Berezin Number Inequalities via Convex Functions

Mualla Birgül Huban\textsuperscript{a}, Hamdullah Başaran\textsuperscript{b}, Mehmet Gürdal\textsuperscript{b}

\textsuperscript{a}Isparta University of Applied Sciences, Isparta, Turkey
\textsuperscript{b}Department of Mathematics, Suleyman Demirel University, 32260, Isparta, Turkey

Abstract. The Berezin symbol \( \tilde{A} \) of an operator \( A \) on the reproducing kernel Hilbert space \( \mathcal{H}(\Omega) \) over some set \( \Omega \) with the reproducing kernel \( k_\xi \) is defined by
\[
\tilde{A}(\xi) = \left( \frac{k_\xi}{\|k_\xi\|}, \frac{k_\xi}{\|k_\xi\|} \right), \quad \xi \in \Omega.
\]
The Berezin number of an operator \( A \) is defined by
\[
\text{ber}(A) := \sup_{\xi \in \Omega} |\tilde{A}(\xi)|.
\]

We study some problems of operator theory by using this bounded function \( \tilde{A} \), including treatments of inner product inequalities via convex functions for the Berezin numbers of some operators. We also establish some inequalities involving the Berezin inequalities.

1. Introduction

Let \( \Omega \) be a subset of a topological space \( X \) such that the boundary \( \partial \Omega \) is nonempty. Let \( \mathcal{H} \) be an infinite-dimensional Hilbert space complex-valued functions defined on \( \Omega \). We say that \( \mathcal{H} \) is a reproducing kernel Hilbert space if the following two conditions are satisfied:

(i) for any \( \xi \in \Omega \), the evaluation functionals \( f \mapsto f(\xi) \) are continuous on \( \mathcal{H} \);
(ii) for any \( \xi \in \Omega \), there exists \( f_\xi \in \mathcal{H} \) such that \( f_\xi(\xi) \neq 0 \) (or equivalently, there is no \( \xi_0 \in \Omega \) such that \( f(\xi_0) = 0 \) for every \( f \in \mathcal{H} \)).

According to the classical Riesz representation theorem, the assumption (i) implies that, for every \( \xi \in \Omega \) there exists a unique function \( k_\xi \in \mathcal{H} \) such that
\[
f(\xi) = \langle f, k_\xi \rangle_{\mathcal{H}}, \quad f \in \mathcal{H}.
\]
The function \( k_\xi(z) \) is called the reproducing kernel of \( \mathcal{H} \) at point \( \xi \). It is well known that every reproducing kernel Hilbert space is separable. So, if \( \{e_n(z)\}_{n \geq 0} \) is any orthonormal basis of \( \mathcal{H} \), then (see Aronjain [3])
\[
k_\xi(z) = \sum_{n=0}^{\infty} e_n(\xi)e_n(z).
\]

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\textbf{Email addresses:} muallahuban@isparta.edu.tr (Mualla Birgül Huban), hamdullahbasaran@gmail.com (Hamdullah Başaran), gurdalmehmet@sdu.edu.tr (Mehmet Gürdal)
By virtue of assumption (ii), we surely have $k_\xi \neq 0$ and we denote by $\tilde{k}_\xi$ the normalized reproducing kernel, that is $\tilde{k}_\xi := \frac{k_\xi}{\|k_\xi\|}$. Recall that if $\mathcal{B}(\mathcal{H})$ is the Banach algebra of all bounded linear operator on $\mathcal{H}$, then the Berezin symbol $\tilde{A}$ of any operator $A \in \mathcal{B}(\mathcal{H})$ is the complex-valued function defined on the $\Omega$ by the formula (see, Berezin [8, 9])

$$\tilde{A}(\xi) := \left( A\tilde{k}_\xi, \tilde{k}_\xi \right)_\mathcal{H}, \quad \xi \in \Omega.$$ 

