Filomat 36:7 (2022), 2303–2310
https://doi.org/10.2298/FIL2207303M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Infinitely Many Solutions for a Class of Systems Including the $(p_1, \cdots, p_n)$-Biharmonic Operators

Mirkeysaan Mahshid[a], Abdolrahman Razani[b]

[a]*Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.*
[b]*Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, 3414896818, Qazvin, Iran.*

Abstract. In this work, we prove the existence of infinitely many solutions for a general form of an elliptic system involving the $(p_1, \cdots, p_n)$-biharmonic operators via variational methods.

1. Introduction

The investigation of existence of solutions for problems at resonance has drawn the attention of many authors, (see for example [2–4, 13–16]) as well as the existence of infinity many solutions for elliptic systems (see [1, 6–8, 10, 11, 17, 18] and the references therein). Here, we consider the following system with Navier boundary conditions

$$\begin{cases} -\Delta^2_{p_i} u_i - \theta_i(x)\Delta_{p_i} u_i = \mu F_{u_i}(x, u_1, \cdots, u_n) & \text{in } \Omega, \\ \Delta u_i = u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

for $1 \leq i \leq n$. Where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded domain with smooth boundary and $\mu > 0$ is a real parameter. For each $1 \leq i \leq n, p_i > max\{1, \frac{N}{2}\}$, $\Delta_{p_i} u = div(|\nabla u|^{p_i - 2}\nabla u)$ and $\Delta^2_{p_i} u = \Delta(|\Delta u|^{p_i-2}\Delta u)$ denote p_i-Laplacian and p_i-biharmonic operators, respectively, where $u : \Omega \to \mathbb{R}$. Note that for $p = 2$, the linear operator $\Delta^2_2 = \Delta^2 = \Delta.\Delta$ is the iterated Laplacian that multiplied with positive constant appears often in Navier-Stokes equations as being a viscosity coefficient. $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, and besides, F_{u_i} is the partial derivative of F with respect to $u_i, i = 1, \cdots, n$. Plus that $\theta_i(x)$ hold the following condition:

(Θ) $\qquad \theta_i \in L^\infty(\Omega)$ such that $ess \inf_{x\in\bar{\Omega}} \theta_i(x) > 0, 1 \leq i \leq n$.

We are going to prove the existence of infinitely many weak solutions for system (1) under suitable assumptions on F, whenever the parameter μ belongs to appropriate interval. Here we recall the following theorem [4, 19].

2020 *Mathematics Subject Classification*. Primary 35J35; Secondary 35J92, 45G15
Keywords. $(p_1, \cdots, p_n)$-biharmonic systems, variational methods, infinitely many solutions
Received: 12 June 2021; Revised: 15 July 2021; Accepted: 28 July 2021
Communicated by Maria Alessandra Ragusa
Email addresses: Keysaan.Mahshid@kiau.ac.ir (Mirkeysaan Mahshid), razani@sci.ikiu.ac.ir (Abdolrahman Razani)

Theorem 1.1. *Let X be a reflexive real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semi-continuous, and coercive, and Ψ is sequentially weakly upper-semi-continuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)}$$

$$\kappa := \liminf_{r\to+\infty} \phi(r), \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \phi(r).$$

Then

- *(a) If $\kappa < +\infty$ then, for each $\mu \in (0, \frac{1}{\kappa})$, the following alternative holds: either*
- *(a1) $I_\mu := \Phi - \mu\Psi$ possesses a global minimum, or*
- *(a2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_μ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$.*
- *(b) If $\delta < +\infty$ then, for each $\mu \in (0, \frac{1}{\delta})$, the following alternative holds: either*
- *(b1) there is a global minimum of Φ that is a local minimum of I_μ, or*
- *(b2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_μ that weakly converges to a global minimum of Φ with*

$$\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi.$$

2. Preliminaries

In this section we prepare some definitions and notations. Throughout this paper Ω is an open bounded subset of $\mathbb{R}^N, N > 2$ with smooth boundary. At first we define Carathéodory function on $\Omega \times \mathbb{R}^n$.

