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Some Applications of $\eta-$Ricci Solitons to Contact Riemannian Submersions

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Abstract. The aim of this paper is to study a contact Riemannian submersion $\pi : M \to B$ between almost contact metric manifolds such that its total space M admits an $\eta-$Ricci soliton. Here, we obtain some necessary conditions for which any fiber of π and the manifold B are $\eta-$Ricci soliton, Ricci soliton, generalized quasi-Einstein, quasi-Einstein, $\eta-$Einstein or Einstein. Finally, we study the total space M of π equipped with a torqued vector field and give some characterizations for any fiber and the manifold B of such a submersion π.

1. Introduction

One of the current theories in modern physics is the study of Einstein's theory of general relativity. Besides, quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations.

A non-flat Riemannian manifold (M, g) is said to be a generalized quasi-Einstein, if the Ricci tensor of (M, g) satisfies

$$Ric(E,F) = ag(E,F) + b\alpha(E)\alpha(F) + c\beta(E)\beta(F), \tag{1}$$

where a, b, c are the functions and α, β non-zero 1-forms, such that $g(E, U) = \alpha(E)$ and $g(E, V) = \beta(E)$, for unit vector fields U, V, tangent to M. For the equation (1), if the scalar b or c is zero, then M becomes a quasi-Einstein manifold. Also, if both of the scalars b and c are zero in (1), M becomes an Einstein (for more details, we refer to [2, 5, 8]).

On the other hand, the concept of Ricci flow was introduced by R. S. Hamilton in 1982 to obtain a canonical metric on a smooth manifold. For the metrics on a manifold, the Ricci flow is an evolution equation

$$\frac{\partial}{\partial t} g(t) = -2Ric$$

which is called the heat equation. Also, he showed that the self similar solutions of such a flow are Ricci solitons which are as natural generalizations of Einstein metrics [14].

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A Riemannian manifold (M, g) is said to be a Ricci soliton, if there exists a smooth vector field (so-called potential field) v, such that

$$\frac{1}{2}\mathscr{L}_v g + Ric + \lambda g = 0$$

is satisfied. Here, $\mathscr{L}_v g$ is the Lie-derivative of the metric tensor g with respect to v, Ric is the Ricci tensor of M and λ is a constant. A Ricci soliton is denoted by (M, g, v, λ) and called shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

A more general notion of $\eta-$Ricci soliton was introduced by J.T. Cho and M. Kimura in [10]. According to their definition, a Riemannian manifold (M, g) is called $\eta-$Ricci soliton if there exists a smooth vector field v which satisfies

$$\frac{1}{2}(\mathscr{L}_v g)(E, F) + Ric(E, F) + \lambda g(E, F) + \mu\eta(E)\eta(F) = 0, \tag{2}$$

for any $E, F \in \Gamma(TM)$. Here λ and μ are functions and η is a 1-form. Note that if $\mu = 0$, then the $\eta-$Ricci soliton becomes a Ricci soliton.

Considering the geometric importance of these notions, the study of $\eta-$Ricci solitons has considerably increased in many context for the last decades: on paracontact manifolds [1, 17], on Sasakian manifolds [16], on Kenmotsu manifolds [20], on warped product manifolds [3], etc.

In the present paper, our goal is to classify any fiber and the manifold B of contact Riemannian submersion π. First, we give the Ricci tensors on the distributions $\mathscr{H}$ and $\mathscr{V}$ for such a submersion and by taking the potential field of $\eta-$Ricci soliton horizontal or vertical, we obtain that such a fiber or B is Einstein, $\eta-$Einstein, generalized quasi-Einstein, Ricci soliton or $\eta-$Ricci soliton. In the last section, we study the total space M of π equipped with a torqued vector field $\mathscr{T}$ and obtain some characterizations for contact Riemannian submersions.

2. Preliminaries

The authors recall the following notations from [4, 12, 13, 15]:

A Riemannian manifold M of dimension $(2m + 1)$ has an almost contact structure (ϕ, ξ, η) if it admits a vector field ξ (the so-called characteristic vector field), a 1-form η and a field ϕ of endomorphisms of the tangent spaces satisfying:

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi. \tag{3}$$

As a consequence of (3), we note that $\phi(\xi) = 0$ and $\eta \circ \phi = 0$.

If M is endowed with an almost contact structure (ϕ, ξ, η), then it is called an almost contact manifold. Also, a Riemannian metric g on M which satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{4}$$

for any vector fields X, Y. In this case, M has an almost contact metric structure and g is said to be a metric compatible with the almost contact structure (ϕ, ξ, η) and the almost contact metric manifold is denoted by (M, ϕ, ξ, η, g).

On the other hand, the concept of Riemannian submersion between Riemannian manifolds is very popular in Theoretical Physics as well as Differential Geometry, particularly, in general relativity and Kaluza-Klein theory. For this reason, Riemannian submersions have been studied intensively (see [18, 19]).

Now, we recall the following concepts:

Let (M^m, g) and (B^n, g') be Riemannian manifolds and $\pi : (M, g) \to (B, g')$ be a surjective C^∞-map. If π has maximal rank at any point of M, then π is called a C^∞-submersion. A fiber over any $x \in B$, $\pi^{-1}(x)$, is a closed

$r-$dimensional submanifold of M, $r = m - n$. For any $p \in M$, putting $\mathscr{V}_p = ker\pi_{*p}$, we have an integrable distribution $\mathscr{V}$ which corresponds to the foliation of M determined by the fibers of π. Therefore, one has $\mathscr{V}_p = T_p\pi^{-1}(x)$ such that $\mathscr{V}$ is called the vertical distribution. Also, $\mathscr{H}$ be the complementary distribution of $\mathscr{V}$ determined by g. Then, we have the orthogonal decomposition $T_p(M) = \mathscr{V}_p \oplus \mathscr{H}_p$, $p \in M$, such that $\mathscr{H}$ is called the horizontal distribution. We note that for any $X' \in \Gamma(TB)$, the basic vector field $\pi-$related to $X^{'}$ is named the horizontal lift of $X^{'}$. Here, $\pi_* X$ is denoted by the vector field $X^{'}$ to which X is $\pi-$related.

A map π between Riemannian manifolds M and B is called a Riemannian submersion, if the following conditions hold:

(i) π has a maximal rank,

(ii) The differential π_{*p} preserves the length of the horizontal vector fields at each point of M.

For any $E \in \Gamma(TM)$, we denote vE and hE the vertical and horizontal components of E, respectively. A Riemannian submersion $\pi : (M, g) \to (B, g^{'})$ has the following properties:

(i) $g(X, Y) = g^{'}(X^{'}, Y^{'}) \circ \pi$,

(ii) $h[X, Y]$ is the basic vector field π-related to $[X^{'}, Y^{'}]$,

(iii) $h(\nabla_X Y)$ is the basic vector field π-related to $\nabla^{'}_{X'} Y^{'}$,

(iv) for any vertical vector field V, $[X, V]$ is the vertical,

where ∇ and $\nabla^{'}$ denote the Levi-Civita connections of M and B, respectively and X, Y are the basic vector fields, π-related to $X^{'}, Y^{'}$.

