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Existence of Solutions for a Class of Generalized Quasilinear Schrödinger Equations with p-Laplacian

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Abstract. In this paper we establish the existence of solutions for a class of generalized quasilinear Schrödinger equations with p-Laplacian. Our results cover some typical physics models. The main technique we use is the methods of change of variables and G-link theorem in critical point theory.

1. Introduction

In this paper we study the following generalized quasilinear elliptic equation

$$\begin{cases} \Delta_p u + uh'(u^2)[\Delta_p h(u^2)] + f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $1 < p < N$ and f is a Caratheodory function on $\Omega \times \mathbb{R}$. Moreover, h is a uncertain real function, which corresponds to some mathematical models of physical phenomena, e.g. $h(s) = s^{\frac{\sigma}{2}}$ with $\sigma \geq 1$.

Obviously, problem (1) is a classical p-Laplacian equation when h is a constant. Meanwhile, problem (1) is also viewed as a generalized quasilinear Schrödinger equation with p-Laplacian. Indeed, solutions of problem (1) are related to the existence of solutions for the following quasilinear Schrödinger equations with p-Laplacian

$$i\frac{\partial \psi}{\partial t} = -\Delta_p \psi + W(x)\psi - f(x, |\psi|^2)\psi - \kappa[\Delta_p h(|\psi|^2)]h'(|\psi|^2)\psi, \tag{2}$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ is a wave function, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential and $\kappa > 0$. Let $h(s) = s$ and $\psi(t,x) = exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and u is a positive real function. Then ψ satisfies (2) if and only if the function $u(x)$ solves the following elliptic type equation

$$-\Delta_p u + (W(x) - E)u - \kappa[\Delta_p(u^2)]u = f(x,u), \tag{3}$$

where f is a new nonlinearity. Apparently, if $W(x) = E$ and $\kappa = 1$, then (3) is just the equation in (1). It is learned that problem (1) was used for a superfluid film equation in plasma physics when $p = 2$ and $h(s) = s$,

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see [1]. Moreover, when $p = 2$ and $h(s) = \sqrt{1+s}$, problem (1) models the phenomena of the self-channeling of a high-power ultrashort laser in matter [2, 3] as well as the theory of Heidelberg ferromagnetism and magnus, dissipative quantum mechanics, condensed matter theory and fluid mechanics, see [4–7] and the references therein.

In recent years, problem (1) were studied primarily in the context $p = 2$ and $h(s) = s$. In this connection, we refer the readers to [8–16]. Recently, there appeared some works dealing with problem (1) when $p \neq 2$ and $h(s) = s$. Such as Liu [17], Liu and Zhao [18], Liu and Liu [19] proved that the existence and multiplicity of solutions by using the Morse theory when problem (1) is resonant or almost resonant near infinity at λ_1 (the first eigenvalue of p-Laplacian) from left or right side, but they did not obtain a similar result at higher variational eigenvalues of p-Laplacian. In addition, to our best knowledge, so far there is no any result on the existence of solutions for problem (1) when $p \neq 2$ and $h(s) \neq s$.

We were motivated by the above results, we devote this paper to proving the existence of weak solutions when problem (1) is resonant or almost resonant near infinity at λ_{k+1} from left side, where λ_{k+1} is a variational eigenvalue of p-Laplacian (see Section 2). In this paper, there are mainly two aspects of difficulties which need to be overcome. On one hand, h is a general uncertain function, which lead to the fact that there may be two different scales in the equation and hence the minimization argument in [9] can be not applied. On the other hand, there is no natural functions spaces for the associated energy functional to be well defined. To overcome these difficulties, we reformulate equation (1) into the standard p-Laplacian equation by using a generalized change of variables and make a slight different definition of weak solutions. The main tool we use is the G-link theorem in critical point theory.