Definition 2.1. *We say that $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, if*

- *$x \to f(x, \tau_1, \cdots, \tau_n)$ is measurable for every $(\tau_1, \cdots, \tau_n) \in \mathbb{R}^n$.*
- *$(\tau_1, \cdots, \tau_n) \to f(x, \tau_1, \cdots, \tau_n)$ is continuous for a.e. $x \in \Omega$.*

The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) := \{u \in L^p(\Omega); |\nabla u| \in L^p(\Omega)\},$$

and norm in $W^{1,p}(\Omega)$ is $||u||_{1,p} := |u|_p + |\ |\nabla u|\ |_p$, where $|.|_p$ denotes the norm on $L^p(\Omega)$ and the vector $\nabla u = (\frac{\partial u}{\partial x_1}(x), \cdots, \frac{\partial u}{\partial x_n}(x))$ is the gradient of u at $x = (x_1, \cdots, x_n)$. Also, we set

$$W_0^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega);\ u|_{\partial\Omega} = 0\},$$

equipped with the norm $||u||_* := |\ |\nabla u|\ |_p$. Similarly,

$$W^{2,p}(\Omega) := \{u \in L^p(\Omega); |\nabla u|, |\Delta u| \in L^p(\Omega)\},$$

equipped with the norm

$$||u||_{2,p} := |u|_p + |\ |\nabla u|\ |_p + |\Delta u|_p.$$

And we set

$$W_0^{2,p}(\Omega) := \{u \in W^{2,p}(\Omega);\ u|_{\partial\Omega} = 0\}$$

endowed with the norm $\|u\|_{**} := |\Delta u|_p$. For $1 \leq i \leq n$, we set

$$X_i := W_0^{1,p_i}(\Omega) \bigcap W_0^{2,p_i}(\Omega),$$

where by Poincaré inequality:

$$\int_\Omega u^{p_i} dx \leq \hat{c} \int_\Omega |\nabla u|^{p_i} dx,$$

X_i can be endowed with the norm $\|u\|_{X_i} = |\Delta u|_{p_i}$, where $\hat{c} > 0$ is the best possible. Let us point out for the given $\theta_i \in L^\infty(\Omega)$ satisfied (Θ) condition, the following norm is a norm on X_i which is equivalent to $\|u\|_{X_i}$:

$$\|u\|_{p_i} = \Big(\int_\Omega (|\Delta u(x)|^{p_i} + \theta_i(x)|\nabla u(x)|^{p_i}) dx \Big)^{\frac{1}{p_i}}.$$

More precisely, we have

$$\|u\|_{X_i} \leq \|u\|_{p_i} \leq \hat{c}_* \|u\|_{X_i}, \tag{2}$$

where $\hat{c}_* = (1 + \hat{c}|\theta_i|_\infty)$. We let $\mathcal{X}$ be the Cartesian product of the n Sobolev spaces X_i for $1 \leq i \leq n$, i.e.,

$$\mathcal{X} = \prod_{i=1}^n X_i,$$

endowed with the norm

$$\|u\|_{\mathcal{X}} := \sum_{i=1}^n \|u_i\|_{p_i}, \quad u = (u_1, \cdots, u_n) \in \mathcal{X}.$$

We set

$$C := \max_{1 \leq i \leq n} \sup_{u_i \in X_i \setminus \{0\}} \frac{\max_{x\in\Omega} |u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}},$$

that according to the Rellich Kondrachov theorem $X_i \hookrightarrow C(\bar{\Omega})$, $1 \leq i \leq n$, is compact and hence $C < +\infty$ (see more details in [5, p.290] and [20, p.286]). Moreover, for any $u_i \in X_i, i = 1, \cdots, n$, we have

$$\sup_{x\in\Omega} |u_i(x)|^{p_i} \leq C \|u_i\|_{p_i}^{p_i}. \tag{3}$$

We end up this section by the next definition which is the meaning of weak solution for the problem (1):

Definition 2.2. *We say that $u = (u_1, \cdots, u_n) \in \mathcal{X}$ is a weak solution of the system (1) if*

$$\int_\Omega \sum_{i=1}^n \Big(|\Delta u_i(x)|^{p_i-2} \Delta u_i(x) \Delta v_i(x) + \theta_i(x) |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla v_i(x) \Big) dx - \mu \int_\Omega \sum_{i=1}^n F_{u_i}(x, u_1(x), \cdots, u_n(x)) v_i(x) dx = 0,$$

for all $v = (v_1, \cdots, v_n) \in \mathcal{X}$.