Moreover, the tensor fields $\mathcal{T}$ and $\mathcal{A}$ are said to be the fundamental tensor fields on the manifold M which are defined by

$$\mathcal{T}(E, F) = \mathcal{T}_E F = h(\nabla_{vE} vF) + v(\nabla_{vE} hF),$$
$$\mathcal{A}(E, F) = \mathcal{A}_E F = v(\nabla_{hE} hF) + h(\nabla_{hE} vF),$$

for any $E, F \in \Gamma(TM)$.

The fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$ on M satisfy the following properties:

$$g(\mathcal{T}_E F, G) = -g(\mathcal{T}_E G, F) \tag{5}$$
$$g(\mathcal{A}_E F, G) = -g(\mathcal{A}_E G, F) \tag{6}$$

and

$$\mathcal{T}_V W = \mathcal{T}_W V, \tag{7}$$
$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2} v[X, Y], \tag{8}$$

for any $E, F, G \in \Gamma(TM)$, $V, W \in \mathscr{V}_p$ and $X, Y \in \mathscr{H}_p$, $p \in M$.

Note the fact that the vanishing of the tensor field $\mathcal{T}$ or $\mathcal{A}$ has some geometric meanings. For instance, if the tensor $\mathcal{A}$ vanishes identically on M, the horizontal distribution $\mathscr{H}$ is integrable. If the tensor $\mathcal{T}$ vanishes identically, any fiber of π is a totally geodesic submanifold of M.

Using the fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$, one can see that

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{9}$$
$$\nabla_V X = h(\nabla_V X) + \mathcal{T}_V X, \tag{10}$$
$$\nabla_X V = \mathcal{A}_X V + v(\nabla_X V), \tag{11}$$
$$\nabla_X Y = h(\nabla_X Y) + \mathcal{A}_X Y, \tag{12}$$

where ∇ and $\hat{\nabla}$ are the Levi-Civita connections of M and any fiber of π respectively, for any $V, W \in \mathscr{V}$ and $X, Y \in \mathscr{H}$.

The mean curvature vector field H on any fiber of Riemannian submersion π is given by

$$\mathcal{N} = rH, \tag{13}$$

such that

$$\mathcal{N} = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j \tag{14}$$

where r denotes the dimension of any fiber of π and $\{U_1, U_2, ..., U_r\}$ is an orthonormal basis of $\mathscr{V}$.

Using the equality (14), we get

$$g(\nabla_E \mathcal{N}, X) = \sum_{j=1}^{r} g\Big((\nabla_E \mathcal{T})(U_j, U_j), X\Big)$$

for any $E \in \Gamma(TM)$ and $X \in \mathscr{H}$.

Denote the horizontal divergence of the horizontal vector field X by $\check{\delta}(X)$ given by

$$\check{\delta}(X) = \sum_{i=1}^{n} g(\nabla_{X_i} X, X_i), \tag{15}$$

where $\{X_i\}_{1 \leq i \leq n}$ is an orthonormal frame of $\mathscr{H}$, such that n is also the dimension of B.

On the other hand, any fiber of π is totally umbilical, if

$$\mathcal{T}_U W = g(U, W) H, \tag{16}$$

is satisfied. Here, H is the mean curvature vector field of π in M, for any $U, W \in \mathscr{V}$.

Furthermore, the Ricci tensor Ric on M satisfies

$$\begin{aligned} Ric(X, Y) &= Ric^{'}(X^{'}, Y^{'}) \circ \pi - \frac{1}{2}\Big\{g(\nabla_X \mathcal{N}, Y) + g(\nabla_Y \mathcal{N}, X)\Big\} \\ &\quad + 2\sum_{i=1}^{n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + \sum_{j=1}^{r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y) \end{aligned} \tag{17}$$

$$\begin{aligned} Ric(U, W) &= \hat{Ric}(U, W) + g(\mathcal{N}, \mathcal{T}_U W) - \sum_{i=1}^{n} g\Big((\nabla_{X_i} \mathcal{T})(U, W), X_i\Big) \\ &\quad - \sum_{i=1}^{n} g(\mathcal{A}_{X_i} U, \mathcal{A}_{X_i} W) \end{aligned} \tag{18}$$

where $\{X_i\}$ and $\{U_j\}$ are the orthonormal basis of $\mathscr{H}$ and $\mathscr{V}$ respectively, for any $X, Y \in \mathscr{H}$ and $U, V \in \mathscr{V}$.

2.1. *Contact Riemannian submersions*

Let M^{2m+1} and B^{2n+1} be $C^{\infty}-$Riemannian manifolds with the almost contact metric structures (ϕ, ξ, η, g) and $(\phi^{'}, \xi^{'}, \eta^{'}, g^{'})$ respectively.

A Riemannian submersion $\pi : (M^{2m+1}, g) \rightarrow (B^{2n+1}, g^{'})$ is called a contact Riemannian submersion if the following conditions hold:

a) $\pi_* \xi = \xi^{'}$,

b) $\pi_* \circ \phi = \phi^{'} \circ \pi_*$.

For the contact Riemannian submersion $\pi : (M^{2m+1}, g) \to (B^{2n+1}, g^{'})$, the following properties are satisfied:

1. ϕX is the basic vector field $\pi-$related to $\phi^{'} X^{'}$,

2. $h(S(X, Y))$ is the basic vector field $\pi-$related to $S^{'}(X^{'}, Y^{'})$,

3. $h((\nabla_X \phi)Y)$ is the basic vector field $\pi-$related to $(\nabla^{'}_{X^{'}} \phi^{'})Y^{'}$,

where $S = N_\phi + 2d\eta \otimes \xi$ and $S^{'} = N^{'}_{\phi^{'}} + 2d\eta^{'} \otimes \xi^{'}$ are the normality tensor fields of the manifolds M, B such that N_ϕ and $N^{'}_{\phi^{'}}$ are the Nijenhius tensors of ϕ and $\phi^{'}$, respectively and X, Y are basic vector fields on M, $\pi-$related to $X^{'}, Y^{'}$ on B, respectively. Also, we here note that the vertical distribution $\mathscr{V}$ and horizontal distribution $\mathscr{H}$ of dimension $2r$ and $2n + 1$, respectively, such that $r = m - n$.

Considering above properties, note that the followings are satisfied:

(i) The distributions $\mathscr{V}$ and $\mathscr{H}$ are $\phi-$invariant,

(ii) The characteristic vector field ξ is horizontal,

(iii) Since ξ is horizontal, we have $\eta(U) = 0$ for any $U \in \mathscr{V}$ and this implies $\mathscr{V}_p \subset ker\eta_p$, for any $p \in M$ (for details, see [12]).

3. Contact Riemannian Submersions whose total space admits an $\eta-$Ricci Soliton

Now, we recall the following lemma from [11]:

Lemma 3.1. *Let $\pi : (M, g) \to (B, g')$ be a Riemannian submersion between Riemannian manifolds.The followings are equivalent to each other:*

(i) the vertical distribution $\mathscr{V}$ is parallel,

(ii) the horizontal distribution $\mathscr{H}$ is parallel,

(iii) the fundamental tensor fields $\mathcal{T}$ and $\mathcal{A}$ vanish, identically.

