Infinitely Many Solutions for a Class of Systems Including the $(p_1, \cdots, p_n)$-Biharmonic Operators

Mirkeysaan Mahshid\textsuperscript{a}, Abdolrahman Razani\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
\textsuperscript{b}Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, 3414896818, Qazvin, Iran.

Abstract. In this work, we prove the existence of infinitely many solutions for a general form of an elliptic system involving the $(p_1, \cdots, p_n)$-biharmonic operators via variational methods.

1. Introduction

The investigation of existence of solutions for problems at resonance has drawn the attention of many authors, (see for example [2–4, 13–16]) as well as the existence of infinity many solutions for elliptic systems (see [1, 6–8, 10, 11, 17, 18] and the references therein). Here, we consider the following system with Navier boundary conditions

\[
\begin{aligned}
-\Delta_{p_i}^2 u_i - \theta_i(x)\Delta_{p_i} u_i &= \mu F_{u_i}(x, u_1, \cdots, u_n) \quad \text{in } \Omega, \\
\Delta u_i &= u_i = 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

for $1 \leq i \leq n$. Where $\Omega \subset \mathbb{R}^N, N \geq 2$ is a bounded domain with smooth boundary and $\mu > 0$ is a real parameter. For each $1 \leq i \leq n$, $p_i > \max\{1, \frac{N}{2}\}$, $\Delta_{p_i} u = \text{div}(\nabla |\nabla u|^{p_i-2} \nabla u)$ and $\Delta_{p_i}^2 u = \Delta(\Delta |\Delta u|^{p_i-2} \Delta u)$ denote $p_i$-Laplacian and $p_i$-biharmonic operators, respectively, where $u : \Omega \to \mathbb{R}$. Note that for $p = 2$, the linear operator $\Delta_2^2 = \Delta^2 = \Delta \Delta$ is the iterated Laplacian that multiplied with positive constant appears often in Navier-Stokes equations as being a viscosity coefficient. $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, and besides, $F_{u_i}$ is the partial derivative of $F$ with respect to $u_i, i = 1, \cdots, n$. Plus that $\theta_i(x)$ hold the following condition:

\[\theta_i \in L^{\infty}(\Omega) \quad \text{such that } \text{ess inf}_{x \in \Omega} \theta_i(x) > 0, 1 \leq i \leq n.\]

We are going to prove the existence of infinitely many weak solutions for system (1) under suitable assumptions on $F$, whenever the parameter $\mu$ belongs to appropriate interval. Here we recall the following theorem [4, 19].
The Sobolev space

Definition 2.1. We say that \( f \in W \) and norm in \( \nabla \subset \mathbb{R} \subset \mathbb{R} \) is a Carathéodory function if

\[
\Phi := \frac{\sup_{v \in \Phi^{-1}(\inf \Phi)} |\Psi(v)| - \Psi(u)}{r - \Phi(u)}
\]

\[
\kappa := \lim_{r \to \infty} \inf \Phi(r),
\]

\[
\delta := \lim_{r \to (\inf \Phi)^{+}} \inf \Phi(r).
\]

Then

(a) If \( \kappa < +\infty \) then, for each \( \mu \in (0, \frac{1}{\delta}) \), the following alternative holds: either

(b) there is a global minimum of \( I \), or

(b2) there is a sequence \( \{u_n\} \) of pairwise distinct critical points (local minima) of \( I \) such that \( \lim_{n \to +\infty} \Phi(u_n) = +\infty \).

(b) If \( \delta < +\infty \) then, for each \( \mu \in (0, \frac{1}{\delta}) \), the following alternative holds: either

(b1) there is a global minimum of \( \Phi \) that is a local minimum of \( I \), or

(b2) there is a sequence \( \{u_n\} \) of pairwise distinct critical points (local minima) of \( I \) that weakly converges to a global minimum of \( \Phi \) with

\[
\lim_{n \to +\infty} \Phi(u_n) = \inf_{\Omega} \Phi.
\]

2. Preliminaries

In this section we prepare some definitions and notations. Throughout this paper \( \Omega \) is an open bounded subset of \( \mathbb{R}^N, N > 2 \) with smooth boundary. At first we define Carathéodory function on \( \Omega \times \mathbb{R}^n \).

Definition 2.1. We say that \( f : \Omega \times \mathbb{R}^n \to \mathbb{R} \) is a Carathéodory function, if

- \( x \mapsto f(x, \tau_1, \ldots, \tau_n) \) is measurable for every \( (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n \).
- \( (\tau_1, \ldots, \tau_n) \mapsto f(x, \tau_1, \ldots, \tau_n) \) is continuous for a.e. \( x \in \Omega \).

