} \to D$. Then it is clear by the hypothesis of Theorem 2 that $\tilde{H}$ has compact support in $\mathbb{D}$ and belongs to the class $L^p(\mathbb{D})$. Consequently, by Theorem 1 there is a locally Hölder continuous solution $\tilde{U} : \mathbb{D} \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc} \cap W^{1,q}_{\rm loc}(\mathbb{D})$ with some $q > 2$ of the semi-linear Poisson equation

$$\triangle\tilde{U}(z) = \tilde{H}(z)\cdot Q(\tilde{U}(z)) \qquad \text{a.e. in } \mathbb{D} \tag{26}$$

with the angular limits

$$\lim_{z\to\zeta} \tilde{U}(z) = \tilde{\varphi}(\zeta) \qquad \text{a.e. on } \partial\mathbb{D}\,, \tag{27}$$

and, furthermore, this solution $\tilde{U}$ belongs to the class $C^{1,\alpha}_{\rm loc}(\mathbb{D})$ with $\alpha = (p-2)/p$ if $p > 2$ and with any $\alpha \in (0,1)$ if $p = \infty$.

Moreover, by Remark 4, $\tilde{U} = \mathcal{D}^*_{\tilde{G}}$, where $\mathcal{D}^*_{\tilde{G}}$ is the Dirichlet operator described in Section 3, see Lemma 1, and the support of $\tilde{G}$ is in the support of $\tilde{H}$ and the upper bound of $\|\tilde{G}\|_p$ depends only on $\|\tilde{H}\|_p$ and on the function Q. In addition, the source $\tilde{G} : \mathbb{D} \to \mathbb{C}$ is a fixed point of the nonlinear operator $\Omega_{\tilde{G}} := \tilde{H}\cdot Q(\mathcal{D}^*_{\tilde{G}}) : L^p_{\tilde{H}}(\mathbb{D}) \to L^p_{\tilde{H}}(\mathbb{D})$, where $L^p_{\tilde{H}}(\mathbb{D})$ consists of functions $\tilde{G}$ in $L^p(\mathbb{D})$ with supports in the support of $\tilde{H}$.

Next, setting $U = \tilde{U}\circ c$, by simple calculations, see e.g. Section 1.C in [1], we obtain that $\triangle U = |c'|^2\cdot\triangle\tilde{U}\circ c$ and, consequently, the function $U : D \to \mathbb{C}$ is a locally Hölder continuous solution in the class $W^{2,p}_{\rm loc} \cap W^{1,q}_{\rm loc}(D)$ with some $q > 2$ of the equation (24), and, furthermore, this solution U belongs to the class $C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ if $p > 2$ and with any $\alpha \in (0,1)$ if $p = \infty$.

It remains to show that f has the angular limits as $\xi \to \omega \in \partial D$ and satisfies the boundary condition (25) a.e. on ∂D. Indeed, by the Lindelöf theorem, see e.g. Theorem II.C.2 in [12], if ∂D has a tangent at a point ω, then $\arg\,[c_*(\omega) - c(\xi)] - \arg\,[\omega - \xi] \to$ const as $\xi \to \omega$. In other words, the images under the conformal mapping c of sectors in D with a vertex at $\omega \in \partial D$ is asymptotically the same (up to shift and turn) as sectors in $\mathbb{D}$ with a vertex at $\zeta = c_*(\omega) \in \partial\mathbb{D}$. Consequently, nontangential paths in D are transformed under c into nontangential paths in $\mathbb{D}$ and inversely a.e. on ∂D and $\partial\mathbb{D}$, respectively, because the rectifiable boundary ∂D has a tangent a.e. and c_* and c_*^{-1} keep sets of the length zero. Thus, (27) implies (25). □

Remark 5. Moreover, by the proof $\triangle U = G$ a.e., where the support of G is in the support of H and the upper bound of $\|G\|_p$ depends only on $\|H\|_p$, the function Q and the domain D.

In addition, $G = \tilde{G}\circ c$ and $U = \mathcal{D}^*_{\tilde{G}}\circ c$, where c is a conformal mapping of D onto $\mathbb{D}$, $\tilde{G} : \mathbb{D} \to \mathbb{C}$ is a fixed point of the nonlinear operator $\tilde{\Omega}_{G_*} := \tilde{H}\cdot Q(\mathcal{D}^*_{G_*}) : L^p_{\tilde{H}}(\mathbb{D}) \to L^p_{\tilde{H}}(\mathbb{D})$, where $L^p_{\tilde{H}}(\mathbb{D})$ consists of functions G_* in $L^p(\mathbb{D})$ with supports in the support of $\tilde{H} := H \circ C \cdot \overline{C'}$, $C = c^{-1}$, $\mathcal{D}^*_{\tilde{G}}$ is the Dirichlet operator described in Section 3 and associated with $\tilde{\varphi} = \varphi \circ c_*^{-1}$. Here $c_* : \partial D \to \partial\mathbb{D}$ is the homeomorphic boundary correspondence under the mapping c.