Throughout this paper, we assume the following:

Assumption: A contact Riemannian submersion $\pi : (M, g) \to (B, g^{'})$ is defined between almost contact metric manifolds (M, ϕ, ξ, η, g) and $(B, \phi^{'}, \xi^{'}, \eta^{'}, g^{'})$.

We note that $\{X_i, \xi\}_{1 \le i \le 2n}$ and $\{U_j\}_{1 \le j \le 2r}$ are the local orthonormal frames of $\mathscr{H}$ and $\mathscr{V}$, respectively and using (17)-(18), we can give the following:

Lemma 3.2. *Let $\pi : (M, g) \to (B, g^{'})$ be a contact Riemannian submersion between manifolds. Then, the Ricci tensor of M satisfies*

$$\begin{aligned} Ric(U,W) &= \hat{Ric}(U,W) + g(\mathcal{N},\mathcal{T}_U W) - \sum_{i=1}^{2n}\Big\{g\Big((\nabla_{X_i}\mathcal{T})(U,W),X_i\Big) \\ &\quad + g(\mathcal{A}_{X_i}U,\mathcal{A}_{X_i}W)\Big\} - g\Big((\nabla_\xi\mathcal{T})(U,W),\xi\Big) - g(\mathcal{A}_\xi U,\mathcal{A}_\xi W), \end{aligned} \tag{19}$$

$$\begin{aligned} Ric(X,Y) &= Ric'(X',Y')\circ\pi - \frac{1}{2}\Big(\mathscr{L}_{\mathcal{N}}g\Big)(X,Y) \\ &\quad + 2\sum_{i=1}^{2n} g(\mathcal{A}_X X_i,\mathcal{A}_Y X_i) + 2g(\mathcal{A}_X\xi,\mathcal{A}_Y\xi) + \sum_{j=1}^{2r} g(\mathcal{T}_{U_j}X,\mathcal{T}_{U_j}Y), \end{aligned} \tag{20}$$

$$\begin{aligned} Ric(X,\xi) &= Ric'(X',\xi')\circ\pi - \frac{1}{2}\Big(\mathscr{L}_{\mathcal{N}}g\Big)(X,\xi) + 2\sum_{i=1}^{2n} g(\mathcal{A}_X X_i,\mathcal{A}_\xi X_i) \\ &\quad \sum_{j=1}^{2r} g(\mathcal{T}_{U_j}X,\mathcal{T}_{U_j}\xi), \end{aligned} \tag{21}$$

$$\begin{aligned} Ric(\xi,\xi) &= Ric'(\xi',\xi')\circ\pi - g(\nabla_\xi\mathcal{N},\xi) + 2\sum_{i=1}^{2n} g(\mathcal{A}_\xi X_i,\mathcal{A}_\xi X_i) \\ &\quad + \sum_{j=1}^{2r} g(\mathcal{T}_{U_j}\xi,\mathcal{T}_{U_j}\xi), \end{aligned} \tag{22}$$

where Ric' and $\hat{Ric}$ denote the Ricci tensors of B and any fiber of π respectively, for any $U,V\in\mathscr{V}$ and $X,Y\in\mathscr{H}$, $\pi-$related to X',Y'.

Using the equalities (19)-(22) in Lemma 3.2, we have the following characterizations:

Theorem 3.3. *Let (M,g,V,λ) be an $\eta-$Ricci soliton with vertical potential field V and let $\pi:(M,g)\to(B,g')$ be a contact Riemannian submersion. If one of the conditions in Lemma 3.1 is satisfied, then any fiber of π admits a Ricci soliton with potential field V.*

Proof. Since M admits an $\eta-$Ricci soliton with vertical potential field V, from (2), we can write

$$\frac{1}{2}\Big\{g(\nabla_U V,W) + g(\nabla_W V,U)\Big\} + Ric(U,W) + \lambda g(U,W) + \mu\eta(U)\eta(W) = 0, \tag{23}$$

for any $U,W\in\mathscr{V}$. Also $\eta(U)=\eta(W)=0$, because ξ is horizontal. Using (9) in (23), it follows

$$\frac{1}{2}\Big\{g(\hat{\nabla}_U V,W) + g(\hat{\nabla}_W V,U)\Big\} + Ric(U,W) + \lambda g(U,W) = 0. \tag{24}$$

Applying (19) to the equation (24), it gives

$$\begin{aligned} &\frac{1}{2}(\mathscr{L}_V g)(U,W) + \hat{Ric}(U,W) + g(\mathcal{N},\mathcal{T}_U W) - \sum_{i=1}^{2n}\Big\{g((\nabla_{X_i}\mathcal{T})(U,W),X_i) \\ &+ g(\mathcal{A}_{X_i}U,\mathcal{A}_{X_i}W)\Big\} - g((\nabla_\xi\mathcal{T})(U,W),\xi) - g(\mathcal{A}_\xi U,\mathcal{A}_\xi W) \\ &+ \lambda g(U,W) = 0. \end{aligned} \tag{25}$$

Since one of the conditions in Lemma 3.1 is satisfied, the eq. (25) is equivalent to

$$\frac{1}{2}(\mathscr{L}_V\hat{g})(U,W) + \hat{Ric}(U,W) + \lambda\hat{g}(U,W) = 0,$$

which means any fiber of π is a Ricci soliton. □

Example 3.4. *Let (M, J, g) be an almost Hermitian manifold and $(M', \phi', \xi', \eta', g')$ be an almost contact metric manifold. We consider the Riemannian product manifold $M \times M'$ and we set*

$$\bar{\phi}(X, X') = (JX, \phi' X'),$$
$$\bar{\eta}(X, X') = \eta'(X'),$$
$$\bar{\xi} = (0, \xi'),$$

for any $(X, X') \in \Gamma(TM \times TM')$. Then, $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is an almost contact metric structure on $M \times M'$, where

$$\bar{g}((X, X'), (Y, Y') = g(X, Y) + g'(X', Y'),$$

for any $(X, X'), (Y, Y') \in \Gamma(TM \times TM')$.
Now, we consider a projection map

$$\begin{aligned} \pi : M \times M' &\to M' \\ (x, x') &\to x'. \end{aligned}$$

Thus, we have

$$\begin{aligned} \pi_*(\bar{\phi}(X, X')) &= \pi_*(JX, \phi' X') = \phi' X' \\ &= \phi'(\pi_*(X, X')), \end{aligned}$$

for any $(X, X') \in \Gamma(TM \times TM')$. Then, it follows

$$\pi_* \bar{\phi} = \phi' \pi_*$$

(for details, see [9]).

Let $M = S^6(1)$ be a hypersphere with radius 1 centered at the origin O. It is known that S^6 has the canonical nearly Kaehlerian structure. Also, let $M' = S^5(\frac{\sqrt{5}}{2})$ be a hypersphere with radius $\frac{\sqrt{5}}{2}$ centered at the origin O of the almost Hermitian manifold $(\mathbb{R}^6, J, <,>)$, where J and $<,>$ is the standart complex structure and Euclidean metric on $\mathbb{R}^6$, respectively.