3. Weak solution

In this section, our principal result is presented. We assume that functional $I_\mu : \mathcal{X} \to \mathbb{R}$ is given by

$$I_\mu u = \Phi(u) - \mu\Psi(u),$$

for all $u = (u_1, \cdots, u_n) \in \mathcal{X}$, where

$$\Phi(u) = \int_\Omega \sum_{i=1}^{n} \frac{1}{p_i}\Big(|\Delta u_i(x)|^{p_i} + \theta_i(x)|\nabla u_i(x)|^{p_i}\Big)dx, \tag{4}$$

and

$$\Psi(u) = \int_\Omega F(x, u_1(x), \cdots, u_n(x))dx. \tag{5}$$

Since $\mathcal{X}$ is compactly embedded in $C^0(\bar{\Omega}) \times \cdots \times C^0(\bar{\Omega})$, it is well known that Φ and Ψ are well defined and continuously Gâteaux differentiable functionals. Moreover, at the point $u = (u_1, \cdots, u_n) \in \mathcal{X}$ one has

$$\langle \Phi'(u), v\rangle = \int_\Omega \sum_{i=1}^{n}\Big(|\Delta u_i(x)|^{p_i-2}\Delta u_i(x)\Delta v_i(x) + \theta_i(x)|\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x)\Big)dx,$$

and

$$\langle \Psi'(u), v\rangle = \int_\Omega \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \cdots, u_n(x))v_i(x)dx$$

for any $v = (v_1, \cdots, v_n) \in \mathcal{X}$. By the weakly lower semicontinuity of norm, clearly Φ is sequentially weakly lower semicontinuous. Since Ψ has compact derivative, it follows that Ψ is sequentially weakly continuous. From (3), one has

$$\sup_{x\in\Omega} \sum_{i=1}^{n} \frac{1}{p_i}|u_i(x)|^{p_i} \le C \sum_{i=1}^{n} \frac{1}{p_i}\|u_i\|_{p_i}^{p_i}.$$

So, for each $r > 0$

$$\begin{aligned}\Phi^{-1}(]-\infty, r[) : &= \{u = (u_1, \cdots, u_n) \in \mathcal{X} : \Phi(u) < r\}\\ &= \{u = (u_1, \cdots, u_n) \in X : \sum_{i=1}^{n} \frac{1}{p_i}\|u_i\| < r\}\\ &\subseteq \{u = (u_1, \cdots, u_n) \in \mathcal{X} : \sum_{i=1}^{n} \frac{1}{p_i}|u_i(x)|^{p_i} \le Cr, \text{for all } x \in \Omega\}.\end{aligned} \tag{6}$$

Proposition 3.1. *([12]) The functional $\Phi : X \to \mathbb{R}$ is convex and mapping $\Phi' : X \to X^*$ is a strictly monotone and bounded homeomorphism.*

Furthermore, Φ is coercive, since indeed for $u = (u_1, \cdots, u_n)$ we have

$$\Phi(u) = \sum_{i=1}^{n} \frac{1}{p_i}\|u_i\|_{p_i}^{p_i}$$

and when $\|u\|_{\mathcal{X}} \to +\infty$, there exists at least one $\hat{i}$ such that $1 \le \hat{i} \le n$ where $\|u_{\hat{i}}\|_{\mathcal{X}} \to +\infty$ and so $\Phi(u) \to +\infty$ as $\|u\|_{\mathcal{X}} \to +\infty$.

For fixed $x_0 \in \Omega$, set $R_2 > R_1 > 0$ such that $B(x_0, R_2) \subset \Omega$, where $B(x_0, R_2)$ denotes the ball with center at x_0 and radius R_2. Besides, let

$$p^* = \max_{1\le i\le n} p_i \text{ and } p_* = \min_{1\le i\le n} p_i. \tag{7}$$

Define

$$\eta_{p_i} := \frac{\Gamma(1+\frac{N}{2})}{\Big(\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}}\Big)^{p_*}\pi^{\frac{N}{2}}}\Big(\frac{R_2^2-R_1^2}{2N}\Big)^{p_i}\frac{1}{R_2^N-R_1^N}, \tag{8}$$

where Γ denotes the Gamma function. Now we are ready to state the main result of the paper.