The Berezin set of operator $A$ is defined by

$$\text{Ber} (A) = \left( \left( A\tilde{k}_\xi, \tilde{k}_\xi \right)_\mathcal{H} : \xi \in \Omega \right) = \text{Range} (\tilde{A}),$$

and Berezin number $\text{ber} (A)$ of operator $A$ is the following number (see [25, 26])

$$\text{ber} (A) := \sup_{\xi \in \Omega} \left| \tilde{A}(\xi) \right|.$$ 

Since, $\left| \tilde{A}(\xi) \right| \leq ||A||$, Berezin symbol is a bounded function on $\Omega$. Also, it is trivial by Cauchy-Schwarz inequality that $\text{ber} (A) \leq ||A||$. If $A = cI$ with $c \neq 0$, then obviously $\text{ber} (A) = |c| > \frac{|c|}{2} = \frac{1}{2} ||A||$. But Karaev in [26] showed that in general

$$\frac{1}{2} ||A|| \leq \text{ber} (A)$$

is not satisfied for every $A \in \mathcal{B}(\mathcal{H})$.

Berezin set and Berezin number of operators are new numerical characteristics of operators on the RKHS which are introduced by Karaev in [25]. For the basic properties and facts on these new concepts, see [5–7, 26, 32, 34].

It is well-known that

$$\text{ber} (A) \leq w (A) \leq ||A||$$

(1)

and

$$\frac{1}{2} ||A|| \leq w (A) \leq ||A||$$

(2)

for any $A \in \mathcal{B}(\mathcal{H})$. The inequalities in (2) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if $A$ normal. For basic properties of the numerical radius, we refer to [20] and [21]. The inequalities in (2) have been improved considerably by the second author in [29] and [31]. It has been shown in [29] and [31], respectively, that if $A \in \mathcal{B}(\mathcal{H})$, then

$$w (A) \leq \frac{1}{2} ||A|| + ||A^*|| \leq \frac{1}{2} \left( ||A|| + ||A^2||^{1/2} \right),$$

(3)

where $|A| = (A^*A)^{1/2}$ is the absolute value of $A$, and

$$\frac{1}{4} ||A^*A + AA^*|| \leq w^2 (A) \leq \frac{1}{2} ||A^*A + AA^*||.$$

The inequalities in (3), which refine the second inequality in (2), have been utilized in [29] to derive an estimate for the numerical radius of the Frobenius companion matrix (also see [1, 2, 12, 23]).

The purpose of this paper is to establish some inequalities involving of the Berezin number inequalities of operators by using convex function $\tilde{A}$. Usual operator norm inequalities and a related Berezin number inequality of operators are also presented. Related results are contained in [15–19, 22, 35–37].
2. Bereznin Number Inequalities

2.1. Lemmas

In order to prove our results, we need the following sequence of lemmas.

Recall that an operator \( A \in B(H) \) is called positive if \( \langle Ax, x \rangle \geq 0 \) for all \( x \in H \). In this case we will write \( A \geq 0 \). The classical operator Jensen inequality for the positive operators \( A \in B(H) \) is

\[
\langle Ax, x \rangle^r \leq (\langle A^r x, x \rangle)^{\frac{r}{2}}, \quad r \geq 1 \quad (0 \leq r \leq 1)
\]

for any unit vector \( x \in H \).

**Lemma 2.1.** We have the Power-Mean inequality, that reads

\[
a^\lambda b^{1 - \lambda} \leq (\lambda a + (1 - \lambda)b)^\frac{1}{p},
\]

for \( a, b \geq 0 \), \( 0 \leq \lambda \leq 1 \), and \( p \geq 1 \).

The following inequality is the spectral theorem for positive operators and Jensen inequality (see [14]) which states that if \( f \) is a convex function on an interval containing the spectrum of \( A \), then

\[
f(\langle Ax, x \rangle) \leq f(\langle A \rangle x, x)
\]

which \( A \) is a positive operators in \( B(H) \) and \( x \in H \) is an unit vector. If \( f \) is concave, then (6) holds in the reverse direction.