Let N be a unit normal vector field of $S^5(\frac{\sqrt{5}}{2})$. Then, $JN \in \Gamma(TS^5)$ and we set

$$\xi' = -JN$$
$$JX = \phi' X + \eta'(X)N.$$

Therefore, one can see that (ϕ', ξ', η', g') is an almost contact metric structure on $S^5(\frac{\sqrt{5}}{2})$. We here note that g' is the induced metric on $S^5(\frac{\sqrt{5}}{2})$ from $\mathbb{R}^6$.

On the other hand, we consider the Riemannian product

$$M^n = S^{n_1}(r_1) \times S^{n_2}(r_2) \times \ldots \times S^{n_p}(r_p),$$

where $n_1, n_2, \ldots, n_p \geq 2$ and $\frac{n_1 - 1}{r_1^2} = \frac{n_2 - 1}{r_2^2} = \ldots = \frac{n_p - 1}{r_p^2}$. Chen and Deshmukh showed that $(M^n, \bar{g}, x^\top, \lambda)$ is a shrinking Ricci soliton. Here, $x^\top$ is the tangential part of position vector field x with respect to origin (see [6]).

Considering all the above statements, we have

$$\pi : S^6(1) \times S^5(\frac{\sqrt{5}}{2}) \to S^5(\frac{\sqrt{5}}{2})$$

is a contact Riemannian submersion such that the total space $S^6(1) \times S^5(\frac{\sqrt{5}}{2})$ is a Ricci soliton.

Theorem 3.5. *Let (M, g, V, λ) be an $\eta-$Ricci soliton with vertical potential field V and let $\pi : (M, g) \to (B, g')$ be a contact Riemannian submersion. If one of the conditions in Lemma 3.1 is satisfied, then B is an $\eta-$Einstein.*

Proof. Case I. For any horizontal vector fields $X, Y \neq \xi$, we can write

$$\frac{1}{2}\Big\{g(\nabla_X V, Y) + g(\nabla_Y V, X)\Big\} + Ric(X, Y) + \lambda g(X, Y) + \mu\eta(X)\eta(Y) = 0. \tag{26}$$

Also, using (11) in Lie derivative, one has

$$g(\nabla_X V, Y) + g(\nabla_Y V, X) = g(\mathcal{A}_X V, Y) + g(\mathcal{A}_Y V, X) \tag{27}$$

and using the equalities (6) and (8) in (27), then

$$(\mathscr{L}_V g)(X, Y) = 0$$

is found. Using (20) in the Eq. (26), it gives

$$\begin{aligned} Ric^{'}(X^{'}, Y^{'}) \circ \pi - \tfrac{1}{2}(\mathscr{L}_N g)(X, Y) + 2\textstyle\sum_{i=1}^{2n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + 2g(\mathcal{A}_X \xi, \mathcal{A}_Y \xi) \\ + \textstyle\sum_{j=1}^{2r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y) + \lambda g(X, Y) + \mu\eta(X)\eta(Y) = 0. \end{aligned} \tag{28}$$

Since one of the conditions in Lemma 3.1 is satisfied, the Eq. (28) becomes

$$Ric^{'}(X^{'}, Y^{'}) \circ \pi + \lambda g(X, Y) + \mu\eta(X)\eta(Y) = 0.$$

The last equation is equivalent to

$$\Big(Ric^{'}(X^{'}, Y^{'}) + \lambda g^{'}(X^{'}, Y^{'}) + \mu\eta^{'}(X^{'})\eta^{'}(Y^{'})\Big) \circ \pi = 0,$$

which gives

$$Ric^{'}(X^{'}, Y^{'}) + \lambda g^{'}(X^{'}, Y^{'}) + \mu\eta^{'}(X^{'})\eta^{'}(Y^{'}) = 0, \tag{29}$$

where $X, Y \neq \xi \in \mathscr{H}$ are $\pi-$related to $X^{'}, Y^{'} \in \Gamma(TB)$.

Case II. For any horizontal vector field $X \neq \xi$, the Eq. (2) becomes

$$\frac{1}{2}\Big(g(\nabla_X V, \xi) + g(\nabla_\xi V, X)\Big) + Ric(X, \xi) + \lambda g(X, \xi) + \mu\eta(X)\eta(\xi) = 0. \tag{30}$$

Using (11) in (30), one obtains

$$\frac{1}{2}\Big(g(\mathcal{A}_X V, \xi) + g(\mathcal{A}_\xi V, X)\Big) + Ric(X, \xi) + \lambda g(X, \xi) + \mu\eta(X)\eta(\xi) = 0. \tag{31}$$

Considering the equalities (6), (8) in (31), it gives

$$Ric(X, \xi) + \lambda g(X, \xi) + \mu\eta(X)\eta(\xi) = 0.$$

Indeed, applying (21) to the last equality, it follows

$$\begin{aligned} Ric^{'}(X^{'}, \xi^{'}) \circ \pi - \tfrac{1}{2}\Big(\mathscr{L}_N g\Big)(X, \xi) + 2\textstyle\sum_{i=1}^{2n} g(\mathcal{A}_X X_i, \mathcal{A}_\xi X_i) \\ + \textstyle\sum_{j=1}^{2r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} \xi) + \lambda g(X, \xi) + \mu\eta(X)\eta(\xi) = 0. \end{aligned} \tag{32}$$

From Lemma 3.1, the equation (32) is equivalent to

$$Ric^{'}(X^{'}, \xi^{'}) \circ \pi + \lambda g(X, \xi) + \mu\eta(X)\eta(\xi) = 0,$$

which means

$$Ric^{'}(X^{'}, \xi^{'}) + \lambda g^{'}(X^{'}, \xi^{'}) + \mu\eta^{'}(X^{'})\eta^{'}(\xi^{'}) = 0, \tag{33}$$

where $\xi \in \mathscr{H}$, $\pi-$related to $\xi^{'} \in \Gamma(TB)$.

Case III. Finally, choosing $X = Y = \xi$, the Eq. (2) gives

$$g(\nabla_\xi V, \xi) + Ric(\xi, \xi) + \lambda g(\xi, \xi) + \mu\eta(\xi)\eta(\xi) = 0. \tag{34}$$

Using (11) in (34), one has

$$g(\mathcal{A}_\xi V, \xi) + Ric(\xi, \xi) + \lambda g(\xi, \xi) + \mu\eta(\xi)\eta(\xi) = 0. \tag{35}$$

Indeed, using (22) in (35), it follows

$$\begin{aligned} &g(\mathcal{A}_\xi V, \xi) + Ric^{'}(\xi^{'}, \xi^{'}) \circ \pi - g(\nabla_\xi \mathcal{N}, \xi) + 2\textstyle\sum_{i=1}^{2n} g(\mathcal{A}_\xi X_i, \mathcal{A}_\xi X_i) \\ &\quad + \textstyle\sum_{j=1}^{2r} g(\mathcal{T}_{U_j}\xi, \mathcal{T}_{U_j}\xi) + \lambda g(\xi, \xi) + \mu\eta(\xi)\eta(\xi) = 0. \end{aligned} \tag{36}$$

Since $\mathcal{A}_\xi \xi$ vanishes identically and one of the conditions in Lemma 3.1 is satisfied, the Eq. (36) is equivalent to

$$Ric^{'}(\xi^{'}, \xi^{'}) \circ \pi + \lambda g(\xi, \xi) + \mu\eta(\xi)\eta(\xi) = 0,$$

which gives

$$Ric^{'}(\xi^{'}, \xi^{'}) + \lambda g^{'}(\xi^{'}, \xi^{'}) + \mu\eta^{'}(\xi^{'})\eta^{'}(\xi^{'}) = 0. \tag{37}$$