2. Main results

In this section, we will state our existence results. Firstly, from a direct, but a bit of complex computation (we omit the details here), we observe that formally (1) is the Euler-Lagrange equation associated to the natural energy functional

$$J_0(u) = \frac{1}{p}\int_\Omega \left(1 + 2^{p-1}|uh'(u^2)|^p\right)|\nabla u|^p dx - \int_\Omega F(x,u)dx. \tag{4}$$

where $F(x,u) = \int_0^u f(x,s)ds$ is the primitive function of f. But and in general, this functional J_0 may be not well defined in the usual Sobolev spaces $W_0^{1,p}(\Omega)$ equipped with the norm

$$\|u\|^p = \int_\Omega |\nabla u|^p dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

and hence it is difficult to apply variational methods directly. To overcome this difficulty, several ideas and techniques were developed, including the constrained minimization argument [9, 20], the Nehari manifold [15, 21], the method of a change of variables [6, 8] and the perturbation method [22]. Here, we will use the method of a generalized change of variables to overcome that difficulty. To be more precise, we consider a new change of unknown $v = g^{-1}(u)$, where g is defined by

$$g'(t) = \left[1 + 2^{p-1}|g(t)h'(g^2(t))|^p\right]^{-\frac{1}{p}}, \quad \forall t \in [0, +\infty), \tag{5}$$

$$g(t) = -g(-t), \quad \forall t \in (-\infty, 0]. \tag{6}$$

Obviously, it follows from (5) and (6) that g is a nondecreasing function and $g(0) = 0$. Also, after the change of variables, from $I(u)$ we obtain the following functional

$$J(v) = J_0(g(v)) = \frac{1}{p}\int_\Omega |\nabla v|^p dx - \int_\Omega F(x, g(v))dx. \tag{7}$$

In addition, since h is a general uncertain function, we have to add an assumption on g, that is,

(g_0) there exist $\alpha \geq 1$ and $\beta > 0$ such that

$$g(t) \leq \alpha t g'(t) \leq \alpha g(t), \ \ \forall t \in \mathbb{R}^+, \tag{8}$$

$$\lim_{t \to +\infty} \frac{g^{p\alpha}(t)}{t^p} = \beta. \tag{9}$$

It is clear to see that there are many function g satisfying the above conditions (8) and (9), for example, when $h(s) = s^{\frac{\sigma}{2}}$ with $\sigma \geq 1$, the function g defined by (5) and (6) satisfies the condition (g_0), see the proof of Corollary 2.3 in Section 3. On the other hand, from the above conditions on g, we easily get

$$|g(t)| \leq \beta^{\frac{1}{p\alpha}} |t|^{\frac{1}{\alpha}}, \ \ \forall t \in \mathbb{R}. \tag{10}$$

In fact, let $l(t) = \frac{g^{p\alpha}(t)}{t^p}$. By (8), one has $l'(t) \geq 0$ and hence from (9), we know that (10) is true for any $t \in (0, +\infty)$, which together with (6), we conclude that (10) holds for any $t \in \mathbb{R}$.

Now, if we assume that f satisfies the following growth condition:

(f_0) there exist $c_0 > 0$ and $q \in (1, p^*)$ such that for all $(x, s) \in \Omega \times \mathbb{R}$,

$$|f(x, s)| \leq c_0 \left(1 + |s|^{\alpha(q-1)}\right), \tag{11}$$

where α is as in (g_0) and $p^* = \frac{Np}{N-p}$. We note that (11) is exactly the usual subcritical growth condition when $\alpha = 1$. Also, it is interesting to note that α can be greater than one. In other words, the nonlinearity term f is allowed for supercritical growth in the usual Sobolev space $W_0^{1,p}(\Omega)$ and hence J_0 is not well defined in $W_0^{1,p}(\Omega)$ in the general case. However, thanks to (10), it is very easy to see that J is well defined on $W_0^{1,p}(\Omega)$ and $J \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ under the condition (f_0). Moreover, the critical points of J are just weak solutions of the equation

$$\begin{cases} -\Delta_p v = f(x, g(v))g'(v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{12}$$

That is to say, for any $\psi \in C_0^\infty(\Omega)$,

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \psi dx = \int_\Omega f(x, g(v))g'(v)\psi dx. \tag{13}$$

Let $v = g^{-1}(u)$, a direct calculation shows that problem (12) is equivalent to problem (1), which takes $u = g(v)$ as its solution.

Motivated by the above, we will give a slight different definition of weak solutions for problem (1), as described next.