Theorem 3.2. *Assume that*

(i) $F(x,\tau_1,\cdots,\tau_n) \ge 0$ *for every* $(x,\tau_1,\cdots,\tau_n) \in \Omega\times[0,+\infty)^n$*;*

(ii) There exist a point $x_0 \in \Omega$ *and* $R_2 > R_1 > 0$ *such that* $B(x_0,R_2) \subset \Omega$ *and set*

$$\begin{aligned} A &:= \liminf_{\sigma\to+\infty} \frac{\int_\Omega \sup_{\sum_{i=1}^n |\tau_i|\le\sigma} F(x,\tau_1,\cdots,\tau_n)dx}{\sigma^{p_*}},\\ B &:= \limsup_{\tau_1,\cdots,\tau_n\to+\infty} \frac{\int_{B(x_0,R_1)} F(x,\tau_1,\cdots,\tau_n)dx}{\sum_{i=1}^n \frac{\tau_i^{p_i}}{p_i}}. \end{aligned} \tag{9}$$

Then we have $A < \eta B$*, where* $\eta := \min_{1\le i\le n}\eta_{p_i}$*. Then for each*

$$\mu \in \mathcal{M} := \frac{1}{\Big(\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}}\Big)^{p_*}}\Big]\frac{1}{\eta B},\frac{1}{A}\Big[, \tag{10}$$

the problem (1) admits an unbounded sequence of weak solutions.

Proof. To use Theorem 1.1, let Φ and Ψ be as in (4) and (5), respectively. And set

$$\varphi(r) := \inf_{w\in\Phi^{-1}(]-\infty,r[)} \frac{\Big(\sup_{w\in\Phi^{-1}(]-\infty,r[)}\Psi(w)\Big)-\Psi(w)}{r-\Phi(w)},$$

where $w = (w_1,\cdots,w_n)$. By our assumptions $\Phi(0,0) = 0$ and $\Psi(0,0) \ge 0$. Therefore, by (6) for every $r > 0$ and $w = (w_1,\cdots,w_n) \in \mathcal{X}$ one has

$$\begin{aligned} \varphi(r) &\le \frac{\sup_{\Phi^{-1}(]-\infty,r[)}\Psi}{r}\\ &= \frac{1}{r}\sup_{\Phi(w)<r}\int_\Omega F(x,w_1,\cdots,w_n)dx\\ &\le \frac{1}{r}\sup_{\{w\in\mathcal{X}:\sum_{i=1}^n \frac{1}{p_i}|w_i(x)|^{p_i}<Cr,\text{for all } x\in\Omega\}}\int_\Omega F(x,w_1,\cdots,w_n)dx. \end{aligned} \tag{11}$$

Let $\{\sigma_k\}$ be a real sequence of positive numbers such that $\lim_{k\to+\infty}\sigma_k = +\infty$ and

$$\lim_{k\to+\infty} \frac{\int_\Omega \sup_{\sum_{i=1}^n |\tau_i|\le\sigma_k} F(x,\tau_1,\cdots,\tau_n)dx}{\sigma_k^{p_*}} = A < +\infty. \tag{12}$$

And define

$$r_k := \Big(\frac{\sigma_k}{\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}}}\Big)^{p_*}.$$

Let $w = (w_1, \cdots, w_n) \in \Phi^{-1}(]-\infty, r_k[)$, from (6), one has

$$\sum_{i=1}^n \frac{1}{p_i}|w_i(x)|^{p_i} < Cr_k, \text{ for all } x \in \Omega.$$

Then

$$|w_i(x)| \leq (Cr_kp_i)^{\frac{1}{p_i}} \quad \text{for } i = 1, \cdots, n,$$

thus, for each $k \in \mathbb{N}$ large enough ($r_k > 1$),

$$\begin{aligned}\sum_{i=1}^n |w_i(x)| &\leq (Cr_kp_i)^{\frac{1}{p_i}} \\ &\leq \sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}} r_k^{\frac{1}{p_*}} = \sigma_k.\end{aligned}$$

$$\begin{aligned}\varphi(r_k) &\leq \frac{\sup_{\{u\in X:\sum_{i=1}^n |w_i(x)|<\sigma_k, \text{ for all } x\in\Omega\}} \int_\Omega F(x, w_1, \cdots, w_n)dx}{(\frac{\sigma_k}{(\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}}})^{p_*}} \\ &\leq (\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}})^{p_*} \frac{\int_\Omega \sup_{\sum_{i=1}^n |\tau_i|\leq\sigma_k} F(x, \tau_1, \cdots, \tau_n)dx}{\sigma_k^{p_*}}.\end{aligned} \tag{13}$$