As a result of the equalities (29), (33) and (37), it is obtained that the almost contact metric manifold B is an $\eta-$Einstein. □

Theorem 3.6. *Let $(M, g, \mathcal{Z}, \lambda)$ be an $\eta-$Ricci soliton with horizontal potential field $\mathcal{Z}$ and let $\pi : (M, g) \to (B, g^{'})$ be a contact Riemannian submersion with totally umbilical fibers. If the horizontal distribution $\mathscr{H}$ is integrable, then any fiber of π is an Einstein.*

Proof. Since the total space of π admits an $\eta-$Ricci soliton, putting the equality (10) in (2), we have

$$\frac{1}{2}\Big\{g(\mathcal{T}_U \mathcal{Z}, W) + g(\mathcal{T}_W \mathcal{Z}, U)\Big\} + Ric(U, W) + \lambda g(U, W) + \mu\eta(U)\eta(W) = 0,$$

for any $U, W \in \mathscr{V}$. Since ξ is horizontal, we get $\eta(U) = \eta(W) = 0$ and using the equalities (5) and (7), the last equation gives

$$-g(\mathcal{T}_U W, \mathcal{Z}) + Ric(U, W) + \lambda g(U, W) = 0.$$

Applying (19) to the last equation, we get

$$\begin{aligned} &-g(\mathcal{T}_U W, \mathcal{Z}) + \hat{Ric}(U, W) + g(\mathcal{N}, \mathcal{T}_U W) - \sum_{i=1}^{2n} \Big\{g\Big((\nabla_{X_i}\mathcal{T})(U, W), X_i\Big) \\ &+g(\mathcal{A}_{X_i} U, \mathcal{A}_{X_i} W)\Big\} - g((\nabla_\xi \mathcal{T})(U, W), \xi) - g(\mathcal{A}_\xi U, \mathcal{A}_\xi W) + \lambda g(U, W) = 0. \end{aligned}$$

If any fiber is a totally umbilical, we note that

$$\begin{aligned} \sum_{i=1}^{2n} g((\nabla_{X_i}\mathcal{T})(U, W), X_i) + g((\nabla_\xi \mathcal{T})(U, W), \xi) &= \sum_{i=1}^{2n} g(\nabla_{X_i} H, X_i) g(U, W) \\ &\quad + g(\nabla_\xi H, \xi) g(U, W), \end{aligned} \tag{38}$$

where H is the mean curvature vector field, for any $U, W \in \mathscr{V}$. Since $\mathscr{H}$ is integrable and using (38), it follows

$$\begin{aligned} &-g(\mathcal{Z}, H) g(U, W) + \hat{Ric}(U, W) + g(\mathcal{N}, H) g(U, W) - \sum_{i=1}^{2n} g(\nabla_{X_i} H, X_i) g(U, W) \\ &-g(\nabla_\xi H, \xi) g(U, W) + \lambda g(U, W) = 0. \end{aligned} \tag{39}$$

Putting the equality (15) in (39), it is equivalent to

$$\hat{Ric}(U, W) + \Big\{2r\|H\|^2 - g(\mathcal{Z}, H) - \check{\delta}(H) + \lambda\Big\} g(U, W) = 0.$$

Therefore, any fiber of π is an Einstein. □

Particularly, if we choose the potential field $\mathcal{Z} = \xi$, we obtain:

Corollary 3.7. *Let (M, g, ξ, λ) be an $\eta-$Ricci soliton and $\pi : (M, g) \to (B, g')$ be a contact Riemannian submersion with totally umbilical fibers. If the horizontal distribution $\mathscr{H}$ is integrable, then any fiber of π is an Einstein and its Ricci tensor is given by*

$$\hat{Ric} = -(2r\|H\|^2 - \eta(H) - \check{\delta}(H) + \lambda)g,$$

where H is the mean curvature vector field.

Theorem 3.8. *Let $(M, g, \mathcal{Z}, \lambda)$ be an $\eta-$Ricci soliton with horizontal potential field $\mathcal{Z}$ and $\pi : (M, g) \to (B, g')$ be a contact Riemannian submersion. If one of the conditions in Lemma 3.1 is satisfied, then the almost contact metric manifold B admits an $\eta-$Ricci soliton with potential field $\mathcal{Z}'$, such that $\pi_*(\mathcal{Z}) = \mathcal{Z}'$.*

Proof. Case I. Let $X, Y \neq \xi$ be horizontal vectors. From the Eq. (2), we can write

$$\frac{1}{2}\Big(g(\nabla_X \mathcal{Z}, Y) + g(\nabla_Y \mathcal{Z}, X)\Big) + Ric(X, Y) + \lambda g(X, Y) + \mu\eta(X)\eta(Y) = 0.$$

Using (12), it follows

$$\frac{1}{2}\Big(g(h(\nabla_X \mathcal{Z}), Y) + g(h(\nabla_Y \mathcal{Z}), X)\Big) + Ric(X, Y) + \lambda g(X, Y) + \mu\eta(X)\eta(Y) = 0,$$

which gives

$$\frac{1}{2}(\mathscr{L}_{\mathcal{Z}'} g')(X', Y') \circ \pi + Ric(X, Y) + \lambda g(X, Y) + \mu\eta(X)\eta(Y) \quad = \quad 0. \tag{40}$$

Moreover, applying (20) to (40), we get

$$\begin{aligned}
&\frac{1}{2}(\mathscr{L}_{\mathcal{Z}'} g')(X', Y') \circ \pi + Ric'(X', Y') \circ \pi - \frac{1}{2}(\mathscr{L}_N g)(X, Y) \\
&+2\sum_{i=1}^{2n} g(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) + 2g(\mathcal{A}_X \xi, \mathcal{A}_Y \xi) + \sum_{j=1}^{2r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} Y) \\
&+\lambda g(X, Y) + \mu\eta(X)\eta(Y) = 0.
\end{aligned} \tag{41}$$

From Lemma 3.1, the equation (41) is equivalent to

$$\Big(\frac{1}{2}(\mathscr{L}_{\mathcal{Z}'} g')(X', Y') + Ric'(X', Y') + \lambda g'(X', Y') + \mu\eta'(X')\eta'(Y')\Big) \circ \pi = 0,$$

which means

$$\frac{1}{2}(\mathscr{L}_{\mathcal{Z}'} g')(X', Y') + Ric'(X', Y') + \lambda g'(X', Y') + \mu\eta'(X')\eta'(Y') = 0. \tag{42}$$

Case II. For any horizontal vector field $X \neq \xi$, considering the Eq. (12) in (2), it follows

$$\frac{1}{2}(g(h(\nabla_X \mathcal{Z}), \xi) + g(h(\nabla_\xi \mathcal{Z}), X)) + Ric(X, \xi) + \lambda g(X, \xi) + \mu\eta(X)\eta(\xi) = 0. \tag{43}$$