Definition 2.1. *We say u is a weak solution of problem (1) if and only if $v = g^{-1}(u) \in W_0^{1,p}(\Omega)$ is a critical point of the functional J.*

As far as we know, that definition of weak solutions was used by Liu in [17], also see [18]. Next, to state our main theorems, we recall the following p-Laplacian eigenvalue problem

$$\begin{cases} -\Delta_p v = \lambda |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{14}$$

Define

$$\Sigma = \left\{ v \in W_0^{1,p}(\Omega) : \int_\Omega |v|^p dx = 1 \right\}.$$

Then problem (14) has a sequence of eigenvalues with the variational characterization

$$\lambda_k = \inf_{\Lambda \in \Sigma_k} \sup_{v \in \Lambda} \int_\Omega |\nabla v|^p dx, \tag{15}$$

where $\Sigma_k = \{\Lambda \subset \Sigma : \text{there is a odd, continuous and surjective } \gamma : S^{k-1} \to \Lambda\}$ and S^{k-1} is the unit sphere in $\mathbb{R}^k$ (see [23]). Here, we will refer to $\{\lambda_k\}_{k\in\mathbb{N}}$ as the variational eigenvalues of $-\Delta_p$. So far, we do not know whether this represents a complete list of eigenvalues. Fortunately, even without this knowledge, this portion of the spectrum provides enough structure for the G-Linking arguments of the next section. Now we ready to state our main theorems.

Theorem 2.2. *Let $\lambda_k < \lambda_{k+1}$ be two consecutive eigenvalues defined by (15). Assume that the function g satisfies the condition (g_0). In addition, if f satisfies (f_0) and the following assumptions:*

$$\frac{\lambda_k}{\beta} < \liminf_{|t|\to\infty} \frac{pF(x,t)}{|t|^{\alpha p}} \le \limsup_{|t|\to\infty} \frac{pF(x,t)}{|t|^{\alpha p}} \le \frac{\lambda_{k+1}}{\beta} \tag{16}$$

uniformly in $x \in \Omega$, and

$$\lim_{|t|\to\infty} (f(x,t)t - \alpha p F(x,t)) = +\infty \tag{17}$$

uniformly in $x \in \Omega$, where α and β are as in (g_0). Then problem (1) has at least a weak solution.

There are many functions satisfying the conditions (f_0), (16) and (17). For example, let

$$f(x,t) = \frac{\alpha(\lambda_k + \lambda_{k+1})}{2\beta}|t|^{\alpha p-2}t - (\alpha p - 1)|t|^{\alpha p-3}t,$$

where α, β and p are as in Theorem 4. It follows the definition of $F(x,t)$ that

$$F(x,t) = \frac{\lambda_k + \lambda_{k+1}}{2\beta p}|t|^{\alpha p} - |t|^{\alpha p-1}.$$

Obviously, $F(x,t)$ satisfies the condition (16) and

$$\lim_{|t|\to\infty} (f(x,t)t - \alpha p F(x,t)) = \lim_{|t|\to\infty} |t|^{\alpha p-1} = +\infty.$$

In addition, it is also easy to check that f satisfies (f_0) for some α, p and q. This means that f satisfies the assumptions (f_0), (16) and (17).

Next, let $h(s) = s^{\frac{\sigma}{2}}$ in (1) with $\sigma \ge 1$. Then problem (1) becomes

$$\begin{cases} -\Delta_p u - \frac{\sigma}{2}[\Delta_p(|u|^\sigma)]|u|^{\sigma-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

Due to the quasilinear and non-convex term $\frac{\sigma}{2}[\Delta_p(|u|^\sigma)]|u|^{\sigma-2}u$, problem (18) is usually called the Modified Nonlinear Schrödinger Equation by scholars. Meanwhile, as we said in the introduction, problem (18) is related to some important mathematical physical models. Here, by Theorem 2.2, we will conclude the existence of weak solutions for (18). That is,

Corollary 2.3. *Let $\lambda_k < \lambda_{k+1}$ be two consecutive eigenvalues defined by (15). Assume that f satisfies the conditions (11), (16) and (17) with $\alpha = \sigma$ and $\beta = 2$. Then problem (18) has at least a weak solution when $\sigma \ge 1$.*

Remark 2.4. *The condition (17) was firstly introduced by Costa and Magalháes in [24], where $\alpha = 1$, $p = 2$ and the existence of solutions was obtained for an elliptic system. We remark that (16) and (17) imply that problem (1) may be resonant or almost resonant near infinity at λ_{k+1} from left side. Meanwhile, we do not assume that there are no other eigenvalues defined by any other way in the interval $(\lambda_k, \lambda_{k+1})$, and hence problem (1) may be also resonant at some eigenvalues. Moreover, just like we said in the introduction, D. Liu and P. Zhao [18], and J. Liu and D. Liu [19] did not obtain the existence of solutions for problem (18) when $k \neq 1$ and $\sigma \neq 2$. Hence, Corollary 2.3 is a generalization and complement of the results of [18] and [19].*