6. The Dirichlet problem in physical applications

Theorem 2 on the Dirichlet boundary-value problem with arbitrary measurable boundary data over the natural parameter in Jordan domains with rectifiable boundaries can be applied to mathematical models of physical and chemical absorption with diffusion, plasma states, stationary burning etc.

The first circle of such applications is relevant to reaction-diffusion problems. Problems of this type are discussed in [5], p. 4, and, in detail, in [2]. A nonlinear system is obtained for the density U and the temperature T of the reactant. Upon eliminating T the system can be reduced to equations of the type (24),

$$\triangle U \ = \ \sigma \cdot Q(U) \tag{28}$$

with $\sigma > 0$ and, for isothermal reactions, $Q(U) = U^\beta$ where $\beta > 0$ that is called the order of the reaction. It turns out that the density of the reactant U may be zero in a subdomain called a dead core. A particularization of results in Chapter 1 of [5] shows that a dead core may exist just if and only if $\beta \in (0, 1)$ and σ is large enough, see also the corresponding examples in [10]. In this connection, the following statement may be of independent interest.

Corollary 1. *Let D be a Jordan domain in $\mathbb{C}$ with a rectifiable boundary and let a function $\varphi : \partial D \to \mathbb{R}$ be measurable over the natural parameter.*

Suppose that $H : D \to \mathbb{R}$ is a function in the class $L^p(D)$ for $p > 1$ with compact support in D.

Then there is a locally Hölder continuous solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc} \cap W^{1,q}_{\rm loc}(D)$ with some $q > 2$ of the semi-linear Poisson equation

$$\triangle U(\xi) \ = \ H(\xi) \cdot U^\beta(\xi)\,, \quad 0 \ < \ \beta \ < \ 1\,, \qquad \text{a.e. in } D \tag{29}$$

with the angular limits

$$\lim_{\xi \to \omega} U(\xi) \ = \ \varphi(\omega) \qquad \text{a.e. on } \partial D\,. \tag{30}$$

Furthermore, this solution U belongs to the class $C^{1,\alpha}_{\rm loc}(D)$ with $\alpha = (p-2)/p$ if $p > 2$ and with any $\alpha \in (0, 1)$ if $p = \infty$.

Note also that certain mathematical models of a thermal evolution of a heated plasma lead to nonlinear equations of the type (28). Indeed, it is known that some of them have the form $\triangle\psi(u) = f(u)$ with $\psi'(0) = \infty$ and $\psi'(u) > 0$ if $u \neq 0$ as, for instance, $\psi(u) = |u|^{q-1}u$ under $0 < q < 1$, see e.g. [5]. With the replacement of the function $U = \psi(u) = |u|^q \cdot \operatorname{sign} u$, we have that $u = |U|^Q \cdot \operatorname{sign} U$, $Q = 1/q$, and, with the choice $f(u) = |u|^{q^2} \cdot \operatorname{sign} u$, we come to the equation $\triangle U = |U|^q \cdot \operatorname{sign} U = \psi(U)$.

Corollary 2. *Under hypotheses of Corollary 1, there is a locally Hölder continuous solution $U : D \to \mathbb{R}$ in the class $W^{2,p}_{\rm loc} \cap W^{1,q}_{\rm loc}(D)$ with some $q > 2$ of the semi-linear Poisson equation*

$$\triangle U(\xi) \ = \ H(\xi) \cdot |U(\xi)|^{\beta-1} U(\xi)\,, \quad 0 \ < \ \beta \ < \ 1\,, \qquad \text{a.e. in } D \tag{31}$$

such that all the conclusion of Corollary 1 hold, i.e., U is a regular nonclassical solution of the Dirichlet problem for (31) in the given sense.

Finally, we recall that in the combustion theory, see e.g. [3], [18] and the references therein, the following model equation

$$\frac{\partial u(z,t)}{\partial t} \ = \ \frac{1}{\delta} \cdot \triangle u \ + \ e^u\,, \quad t \geq 0,\ z \in D, \tag{32}$$

takes a special place. Here $u \geq 0$ is the temperature of the medium and δ is a certain positive parameter. We restrict ourselves here by the stationary case, although our approach makes it possible to study the parabolic equation (32), see [10]. Namely, the corresponding equation of the type (24) is appeared here with the function $Q(u) = e^{-|u|}$ that is bounded at all.

Corollary 3. *Under hypotheses of Corollary 1, there is a locally Hölder continuous solution* $U : D \to \mathbb{R}$ *in the class* $W^{2,p}_{\rm loc} \cap W^{1,q}_{\rm loc}(D)$ *with some* $q > 2$ *of the semi-linear Poisson equation*

$$\triangle U(\xi) \ = \ H(\xi) \cdot e^{-U(\xi)} \qquad \text{a.e. in } D \tag{33}$$

such that all the conclusion of Corollary 1 hold, i.e., U is a regular nonclassical solution of the Dirichlet problem for (33) in the given sense.

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