Using (21) in above (43)

$$\begin{aligned}
&\tfrac{1}{2}(\mathscr{L}_{\mathcal{Z}'} g')(X', \xi') \circ \pi + Ric'(X', \xi') \circ \pi - \tfrac{1}{2}(\mathscr{L}_N g)(X, \xi) \\
&+2\textstyle\sum_{i=1}^{2n} g(\mathcal{A}_X X_i, \mathcal{A}_\xi X_i) + \sum_{j=1}^{2r} g(\mathcal{T}_{U_j} X, \mathcal{T}_{U_j} \xi) + \lambda g(X, \xi) \\
&\qquad +\mu\eta(X)\eta(\xi) = 0
\end{aligned} \tag{44}$$

is obtained. Also, because Lemma 3.1 is satisfied, the equation (44) gives

$$\Big(\frac{1}{2}(\mathscr{L}_{\mathcal{Z}'}g^{'})(X^{'},\xi^{'})+Ric^{'}(X^{'},\xi^{'})+\lambda g^{'}(X^{'},\xi^{'})+\mu\eta^{'}(X^{'})\eta^{'}(\xi^{'})\Big)\circ\pi=0,$$

which implies

$$\frac{1}{2}(\mathscr{L}_{\mathcal{Z}'}g^{'})(X^{'},\xi^{'})+Ric^{'}(X^{'},\xi^{'})+\lambda g^{'}(X^{'},\xi^{'})+\mu\eta^{'}(X^{'})\eta^{'}(\xi^{'}) = 0. \tag{45}$$

Case III. Choosing $X=Y=\xi$, the Eq. (2) becomes

$$g(\nabla_\xi\mathcal{Z},\xi)+Ric(\xi,\xi)+\lambda g(\xi,\xi)+\mu\eta(\xi)\eta(\xi) = 0. \tag{46}$$

Putting (12) in (46), we have

$$g(h(\nabla_\xi\mathcal{Z}),\xi)+Ric(\xi,\xi)+\lambda g(\xi,\xi)+\mu\eta(\xi)\eta(\xi) = 0.$$

Moreover, using (22), it follows

$$\begin{aligned}&\tfrac{1}{2}(\mathscr{L}_{\mathcal{Z}'}g^{'})(\xi^{'},\xi^{'})\circ\pi+Ric^{'}(\xi^{'},\xi^{'})\circ\pi-g(\nabla_\xi\mathcal{N},\xi)+2\textstyle\sum_{i=1}^{2n}g(\mathcal{A}_\xi X_i,\mathcal{A}_\xi X_i)\\&\qquad+\textstyle\sum_{j=1}^{2r}g(\mathcal{T}_{U_j}\xi,\mathcal{T}_{U_j}\xi)+\lambda g(\xi,\xi)+\mu\eta(\xi)\eta(\xi)=0.\end{aligned} \tag{47}$$

On the other hand, since one of the conditions in Lemma 3.1 is satisfied, the equation (47) gives

$$\Big(\frac{1}{2}(\mathscr{L}_{\mathcal{Z}'}g^{'})(\xi^{'},\xi^{'})+Ric^{'}(\xi^{'},\xi^{'})+\lambda g^{'}(\xi^{'},\xi^{'})+\mu\eta^{'}(\xi^{'})\eta^{'}(\xi^{'})\Big)\circ\pi=0$$

which means

$$(\mathscr{L}_{\mathcal{Z}'}g^{'})(\xi^{'},\xi^{'})+Ric^{'}(\xi^{'},\xi^{'})+\lambda g^{'}(\xi^{'},\xi^{'})+\mu\eta^{'}(\xi^{'})\eta^{'}(\xi^{'})=0. \tag{48}$$

The equalities (42),(45) and (48) give the almost contact metric manifold B is an $\eta-$Ricci soliton with potential field $\mathcal{Z}^{'}$. □

Taking the potential field $\mathcal{N}$ of an $\eta-$Ricci soliton, we obtain a characterization as follows:

Theorem 3.9. *Let $(M,g,\mathcal{N},\lambda)$ be an $\eta-$Ricci soliton with horizontal potential field $\mathcal{N}$ and $\pi:(M,g)\to(B,g^{'})$ be a contact Riemannian submersion with totally umbilical fibers. If the horizontal distribution is integrable, then B is a generalized quasi-Einstein manifold.*

Proof. Since the total space M admits an $\eta-$Ricci soliton, putting (20) in (2), we have

$$\begin{aligned}&\frac{1}{2}(\mathscr{L}_{\mathcal{N}}g)(X,Y)+Ric^{'}(X^{'},Y^{'})\circ\pi-\frac{1}{2}(\mathscr{L}_{\mathcal{N}}g)(X,Y)+2\sum_{i=1}^{2n}g(\mathcal{A}_XX_i,\mathcal{A}_YX_i)\\&+2g(\mathcal{A}_X\xi,\mathcal{A}_Y\xi)+\sum_{j=1}^{2r}g(\mathcal{T}_{U_j}X,\mathcal{T}_{U_j}Y)+\lambda g(X,Y)+\mu\eta(X)\eta(Y)=0\end{aligned} \tag{49}$$

for any $X,Y\in\mathscr{H}$. Since the horizontal distribution $\mathscr{H}$ is integrable, the tensor field $\mathcal{A}\equiv0$. Then (49) is equivalent to

$$Ric^{'}(X^{'},Y^{'})\circ\pi+\sum_{j=1}^{2r}g(\mathcal{T}_{U_j}X,\mathcal{T}_{U_j}Y)+\lambda g(X,Y)+\mu\eta(X)\eta(Y)=0. \tag{50}$$

On the other hand, using (5) we can express

$$\begin{aligned}\sum_{j=1}^{2r} g(\mathcal{T}_{U_j}X, \mathcal{T}_{U_j}Y) &= \sum_{i,j,k=1}^{2r} g(\mathcal{T}_{U_j}X, U_i)g(\mathcal{T}_{U_j}Y, U_k)g(U_i, U_k)\\ &= \sum_{i,j=1}^{2r} g(\mathcal{T}_{U_j}U_i, X)g(\mathcal{T}_{U_j}U_i, Y)\\ &= \sum_{i,j=1}^{2r} g(\mathcal{T}_{U_j}X, U_i)g(\mathcal{T}_{U_j}Y, U_i), \end{aligned} \tag{51}$$

where $\{U_1, ..., U_{2r}\}$ denotes a local orthonormal frame of $\mathscr{V}$. Since π has totally umbilical fibers, applying (16) to (51) and using (13), we have

$$\begin{aligned}\sum_{j=1}^{2r} g(\mathcal{T}_{U_j}X, \mathcal{T}_{U_j}Y) &= g(2rH, X)g(2rH, Y)\\ &= g(\mathcal{N}, X)g(\mathcal{N}, Y). \end{aligned} \tag{52}$$

Denoting the dual 1-from of $\mathcal{N}$ by σ, then (52) yields

$$\sum_{j=1}^{2r} g(\mathcal{T}_{U_j}X, \mathcal{T}_{U_j}Y) = \sigma(X)\sigma(Y),$$

for any $X, Y \in \mathscr{H}$. Putting the last equality in (50),

$$Ric^{'}(X^{'}, Y^{'}) \circ \pi + \lambda g(X, Y) + \sigma(X)\sigma(Y) + \mu\eta(X)\eta(Y) = 0 \tag{53}$$

it follows

$$Ric^{'}(X^{'}, Y^{'}) + \lambda g^{'}(X^{'}, Y^{'}) + \sigma^{'}(X^{'})\sigma^{'}(Y^{'}) + \mu\eta^{'}(X^{'})\eta^{'}(Y^{'}) = 0.$$