3. Proof of main results

Let $\|\cdot\|_s$ denote the norm in $L^s(\Omega)$. It is well known that the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$ is compact for $s \in (1, p^*)$. Moreover, for all $s \in (1, p^*]$, there exists $C_s > 0$ such that

$$\|u\|_s \leq C_s\|u\|. \tag{19}$$

For $\lambda_k < \lambda_{k+1}$, define

$$\mathbb{C}_{k+1} = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p dx \geq \lambda_{k+1} \int_\Omega |u|^p dx \right\}.$$

Clearly, $\mathbb{C}_{k+1}$ is a symmetric closed subset of $W_0^{1,p}(\Omega)$. In addition, let X is a Banach space, we say that the function I satisfies the Cerami condition if any sequence $\{v_n\} \subset X$ such that $I(v_n) \to c \in \mathbb{R}$ and $(1 + \|v_n\|)|I'(v_n)|_{X^*} \to 0$ as $n \to \infty$ has a convergent subsequence.

Definition 3.1. *Let Q be a submanifold of a Banach space E with relative boundary ∂Q, S be a closed subset of a Banach space F and G be a subset of $C^0(\partial Q, F \setminus S)$. We say that S and ∂Q are G-linked if for any map $h \in C^0(Q, F)$ such that $h|_{\partial Q} \in G$ there holds $h(Q) \cap S \neq \emptyset$.*

Now, we introduce the G-Linking Theorem in the critical point theory, which will play an essential role in the proof of the main theorems.

Theorem 3.2. *(G-Linking Theorem [25, 26]). Let E, F be two real Banach spaces, $S \subset F$ be a closed subset, $Q \subset E$ be a submanifold with relative boundary ∂Q and G is a subset of $C^0(\partial Q, F \setminus S)$. Set $\Gamma = \{h \in C^0(Q, F) : h|_{\partial Q} \in G\}$. Suppose that S and ∂Q are G-linked and $I \in C^1(F, \mathbb{R})$ satisfies the Cerami condition. In addition, suppose that*

(a) there exists $h_0 \in \Gamma$ such that $\sup_{x\in Q} I(h_0(x)) < +\infty$;

(b) there exist two constants a, b with $b > a$ such that $\inf_{y\in S} I(y) \geq b$ and $\sup_{x\in Q} I(h(x)) \leq a$, $\forall h \in \Gamma$.

Then, the number $c = \inf_{h\in\Gamma} \sup_{x\in Q} I(h(x))$ defines a critical value $c \geq b$ of I.

Proof of Theorem 2.2. The aim is to prove the existence of critical points for the functional J and hence problem (1) has at least a weak solution. In the following, we divide our proof in three steps.

Step 1. We verify that any Cerami sequence for J is bounded in $W_0^{1,p}(\Omega)$.

Indeed, let $\{v_n\} \subset W_0^{1,p}(\Omega)$ be a Cerami sequence for J, that is,

$$J(v_n) \to c \in \mathbb{R} \text{ and } (1 + \|v_n\|)\|J'(v_n)\| \to 0 \tag{20}$$

as $n \to \infty$. Suppose, by contradiction, that $\|v_n\| \to \infty$ as $n \to \infty$. Fixed $\varepsilon_0 > 0$, it follows from (16) and the continuity of F that there exists $M_1 = M_1(\varepsilon_0) > 0$ such that

$$|F(x,t)| \leq \frac{\lambda_{k+1} + \varepsilon_0}{p\beta}|t|^{\alpha p} + M_1, \;\; \forall (x,t) \in \Omega \times \mathbb{R}. \tag{21}$$