The Sobolev space \( W^{1,p}(\Omega) \) is defined by

\[
W^{1,p}(\Omega) := \{ u \in L^p(\Omega); |\nabla u| \in L^p(\Omega) \},
\]

and norm in \( W^{1,p}(\Omega) \) is \( ||u||_{1,p} := ||u||_p + ||\nabla u||_p \), where \( ||\cdot||_p \) denotes the norm on \( L^p(\Omega) \) and the vector

\[
\nabla u = \left( \frac{\partial u}{\partial x_1}(x), \ldots, \frac{\partial u}{\partial x_n}(x) \right)
\]

is the gradient of \( u \) at \( x = (x_1, \ldots, x_n) \). Also, we set

\[
W^{1,p}_0(\Omega) := \{ u \in W^{1,p}(\Omega); u|\partial \Omega = 0 \},
\]

equipped with the norm \( ||u||_* := ||\nabla u||_p \). Similarly,

\[
W^{2,p}(\Omega) := \{ u \in L^p(\Omega); |\nabla u|, |\Delta u| \in L^p(\Omega) \},
\]

equipped with the norm \( ||u||_{2,p} := ||u||_p + ||\nabla u||_p + ||\Delta u||_p \).
We say that $u$ is a weak solution of the system (1) if

$$\int_{\Omega} \sum_{i=1}^{n} (|\Delta u_i(x)|^{p_i-2} \Delta u_i(x) \Delta v_i(x) + \theta_i(x)|\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla v_i(x)) \, dx - \mu \int_{\Omega} \sum_{i=1}^{n} F_i(x, u_1(x), \cdots, u_n(x)) v_i(x) \, dx = 0,$$

for all $v = (v_1, \cdots, v_n) \in X$.
3. Weak solution

In this section, our principal result is presented. We assume that functional \( I_\mu : X \to \mathbb{R} \) is given by

\[
I_\mu u = \Phi(u) - \mu \Psi(u),
\]

for all \( u = (u_1, \ldots, u_n) \in X \), where

\[
\Phi(u) = \int \sum_{i=1}^{n} \frac{1}{p_i} \left( |\Delta u_i(x)|^{p_i} + \theta_i(x)|\nabla u_i(x)|^{p_i} \right) dx,
\]

and

\[
\Psi(u) = \int F(x, u_1(x), \cdots, u_n(x)) dx.
\]

Since \( X \) is compactly embedded in \( C^0(\Omega) \times \cdots \times C^0(\Omega) \), it is well known that \( \Phi \) and \( \Psi \) are well defined and continuously Gâteaux differentiable functionals. Moreover, at the point \( u = (u_1, \cdots, u_n) \in X \) one has

\[
\langle \Phi'(u), v \rangle = \int \sum_{i=1}^{n} \frac{1}{p_i} \left( |\Delta u_i(x)|^{p_i-2} \Delta u_i(x) \Delta v_i(x) + \theta_i(x)|\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla v_i(x) \right) dx,
\]

and

\[
\langle \Psi'(u), v \rangle = \int \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \cdots, u_n(x)) v_i(x) dx
\]

for any \( v = (v_1, \cdots, v_n) \in X \). By the weakly lower semicontinuity of norm, clearly \( \Phi \) is sequentially weakly lower semicontinuous. Since \( \Psi \) has compact derivative, it follows that \( \Psi \) is sequentially weakly continuous. From (3), one has

\[
\sup_{x \in \Omega} \sum_{i=1}^{n} \frac{1}{p_i} |u_i(x)|^{p_i} \leq C \sum_{i=1}^{n} \frac{1}{p_i} \|u_i\|_{p_i}^{p_i}.
\]

So, for each \( r > 0 \)

\[
\Phi^{-1}([-\infty, r]) = \{ u = (u_1, \cdots, u_n) \in X : \Phi(u) < r \}
\]

\[
= \{ u = (u_1, \cdots, u_n) \in X : \sum_{i=1}^{n} \frac{1}{p_i} \|u_i\| < r \}
\]

\[
\subseteq \{ u = (u_1, \cdots, u_n) \in X : \sum_{i=1}^{n} \frac{1}{p_i} |u_i(x)|^{p_i} \leq Cr, \text{ for all } x \in \Omega \}.
\]

Proposition 3.1. (12) The functional \( \Phi : X \to \mathbb{R} \) is convex and mapping \( \Phi' : X \to X^* \) is a strictly monotone and bounded homeomorphism.