Define $\kappa := \liminf_{r\to+\infty} \varphi(r)$, so from (12) and (13), one has

$$\begin{aligned}\kappa &\leq \liminf_{k\to+\infty} \varphi(r_k) \\ &\leq (\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}})^{p_*} \frac{\int_\Omega \sup_{\sum_{i=1}^n |\tau_i|\leq\sigma_k} F(x, \tau_1, \cdots, \tau_n)dx}{\sigma_k^{p_*}} \\ &= A(\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}})^{p_*} < +\infty.\end{aligned}$$

On the other hand, we know that

$$\kappa \leq (\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}})^{p_*} A < \frac{1}{\mu}, \tag{14}$$

then $\mathcal{M} \subseteq]0, \frac{1}{\kappa}[$. For $\mu \in \mathcal{M}$, since $\frac{1}{\mu} < (\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}})^{p_*}\eta B$, the functional $I_\mu = \Phi - \mu\Psi$ is unbounded from below. So, we can consider a sequence $\{\alpha_k\}$ of positive numbers and $\delta > 0$ such that $\alpha_k \to +\infty$ as $k \to \infty$ and

$$\frac{1}{\mu} < \frac{\delta}{\hat{c}_*} < \eta\, (\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}})^{p_*} \frac{\int_{B(x_0,R_1)} F(x, \alpha_k, \cdots, \alpha_k)dx}{\sum_{i=1}^n \frac{\alpha_k^{p_i}}{p_i}}, \tag{15}$$

for k large enough, where $\hat{c}_*$ is as in (2). Let

$$w_k(x) := \begin{cases} 0 & x \in \bar{\Omega} \setminus B(x_0, R_2), \\ \frac{\alpha_k}{R_2 - R_1}\Big(R_2 - \{\Sigma_{i=1}^n (x^i - x_0^i)^2\}^{\frac{1}{2}}\Big) & x \in B(x_0, R_2) \setminus B(x_0, R_1), \\ \alpha_k & x \in B(x_0, R_1). \end{cases} \tag{16}$$

Then $(w_k, \cdots, w_k) \in \mathcal{X}$ and for each $1 \leq i \leq n$ we have

$$\|w_k\|_{X_i}^{p_i} = \frac{\pi^{\frac{N}{2}}}{\Gamma(1 + \frac{N}{2})}\Big(\frac{2N\alpha_k}{R_2^2 - R_1^2}\Big)^{p_i}(R_2^N - R_1^N),$$

see more details in [13]. In accordance with (2) one has

$$\|w_k\|_{X_i} \leq \|w_k\|_{p_i} \leq \hat{c}_* \|w_k\|_{X_i}.$$

Bearing (8) and (4) in mind, we deduce

$$\begin{aligned} \Phi(w_k, \cdots, w_k) &= \int_\Omega \sum_{i=1}^n \frac{1}{p_i}\Big(|\Delta w_k(x)|^{p_i} + \mu_i(x)|\nabla w_k(x)|^{p_i}\Big)dx \\ &= \sum_{i=1}^n \frac{1}{p_i}\|w_k\|_{p_i}^{p_i} \\ &\leq \frac{\hat{c}_*}{(\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}})^{p_*}} \sum_{i=1}^n \frac{1}{p_i}\frac{\alpha_k^{p_i}}{\eta_{p_i}}. \end{aligned} \tag{17}$$

On the other hand, according to our assumptions

$$\Psi(w_k, \cdots, w_k) = \int_\Omega F(x, w_k, \cdots, w_k)dx \geq \int_{B(x_0, R_1)} F(x, \alpha_k, \cdots, \alpha_k)dx. \tag{18}$$

So, it follows from (15), (17) and (18) that

$$\begin{aligned} I_\mu(w_k, \cdots, w_k) &= \Phi(w_k, \cdots, w_k) - \mu\Psi(w_k, \cdots, w_k) \\ &\leq \frac{\hat{c}_*}{(\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}})^{p_*}} \sum_{i=1}^n \frac{1}{p_i}\frac{\alpha_k^{p_i}}{\eta_{p_i}} - \mu \int_{B(x_0, R_1)} F(x, \alpha_k, \cdots, \alpha_k)dx \\ &< \frac{\hat{c}_* - \mu\delta}{\delta\Big(\sum_{i=1}^n (Cp_i)^{\frac{1}{p_i}}\Big)^{p_*}}\Big(\sum_{i=1}^n \frac{1}{p_i}\frac{\alpha_k^{p_i}}{\eta_{p_i}}\Big), \end{aligned} \tag{19}$$

for k large enough, so $\lim_{k\to+\infty} I_\mu(w_k, \cdots, w_k) = -\infty$, and hence the claim has been archived.
The alternative of Theorem 1.1 case (a) assures the existence of unbounded sequence $\{w_k\}$ of critical points of the functional I_μ. This completes the proof in view of the relation between the critical points of I_μ and the weak solutions of problem (1). □