From (9), (10), (20) and (21), for n large enough, one has

$$\begin{aligned} \|v_n\|^p &= pJ(v_n) + p\int_\Omega F(x, g(v_n))dx \\ &\leq p(c+1) + \frac{\lambda_{k+1} + \varepsilon_0}{\beta} \int_\Omega |g(v_n)|^{\alpha p} dx + pM_1|\Omega| \\ &\leq p(c+1) + (\lambda_{k+1} + \varepsilon_0) \int_\Omega |v_n|^p dx + pM_1|\Omega|, \end{aligned} \tag{22}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Let $\widetilde{v}_n = \frac{v_n}{\|v_n\|^p}$, then $\|\widetilde{v}_n\| = 1$ and hence there exists $\widetilde{v} \in W_0^{1,p}(\Omega)$ such that $\widetilde{v}_n \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$ and $\widetilde{v}_n \to v$ strongly in $L^p(\Omega)$. It follows from (22) that $1 \leq (\lambda_{k+1} + \varepsilon_0)\|\widetilde{v}_n\|_{L^p}^p$. This means that there exists $\Omega_0 \subset \Omega$ with positive measure such that $v(x) \neq 0$ for every $x \in \Omega_0$. Thus we have $v_n \chi_{\Omega_0} \to \infty$ as $n \to \infty$, where χ_{Ω_0} denotes the characteristic function of Ω_0. In addition, it follow from (5), (6) and (9) that

$$g(t) \geq \alpha t g'(t) \geq \alpha g(t) \text{ for all } t \leq 0, \tag{23}$$

$$\lim_{t \pm \to \infty} g(t) = \pm\infty, \tag{24}$$

which implies that

$$g(v_n(x)) \to \infty \text{ as } n \to \infty \text{ for } x \in \Omega_0. \tag{25}$$

Note that (16) and (17) imply that

$$\lim_{|t| \to \infty} f(x,t)t = +\infty. \tag{26}$$

In other words, there exists $t_0 > 0$ such that $f(x,t)t \geq 0$ for all $x \in \Omega_0$ and $|t| \geq t_0$. Meanwhile, we also note that $g(t)t$ is nonnegative for all $t \in \mathbb{R}$. Hence, from (8), (23), (25), (26) and (17), we have

$$\begin{aligned} f(x, g(v_n))g'(v_n)v_n - pF(x, g(v_n)) &\geq \frac{1}{\alpha} f(x, g(v_n))g(v_n) - pF(x, g(v_n)) \\ &= \frac{1}{\alpha}\left[f(x, g(v_n))g(v_n) - \alpha p F(x, g(v_n))\right] \\ &\to +\infty \end{aligned} \tag{27}$$

as $n \to \infty$ for all $x \in \Omega_0$. It follows from (27) and Fatou's lemma that

$$\begin{aligned} &\liminf_{n\to\infty} \int_\Omega [f(x, g(v_n))g'(v_n)v_n - pF(x, g(v_n))]dx \\ &\geq \frac{1}{\alpha} \int_{\Omega_0} \liminf_{n\to\infty} [f(x, g(v_n))g(v_n) - \alpha p F(x, g(v_n))]dx \\ &= +\infty. \end{aligned} \tag{28}$$

On the other hand, it follows from (20) that there exists $M_2 > 0$ such that

$$M_2 \geq \liminf_{n\to\infty} (pJ(v_n) - \langle J'(v_n), v_n\rangle) = \liminf_{n\to\infty} \int_\Omega [f(x, g(v_n))g'(v_n)v_n - pF(x, g(v_n))]dx,$$

which contradicts with (28) and hence this completes the proof of Step 1.

Step 2. We verify that J satisfies the Cerami condition.

Indeed, let $\{v_n\} \subset W_0^{1,p}(\Omega)$ be a Cerami sequence for J. It follows from Step 1 that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$, that is, there exists $M_3 > 0$ such that

$$\|v_n\| \leq M_3, \quad \forall n \in \mathbb{N}. \tag{29}$$

Hence, we can assume that there exists a function $v \in W_0^{1,p}(\Omega)$ such that

$$\begin{cases} v_n \rightharpoonup v & \text{weakly in } W_0^{1,p}(\Omega), \\ v_n \to v & \text{strongly in } L^p(\Omega). \end{cases}$$

It follows from (20) that

$$\lim_{n\to\infty} \langle J'(v_n), v_n - v\rangle = 0. \tag{30}$$

Moreover, from (11), (5), (10), (19) and (29), one has

$$\begin{aligned}\left|\int_\Omega f(x,g(v_n))g'(v_n)(v_n-v)dx\right| &\le c_0\int_\Omega (1+|g(v_n)|^{\alpha(q-1)})|g'(v_n)||v_n-v|dx\\ &\le c_0\int_\Omega (1+\beta^{\frac{q-1}{p}}|v_n|^{q-1})|v_n-v|dx\\ &\le c_0\left(|\Omega|^{\frac{q-1}{q}}+\beta^{\frac{q-1}{p}}\|v_n\|_{L^q}^{q-1}\right)\|v_n-v\|_{L^q}\\ &\le c_0\left(|\Omega|^{\frac{q-1}{q}}+(C_qM_3\beta^{\frac{1}{p}})^{q-1}\right)\|v_n-v\|_{L^q}\\ &\to 0 \end{aligned} \tag{31}$$

as $n\to\infty$. It follows from (30) and (31) that

$$\lim_{n\to\infty}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla(v_n-v)dx=0. \tag{32}$$