Furthermore, \( \Phi \) is coercive, since indeed for \( u = (u_1, \cdots, u_n) \) we have

\[
\Phi(u) = \sum_{i=1}^{n} \frac{1}{p_i} \|u_i\|_{p_i}^{p_i}
\]

and when \( \|u\|_X \to +\infty \), there exists at least one \( \hat{i} \) such that \( 1 \leq \hat{i} \leq n \) where \( \|u\|_X \to +\infty \) and so \( \Phi(u) \to +\infty \) as \( \|u\|_X \to +\infty \).
For fixed $x_0 \in \Omega$, set $R_2 > R_1 > 0$ such that $B(x_0, R_2) \subset \Omega$, where $B(x_0, R_2)$ denotes the ball with center at $x_0$ and radius $R_2$. Besides, let
\[
p' = \max_{1 \leq i \leq n} p_i \text{ and } p_i = \min_{1 \leq i \leq n} p_i.
\]

Define
\[
\eta_{p_i} := \frac{\Gamma(1 + \frac{n}{2})}{\left(\sum_{i=1}^n (Cp_i)^{\frac{n}{2}}\right)^{\frac{p_i}{n}} \frac{1}{p_i} \left(\frac{R_2^2 - R_1^2}{2N}\right)^{\frac{1}{2}}}.
\]

where $\Gamma$ denotes the Gamma function. Now we are ready to state the main result of the paper.

**Theorem 3.2. Assume that**

(i) $F(x, \tau_1, \ldots, \tau_n) \geq 0$ for every $(x, \tau_1, \ldots, \tau_n) \in \Omega \times [0, +\infty)^n$;

(ii) There exist a point $x_0 \in \Omega$ and $R_2 > R_1 > 0$ such that $B(x_0, R_2) \subset \Omega$ and set
\[
A := \liminf_{\sigma \to +\infty} \int_\Omega \sup_{|\tau| \leq \sigma} F(x, \tau_1, \ldots, \tau_n) dx,
\]
\[
B := \limsup_{|\tau|, \tau_r \to +\infty} \int_{B(x_0, R_1)} F(x, \tau_1, \ldots, \tau_r) dx.
\]

Then we have $A < \eta B$, where $\eta := \min_{1 \leq i \leq n} \eta_{p_i}$. Then for each
\[
\mu \in \mathcal{M} := \left\{ \frac{1}{\sum_{i=1}^n (Cp_i)^{\frac{n}{2}}} \left[ \frac{1}{\eta^{N'}} \frac{1}{A} \right] \right\}
\]
the problem (1) admits an unbounded sequence of weak solutions.

**Proof.** To use Theorem 1.1, let $\Phi$ and $\Psi$ be as in (4) and (5), respectively. And set
\[
\varphi(r) := \inf_{w \in \Phi^{-1}([0, +\infty)]} \left( \frac{\sup_{w \in \Phi^{-1}([0, +\infty])} \Psi(w) - \Psi(w)}{r - \Phi(w)} \right),
\]
where $w = (w_1, \ldots, w_n)$. By our assumptions $\Phi(0, 0) = 0$ and $\Psi(0, 0) \geq 0$. Therefore, by (6) for every $r > 0$ and $w = (w_1, \ldots, w_n) \in X$ one has
\[
\varphi(r) \leq \frac{\sup_{w \in \Phi^{-1}([0, +\infty])} \Psi}{r} = \frac{1}{r} \sup_{w \in \Phi^{-1}([0, +\infty])} \int_{\Omega} F(x, w_1, \ldots, w_n) dx \leq \frac{1}{r} \sup_{w \in X} \sum_{i=1}^n \int_{\Omega} F(x, w_1, \ldots, w_n) dx.
\]

Let $\{\sigma_k\}$ be a real sequence of positive numbers such that $\lim_{k \to +\infty} \sigma_k = +\infty$ and
\[
\lim_{k \to +\infty} \sup_{|\tau|, \tau_r \leq \sigma_k} \frac{F(x, \tau_1, \ldots, \tau_n) dx}{\sigma_k^{p_i}} = A < +\infty.
\]
And define 
\[ r_k := \left( \frac{\sigma_k}{\sum_{i=1}^{n}(Cp_i)^{\frac{1}{p_i}}} \right)^{p_i}. \]

Let \( w = (w_1, \ldots, w_n) \in \Phi^{-1}([-\infty, r_k]) \), from (6), one has
\[ \sum_{i=1}^{n} \frac{1}{p_i} |w_i(x)|^{p_i} < C_{r_k} \text{ for all } x \in \Omega. \]

Then
\[ |w_i(x)| \leq (C_{r_k} p_i)^{\frac{1}{p_i}} \text{ for } i = 1, \ldots, n, \]
thus, for each \( k \in \mathbb{N} \) large enough \( (r_k > 1) \),
\[ \sum_{i=1}^{n} |w_i(x)| \leq (C_{r_k} p_i)^{\frac{1}{p_i}} = \sigma_k. \]