Therefore, the manifold B is a generalized quasi-Einstein. □

In a particular case of Theorem 3.9, choosing the potential field $\mathcal{N} = \xi$, we obtain:

Corollary 3.10. *Let (M, g, ξ, λ) be an η-Ricci soliton and $\pi : (M, g) \to (B, g^{'})$ be a contact Riemannian submersion with totally umbilical fibers. If the horizontal distribution $\mathscr{H}$ is integrable, then B is an η-Einstein manifold.*

As another result of Theorem 3.9, we get:

Corollary 3.11. *Let $(M, g, \mathcal{N}, \lambda)$ be a Ricci soliton and $\pi : (M, g) \to (B, g^{'})$ be a contact Riemannian submersion with totally umbilical fibers. If the horizontal distribution $\mathscr{H}$ is integrable, then B is a quasi-Einstein manifold.*

4. Contact Riemannian submersions whose total space is endowed with a torqued vector field

A vector field $\mathscr{T}$ on a Riemannian manifold M is said to be a torqued, if the following equalities are satisfied

$$\nabla_E \mathscr{T} = fE + \gamma(E)\mathscr{T}, \quad \gamma(\mathscr{T}) = 0 \tag{54}$$

for any $E \in \Gamma(TM)$, where f is a function, γ is a 1-form and ∇ is the Levi-Civita connection of M. If the 1-form γ in (54) vanishes identically, then $\mathscr{T}$ is called concircular vector field (see also [7, 8]).

Theorem 4.1. *Let $\pi : (M, g) \to (B, g^{'})$ be a contact Riemannian submersion between almost contact metric manifolds such that the total space M is endowed with a torqued vector field $\mathscr{T}$. Then we have the followings:*

(i) If $\mathscr{T}$ is vertical, $\mathscr{T}$ is also torqued on any fiber of π and $\mathcal{T}_U\mathscr{T}$ vanishes identically, for any $U \in \mathscr{V}$.

(ii) If $\mathscr{T}$ is horizontal, $\mathscr{T}^{'}$ is torqued on B, where $\mathscr{T}$ is the basic vector field, $\pi-$related to $\mathscr{T}^{'}$ and the $\mathcal{A}_X\mathscr{T}$ vanishes identically, for any $X \in \mathscr{H}$.

Proof. If the vector field $\mathscr{T}$ is vertical, using (54), we can write

$$\nabla_U\mathscr{T} = fU + \gamma(U)\mathscr{T}, \quad \gamma(\mathscr{T}) = 0,$$

and combining it with Eq. (9), it follows

$$\hat{\nabla}_U\mathscr{T} + \mathcal{T}_U\mathscr{T} = fU + \gamma(U)\mathscr{T}, \quad \gamma(\mathscr{T}) = 0, \tag{55}$$

for any $U \in \mathscr{V}$. By comparing the horizontal and vertical parts of (55),

$$\hat{\nabla}_U\mathscr{T} = fU + \gamma(U)\mathscr{T}, \quad \gamma(\mathscr{T}) = 0,$$
$$\mathcal{T}_U\mathscr{T} = 0,$$

are obtained. Hence, the first equality above gives that the vector field $\mathscr{T}$ is torqued on any fiber and (*i*) is satisfied.

If the vector field $\mathscr{T}$ is horizontal, using (54), one has

$$\nabla_X\mathscr{T} = fX + \gamma(X)\mathscr{T}, \quad \gamma(\mathscr{T}) = 0,$$

and combining it with Eq. (12), it follows

$$\mathcal{A}_X\mathscr{T} + h(\nabla_X\mathscr{T}) = fX + \gamma(X)\mathscr{T}, \quad \gamma(\mathscr{T}) = 0, \tag{56}$$

for any $X \in \mathscr{H}$. By comparing the horizontal and vertical parts of (56), we obtain

$$h(\nabla_X\mathscr{T}) = fX + \gamma(X)\mathscr{T}, \quad \gamma(\mathscr{T}) = 0, \tag{57}$$
$$\mathcal{A}_X\mathscr{T} = 0.$$

Hence the first equation gives

$$\nabla^{'}_{X'}\mathscr{T}^{'} = fX^{'} + \gamma^{'}(X^{'})\mathscr{T}^{'}, \quad \gamma^{'}(\mathscr{T}^{'}) = 0,$$

which means the vector field $\mathscr{T}^{'}$ is torqued on the manifold B, such that $\mathscr{T}$ is the basic vector field π-related to $\mathscr{T}^{'}$. Therefore, the condition (*ii*) is obtained. □

From now on, we suppose that the total space M of contact Riemannian submersion π is equipped with the torqued vector field ξ.

Lemma 4.2. *Let $\pi : (M, g) \to (B, g^{'})$ be a contact Riemannian submersion with totally umbilical fibers and (M, g, ξ, λ) be an $\eta-$Ricci soliton. If the horizontal distribution $\mathscr{H}$ is integrable, then the Ricci tensor $\hat{Ric}$ of any fiber of π is given by*

$$\hat{Ric} = -\Big((\lambda + f) - \check{\delta}(H) + 2r\|H\|^2\Big)g, \tag{58}$$

where H is the mean curvature vector field.

Proof. Firstly, since ξ is a torqued on M, using (54) we can write

$$\nabla_U \xi = fU + \gamma(U)\xi, \quad \gamma(\xi) = 0, \tag{59}$$

for any $U \in \mathscr{V}$. From (10), it follows

$$\mathcal{T}_U \xi + h(\nabla_U \xi) = fU + \gamma(U)\xi, \quad \gamma(\xi) = 0$$

and then

$$\begin{aligned} &\mathcal{T}_U \xi = fU, \\ &h(\nabla_U \xi) = \gamma(U)\xi, \end{aligned} \tag{60}$$

are found.