Similarly, we also have

$$\lim_{n\to\infty}\int_\Omega |\nabla v|^{p-2}\nabla v\nabla(v_n-v)dx=0. \tag{33}$$

From (32) and (33), one has

$$\lim_{n\to\infty}\int_\Omega |\nabla v_n-\nabla v|^p\,dx\le\lim_{n\to\infty}\int_\Omega\left(|\nabla v_n|^{p-2}\nabla v_n-|\nabla v|^{p-2}\nabla v\right)\nabla(v_n-v)dx=0.$$

That is, $v_n\to v$ strongly in $W_0^{1,p}(\Omega)$. This completes the proof of Step 2.

Step 3. We prove that J satisfies the conditions in Theorem 3.2.

Indeed, by the continuity of F and the left hand side inequality of (18), for any $\varepsilon>0$, there exists $M_4=M_4(\varepsilon)>0$ such that

$$F(x,t)\ge\frac{\lambda_k+2\varepsilon}{p\beta}|t|^p-M_4,\ \ \forall(x,t)\in\Omega. \tag{34}$$

By (15), there exists $\Lambda\in\Sigma_k$ such that

$$\sup_{v\in\Lambda}\int_\Omega|\nabla v|^pdx\le\lambda_k+\varepsilon. \tag{35}$$

It follows from (34), (35) and (10) that

$$\begin{aligned} J(tv) &= \frac{1}{p}\|tv\|^p-\int_\Omega F(x,g(tv))dx\le\frac{1}{p}\|tv\|^p-\frac{\lambda_k+2\varepsilon}{p\beta}\int_\Omega|g(tv))|^{\alpha p}dx+M_4|\Omega|\\ &\le \frac{1}{p}\|tv\|^p-\frac{\lambda_k+2\varepsilon}{p}\|tv\|_{L^p}^p dx+M_4|\Omega|\\ &\le -\frac{\varepsilon}{p}t^p+M_4|\Omega|\\ &\to -\infty \end{aligned} \tag{36}$$

as $t\to+\infty$. In the following, letting

$$G(x,t)=F(x,t)-\frac{\lambda_{k+1}}{p\beta}|t|^{\alpha p}. \tag{37}$$

Thus we have

$$G'(x,t)t-\alpha pG(x,t)=f(x,t)t-\alpha pF(x,t).$$

From this and (17), for any $M > 0$, there exists $c_M > 0$ such that if $|t| \geq c_M$,

$$G'(x,t)t - \alpha p G(x,t) \geq M, \quad \forall x \in \Omega. \tag{38}$$

So, we obtain for $s > 0$,

$$\frac{d}{ds}\left(\frac{G(x,s)}{s^{\alpha p}}\right) = \frac{G'(x,s)s - \alpha p G(x,s)}{s^{\alpha p+1}} \geq \frac{M}{s^{\alpha p+1}}. \tag{39}$$

Integrating (39) over the interval $[t,T] \subset [c_M, +\infty)$, we get

$$\frac{G(x,t)}{t^{\alpha p}} \leq \frac{G(x,T)}{T^{\alpha p}} + \frac{M}{\alpha p}\left(\frac{1}{T^{\alpha p}} - \frac{1}{t^{\alpha p}}\right). \tag{40}$$

Now, we claim that there exists $M_5 > 0$ such that

$$G(x,t) \leq M_5 \text{ for all } t \in \mathbb{R} \text{ and } x \in \Omega. \tag{41}$$

In fact, by the right side inequality of (16), we have

$$\begin{aligned}
\limsup_{T\to+\infty} G(x,T) &= \limsup_{T\to+\infty}\left(F(x,T) - \frac{\lambda_{k+1}}{p\beta}|T|^{\alpha p}\right) \\
&\leq \limsup_{T\to+\infty} \frac{|T|^{\alpha p}}{p}\left(\frac{pF(x,T)}{|T|^{\alpha p}} - \frac{\lambda_{k+1}}{\beta}\right) \\
&\leq 0.
\end{aligned} \tag{42}$$