Define \( \kappa := \lim \inf_{r \to +\infty} \varphi(r) \), so from (12) and (13), one has
\[ \kappa \leq \lim \inf_{k \to +\infty} \varphi(r_k) \leq \left( \int_{\Omega} F(x, w_1, \ldots, w_n) dx \right) \frac{\sigma_k}{\left( \sum_{i=1}^{n}(Cp_i)^{\frac{1}{p_i}} \right)^{p_i}} \]
\[ \leq \left( \sum_{i=1}^{n}(Cp_i)^{\frac{1}{p_i}} \right)^{p_i} \int_{\Omega} \sup_{\sum_{i=1}^{n}|\tau_i| \leq \delta} F(x, \tau_1, \ldots, \tau_n) dx. \]

Define \( \kappa := \lim \inf_{r \to +\infty} \varphi(r) \), so from (12) and (13), one has
\[ \kappa \leq \lim \inf_{k \to +\infty} \varphi(r_k) \leq \left( \int_{\Omega} F(x, w_1, \ldots, w_n) dx \right) \frac{\sigma_k}{\left( \sum_{i=1}^{n}(Cp_i)^{\frac{1}{p_i}} \right)^{p_i}} \]
\[ \leq \left( \sum_{i=1}^{n}(Cp_i)^{\frac{1}{p_i}} \right)^{p_i} \int_{\Omega} \sup_{\sum_{i=1}^{n}|\tau_i| \leq \delta} F(x, \tau_1, \ldots, \tau_n) dx. \]

On the other hand, we know that
\[ \kappa \leq \left( \sum_{i=1}^{n}(Cp_i)^{\frac{1}{p_i}} \right)^{p_i} A < \frac{1}{\mu}. \]
for $k$ large enough, where $\hat{c}$ is as in (2). Let

$$w_k(x) := \begin{cases} 0 & x \in \Omega \setminus B(x_0, R_2), \\ \frac{\alpha_k}{R_2 - R_1} \left( R_2 - (\sum_{i=1}^n (x^i - x_0^i)^2)^{1/2} \right) & x \in B(x_0, R_2) \setminus B(x_0, R_1), \\ \alpha_k & x \in B(x_0, R_1). \end{cases}$$

Then $(w_k, \ldots, w_k) \in X$ and for each $1 \leq i \leq n$ we have

$$\|w_k\|_{X_i} = \frac{\pi^{\frac{n}{2}}}{\Gamma(1 + \frac{n}{2})} \left( 2N\alpha_k \right)^{\frac{n}{2}} (R_N^N - R_i^N),$$

see more details in [13]. In accordance with (2) one has

$$\|w_k\|_{X_i} \leq \|w_k\|_{C_{\hat{c}}} \leq \hat{c}_i \|w_k\|_{X_i}.$$

Bearing (8) and (4) in mind, we deduce

$$\Phi(w_k, \ldots, w_k) = \int_{\Omega} \sum_{i=1}^n \frac{1}{p_i} (|\Delta w_k(x)|^p + \mu_i(x)|\nabla w_k(x)|^p) \, dx$$

$$= \sum_{i=1}^n \frac{1}{p_i} \|w_k\|_{p_i}^p$$

$$\leq \frac{\hat{c}_i}{(\sum_{i=1}^n (C_{p_i})^\frac{1}{p_i})^p} \sum_{i=1}^n \frac{1}{p_i} \alpha_k^{p_i}.$$ \hspace{1cm} (17)

On the other hand, according to our assumptions

$$\Psi(w_k, \ldots, w_k) = \int_{\Omega} F(x, w_k, \ldots, w_k) \, dx \geq \int_{B(x_0, R_1)} F(x, w_k, \ldots, w_k) \, dx.$$ \hspace{1cm} (18)

So, it follows from (15), (17) and (18) that

$$I_\mu(w_k, \ldots, w_k) = \Phi(w_k, \ldots, w_k) - \mu \Psi(w_k, \ldots, w_k)$$

$$\leq \frac{\hat{c}_i}{(\sum_{i=1}^n (C_{p_i})^\frac{1}{p_i})^p} \sum_{i=1}^n \frac{1}{p_i} a_i \alpha_k^{p_i} - \mu \int_{B(x_0, R_1)} F(x, \alpha_k, \ldots, \alpha_k) \, dx$$

$$< \frac{\hat{c}_i}{\delta (\sum_{i=1}^n (C_{p_i})^\frac{1}{p_i})^p} \left( \sum_{i=1}^n \frac{1}{p_i} a_i \alpha_k^{p_i} \right).$$ \hspace{1cm} (19)

for $k$ large enough, so $\lim_{k \to +\infty} I_\mu(w_k, \ldots, w_k) = -\infty$, and hence the claim has been archived.

The alternative of Theorem 1.1 case (a) assures the existence of unbounded sequence $\{w_k\}$ of critical points of the functional $I_\mu$. This completes the proof in view of the relation between the critical points of $I_\mu$ and the weak solutions of problem (1). \hfill \Box

References


[17] A. Razani, Game-theoretic \( p \)-Laplace operator involving the gradient, Miskolc Mathematical Notes Accepted (2020).