On the other hand, since M admits an $\eta-$Ricci soliton, one has

$$\frac{1}{2}(\mathscr{L}_\xi g)(U,W) + Ric(U,W) + \lambda g(U,W) + \mu\eta(U)\eta(W) = 0. \tag{61}$$

Here we note that $\eta(U) = \eta(W) = 0$.Then, applying (10) to (61), we have

$$\frac{1}{2}\Big\{g(\mathcal{T}_U\xi, W) + g(\mathcal{T}_W\xi, U)\Big\} + Ric(U,W) + \lambda g(U,W) = 0.$$

Putting (19) and (60) in the last equality gives

$$\begin{aligned} &fg(U,W) + \hat{Ric}(U,W) + g(\mathcal{N}, \mathcal{T}_U W) - \sum_{i=1}^{2n} g((\nabla_{X_i}\mathcal{T})(U,W), X_i) \\ &-g((\nabla_\xi \mathcal{T})(U,W),\xi) - \sum_{i=1}^{2n} g(\mathcal{A}_{X_i}U, \mathcal{A}_{X_i}W) - g(\mathcal{A}_\xi U, \mathcal{A}_\xi W) + \lambda g(U,W) = 0. \end{aligned} \tag{62}$$

Since π has totally umbilical fibers and $\mathscr{H}$ is integrable, the Eq. (62) is equivalent to

$$\begin{aligned} &\hat{Ric}(U,W) + 2r\|H\|^2 g(U,W) - \sum_{i=1}^{2n}\Big\{(\nabla_{X_i}g)(U,W)g(H,X_i) + g(\nabla_{X_i}H, X_i)g(U,W)\Big\} \\ &-(\nabla_\xi g)(U,W)g(H,\xi) - g(\nabla_\xi H,\xi)g(U,W) + (\lambda + f)g(U,W) = 0, \end{aligned} \tag{63}$$

for any $U, W \in \mathscr{V}$. Applying (15) to (63), we obtain

$$\hat{Ric}(U,W) + 2r\|H\|^2 g(U,W) - \check{\delta}(H)g(U,W) + (\lambda + f)g(U,W) = 0,$$

which gives (58). □

The proof of the next lemma is given by the similar way of Lemma 4.2:

Lemma 4.3. *Let $\pi : (M,g) \to (B,g')$ be a contact Riemannian submersion with totally umbilical fibers and let (M,g,ξ,λ) be an $\eta-$Ricci soliton. If $\mathscr{H}$ is integrable, then B is Einstein.*

Theorem 4.4. *Let $\pi : (M,g) \to (B,g')$ be a contact Riemannian submersion between almost contact metric manifolds. Then, we have the following:*

(i) The vector field ξ' is a torqued on the distribution $\mathcal{D}'$, such that $TB = \mathcal{D}' \oplus Span\{\xi'\}$ and $\pi_*\xi = \xi'$.
(ii) The vector field ξ' is a concircular on the distribution $Span\{\xi'\}$.

Proof. Since ξ is a torqued on M, we get

$$\nabla_X\xi = fX + \gamma(X)\xi, \quad \gamma(\xi) = 0,$$

for any $X \in \mathscr{H}$. Also, using (12) in the last equality, it gives

$$\mathcal{A}_X\xi + h(\nabla_X\xi) = fX + \gamma(X)\xi, \quad \gamma(\xi) = 0. \tag{64}$$

If we choose the horizontal vector field $X \neq \xi$ and compare of the horizontal and vertical components of (64), it gives

$$\mathcal{A}_X\xi = 0,$$
$$h(\nabla_X\xi) = fX + \gamma(X)\xi. \quad \gamma(\xi) = 0,$$

Hence, the last equality follows

$$\nabla'_{X'}\xi' = fX' + \gamma'(X')\xi', \quad \gamma'(\xi') = 0,$$

which means ξ' is a torqued vector field on the distribution $\mathcal{D}'$.

On the other hand, if we take $X = \xi$ in (64), one has

$$\nabla_\xi\xi = \mathcal{A}_\xi\xi + h(\nabla_\xi\xi) = f\xi + \gamma(\xi)\xi, \quad \gamma(\xi) = 0$$

and it follows

$$h(\nabla_\xi\xi) = f\xi. \tag{65}$$

Since $h(\nabla_\xi\xi)$ is the basic vector field $\pi-$related to $\nabla'_{\xi'}\xi'$, the Eq. (65) is equivalent to

$$\nabla'_{\xi'}\xi' \quad = \quad f\xi',$$

which is nothing but ξ' is a concircular on the distribution $Span\{\xi'\}$. □

Theorem 4.5. *Let $\pi : (M, g) \to (B, g')$ be a contact Riemannian submersion and let (M, g, ξ, λ) be an $\eta-$Ricci soliton. If any condition in Lemma 3.1 is satisfied, then the Ricci tensor on $\mathcal{D}'$ is given by*

$$Ric' \quad = \quad -\Big((\lambda + f) + \frac{1}{2}(\gamma' \otimes \eta' + \eta' \otimes \gamma') - \mu\eta' \otimes \eta'\Big)g'.$$

Proof. Since M admits an $\eta-$Ricci soliton, we can write

$$\frac{1}{2}(\mathscr{L}_\xi g)(X, Y) + Ric(X, Y) + \lambda g(X, Y) + \mu\eta(X)\eta(Y) \quad = \quad 0, \tag{66}$$

for any $X, Y \in \mathscr{H}$. Using (54) the Lie-derivative of (66), it gives

$$\begin{aligned}
\frac{1}{2}(\mathscr{L}_\xi g)(X, Y) &= \frac{1}{2}\Big\{g(\nabla_X\xi, Y) + g(\nabla_Y\xi, X)\Big\} \\
&= \frac{1}{2}\Big\{g(h(\nabla_X\xi), Y) + g(h(\nabla_Y\xi), X)\Big\} \\
&= \frac{1}{2}\Big\{g(fX + \gamma(X)\xi, Y) + g(fY + \gamma(Y)\xi, X)\Big\} \\
&= fg(X, Y) + \frac{1}{2}\Big\{\gamma(X)\eta(Y) + \eta(X)\gamma(Y)\Big\}.
\end{aligned}$$

Putting the last statement in (66), we have

$$fg(X, Y) + \frac{1}{2}\Big\{\gamma(X)\eta(Y) + \eta(X)\gamma(Y)\Big\} + Ric(X, Y) + \lambda g(X, Y) + \mu\eta(X)\eta(Y) = 0.$$

Applying the Eq. (20) to the last equation, it gives

$$Ric^{'}(X^{'},Y^{'})\circ\pi-\frac{1}{2}(\mathscr{L}_{N}g)(X,Y)+2\sum_{i=1}^{2n}g(\mathcal{A}_{X}X_{i},\mathcal{A}_{Y}X_{i})+2g(\mathcal{A}_{X}\xi,\mathcal{A}_{Y}\xi)$$
$$+\sum_{j=1}^{2r}g(\mathcal{T}_{U_{j}}X,\mathcal{T}_{U_{j}}Y)+fg(X,Y)+\frac{1}{2}\Big\{\gamma(X)\eta(Y)+\eta(X)\gamma(Y)\Big\}+\lambda g(X,Y)$$
$$+\mu\eta(X)\eta(Y)=0.$$

Since one of the conditions of Lemma 3.1 is satisfied, we get

$$Ric^{'}(X^{'},Y^{'})\circ\pi+(\lambda+f)g(X,Y)+\frac{1}{2}\Big\{\gamma(X)\eta(Y)+\eta(X)\gamma(Y)\Big\}+\mu\eta(X)\eta(Y)=0,$$

for any horizontal vectors $X,Y\neq\xi$. Therefore, the last equation is equivalent to

$$Ric^{'}(X^{'},Y^{'})+(\lambda+f)g^{'}(X^{'},Y^{'})+\frac{1}{2}\Big\{\gamma^{'}(X^{'})\eta^{'}(Y^{'})+\eta^{'}(X^{'})\gamma^{'}(Y^{'})\Big\}$$
$$+\mu\eta^{'}(X^{'})\eta^{'}(Y^{'})=0,$$

for any vector fields $X^{'},Y^{'}\neq\xi^{'}$ and the proof is completed. □

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