It follows from (40) and (42) that

$$G(x,t) \leq -\frac{M}{\alpha p} \tag{43}$$

for all $t \in (c_M, +\infty)$ and $x \in \Omega$. Similarly, we can also conclude that (43) holds for all $t \in (-\infty, -c_M)$ and $x \in \Omega$. Then, from the arbitrariness of M, one gets $\lim_{|t|\to\infty} G(x,t) \to -\infty$ uniformly for $x \in \Omega$. This means that (41) is true. Therefore, from (37), (10) and (41), we have for any $v \in \mathbb{C}_{k+1}$,

$$\begin{aligned}
J(v) &= \frac{1}{p}\|v\|^p - \int_\Omega F(x,g(v))dx \\
&= \frac{1}{p}\|v\|^p - \int_\Omega \left(G(x,g(v)) + \frac{\lambda_{k+1}}{p\beta}|g(v)|^{\alpha p}\right)dx \\
&\geq \frac{1}{p}(\|v\|^p - \lambda_{k+1}\|v\|^p_{L^p}) - \int_\Omega G(x,g(v))dx \\
&\geq -M_5|\Omega|.
\end{aligned} \tag{44}$$

It follows from (36) and (44) that there exists $\rho_0 > 0$ such that

$$\alpha_0 = \max_{v\in\Lambda, t\geq\rho_0} \varphi(tv) < -M_5|\Omega| = \beta_0. \tag{45}$$

This meams that the condition (b) in Theorem 3.2 is satisfied.

Next, we claim that $\mathbb{C}_{k+1}$ and S^{k-1} are G-linked. Indeed, the proof of the claim is quite similar to that given in [25, 26] and so is omitted. In addition, from the compactness of the closed unit ball B^k in $\mathbb{R}^k$, we easily conclude that the condition (a) in Theorem 3.2 is also true. In conclusion, the functional J satisfies all of the conditions in Theorem 3.2. Hence, problem (12) has at least a weak solution, so is problem (1).

Proof of Corollary 2.3. In fact, when $h(s) = s^{\frac{\sigma}{2}}$ with $\sigma \geq 1$, we have

$$g'(t) = \left[1 + \frac{\sigma^p}{2}|g(t)|^{p(\sigma-1)}\right]^{-\frac{1}{p}}. \tag{46}$$

By Theorem 2.2, it suffices to check g defined by (6) and (46) satisfies the condition (g_0) with $\alpha = \sigma$ and $\beta = 2$. Indeed, let $l : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$l(t) = \sigma t - \left[1 + \frac{\sigma^p}{2}|g(t)|^{p(\sigma-1)}\right]^{\frac{1}{p}} g(t).$$

It follows from (6) that $l(0) = 0$, and since $\sigma \geq 1$,

$$l'(t) = (\sigma - 1)\left[1 - \frac{\frac{\sigma^p}{2}|g(t)|^{p(\sigma-1)}}{1 + \frac{\sigma^p}{2}|g(t)|^{p(\sigma-1)}}\right] \geq 0, \tag{47}$$

which means that $l(t) \geq l(0)$ for all $t \in \mathbb{R}^+$. This proves that the left hand side inequality of (8). The right hand side inequality can be proved in a similar way. Now we begin to prove (9). Indeed, it follow from (8) that $g(t) \to +\infty$ as $t \to +\infty$. By the principle of L'Hospital and (46), we have

$$\begin{aligned} \lim_{t\to+\infty} \frac{g^{\sigma p}(t)}{t^p} &= \left[\lim_{t\to+\infty} \frac{g^{\sigma}(t)}{t}\right]^p = \left[\lim_{t\to+\infty} \sigma g^{\sigma-1}(t) g'(t)\right]^p \\ &= \lim_{t\to+\infty} \frac{\left[\sigma g^{\sigma-1}(t)\right]^p}{1 + \frac{\sigma^p}{2}|g(t)|^{p(\sigma-1)}} \\ &= 2. \end{aligned}$$

Hence, the function g defined by (6) and (46) satisfies the condition (g_0) with $\alpha = \sigma$ and $\beta = 2$. Hence, it follows from Theorem 2.2 that (18) has at least a weak solution.

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