References

[1] F. Abdolrazaghi, A. Razani, R. Mirzaei, Multiple weak solutions for a kind of time-dependent equation involving singularity, Filomat 34 (2020) no. 13, 4567–4574.

[2] G. A. Afrouzi, Q. Zhang, A. Hadjian, Multiple solutions for a class of Neumann doubly eigenvalue boundary value systems involving the $(p_1(x), \ldots, p_n(x))$-Laplacian, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas 108 (2014) no. 2, 1055–1064.

[3] A. Anane, O. Chakrone, B. Karim, A. Zerouali, Existence of solution for a resonant Steklov Problem, Bol. Soc. Paranaense de Mat. 1 (2009) 87–90.

[4] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, Boundary Value Problems 2009 (2009), 1–20.

[5] L. C. Evans, Partial Differential Equations, (2nd ed.), Graduate Studies in Mathematics, Vol. 19, American Mathematical Society, Providence, RI, 2010.

[6] S. Heidari, A. Razani, Infinitely many solutions for nonlocal elliptic systems in Orlicz–Sobolev spaces, Georgian Mathematical Journal, 29 (2021) no. 1, 45–54.

[7] S. Heidari, A. Razani, Infinitely many solutions for $(p(x), q(x))$- Laplacian-like systems, Communications of the Korean Mathematical Society 36 (2021) no. 1, 51–62.

[8] S. M. Khalkhali, S. Heidarkhani, A. Razani, Infinitely many solutions for a fourth-order boundary-value problem, Electronic Journal of Differential Equations 164 (2012) 1–14.

[9] R. Mahdavi Khanghahi, A. Razani, Solutions for a singular elliptic problem involving the $p(x)$-Laplacian, Filomat 32 (2018) 4841–4850.

[10] M. Makvand Chaharlang, A. Razani, Infinitely many solutions for a fourth order singular elliptic problem, Filomat 32 (2018) 5003–5010.

[11] M. Makvand Chaharlang, A. Razani, Existence of infinitely many solutions for a class of nonlocal problems with Dirichlet boundary condition, Communications of the Korean Mathematical Society 34 (2019) no. 1, 155–167,

[12] M. M. Rodrigues, Multiplicity of solutions on a Nonlinear eigenvalue problem for $p(x)$-Laplacian-like operators, Mediterranean Journal of Mathematics 9 (2012) 211–223.

[13] M. Massar, EL. M. Hssini, N. Tsouli, Infinitely many solutions for class of Navier boundary (p, q)-biharmonic systems, Electronic Journal of Differential Equations 2012 (2012) no. 163, 1–9.

[14] M.A. Ragusa, Elliptic boundary value problem in vanishing mean oscillation hypothesis, Commentationes Mathematicae Universitatis Carolinae 40 (1999) no. 4, 651–663.

[15] M.A. Ragusa, Hölder regularity results for solutions of parabolic equations, Variational Analysis and Applications, Nonconvex Optimization and Its Applications, Springer, Boston, MA. Vol.79 (2005), 921–934.

[16] M.A. Ragusa and A. Tachikawa, Boundary regularity of minimizers of $p(x)$-energy functionals, Annales de l'Institut Henri Poincaré, Analyse Non linéaire 33 (2016) no. 2, 451–476.

[17] A. Razani, Game-theoretic p-Laplace operator involving the gradient, Miskolc Mathematical Notes Accepted (2020).

[18] A. Razani, Two weak solutions for fully nonlinear Kirchhoff-type problem, Filomat, accepted (2020).

[19] B. Ricceri, A general variational principle and some of its applications, Journal of Computational and Applied Mathematics 113 (2000) 401–410.

[20] M. Taylor, Partial Differential Equations I: Basic Theory, Applied Mathematical Sciences, Vol. 115, (2nd ed.), Springer-Verlag New York, 2011.