The mixed Schwarz inequality was introduced in [21], as follows:

**Lemma 2.2.** Let \( A \in B(H) \) and let \( x \in H \) be a unit vector. Then

\[
\|\langle Ax, x \rangle\|^2 \leq \langle|A| x, x \rangle \langle|A'| x, x \rangle.
\]

Another inequality, which can be found by Aujla and Silva [4], gives a norm inequality involving convex function of positive operator which assert

\[
\left\| f\left(\frac{A + B}{2}\right) \right\| \leq \left\| \frac{f(A) + f(B)}{2} \right\|
\]

which \( f \) be a non-negative nondecreasing convex function on \([0, \infty)\) and \( A, B \in B(H) \) be positive operators.

In 2004, Kittaneh [30] has shown follow inequality and this follow inequality is considered as a refined triangle inequality for positive operators.

**Lemma 2.3.** Let \( A \in B(H) \). Then

\[
\|A\|^2 + |A^*|^2 \leq \|A^2\| + ||A||^2.
\]

The following lemma contains a special case of a more general norm inequality that is equivalent to some Löwner–Heinz type inequalities (see [13, 27]).

**Lemma 2.4.** If \( A, B \in B(H) \) are positive operators, then

\[
\|A^{1/2}B^{1/2}\| \leq \|AB\|^{1/2}.
\]

The last lemma contains a recent norm inequality for sums of positive operators that is sharper than the triangle inequality (see [28]).

**Lemma 2.5.** If \( A, B \in B(H) \) are positive operators, then

\[
\|A + B\| \leq \frac{1}{2} (\|A\| + \|B\|) + \sqrt{\|A\| - \|B\|^2} + 4 \|A^{1/2}B^{1/2}\|.
\]
2.2. Main Results

Now, we are ready to state the main results of this section. Our first main result can be stated as follows.

**Theorem 2.6.** If \( A, B \in \mathcal{B}(\mathcal{H}(\Omega)) \) and \( f : [0, \infty) \to \mathbb{R} \) is an increasing convex function, then

\[
 f \left( \left\| \overline{A} (\xi) \overline{B} (\xi) \right\| \right) \leq \frac{1}{2} f \left( \left\| \overline{BA} (\xi) \right\|^2 \right) + \frac{1}{2} \left( (\lambda f (|A|^2) + (1 - \lambda) f (|B'|^2)) \right) k_{\xi} k_{\xi} \tag{10}
\]

for \( 0 \leq \lambda \leq 1 \). Further,

\[
 f \left( \left\| \overline{A} (\xi) \overline{B} (\xi) \right\| \right) \leq \frac{1}{2} f \left( \left\| \overline{BA} (\xi) \right\|^2 \right) + \frac{1}{4} \left( (f (|A|^2) + f (|B'|^2)) \right) k_{\xi} k_{\xi} \tag{11}
\]

**Proof.** Let \( \tilde{k}_{\xi} \) be normalized reproducing kernel. The following refinement of the Cauchy-Schwarz inequality proved by Buzano [11]:

\[
 \|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle|, \tag{12}
\]

for all \( x, y, e \in \mathcal{H} \) and \( \|e\| = 1 \). From inequality (12), we conclude that

\[
 \frac{1}{2} \left( \|x\| \|y\| + |\langle x, y \rangle| \right) \geq |\langle x, e \rangle \langle e, y \rangle|.
\]

Putting \( e = \tilde{k}_{\xi}, x = A \tilde{k}_{\xi} \) and \( y = B \tilde{k}_{\xi} \) in the above, we have

\[
 \frac{1}{2} \left( \|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \| + \|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \| \right) \geq \left| \langle A \tilde{k}_{\xi}, \tilde{k}_{\xi} \rangle \langle B \tilde{k}_{\xi}, \tilde{k}_{\xi} \rangle \right|. \tag{13}
\]

Hence, by the function \( t \to t^2 \) is convex,

\[
 \left| \langle A \tilde{k}_{\xi}, \tilde{k}_{\xi} \rangle \langle B \tilde{k}_{\xi}, \tilde{k}_{\xi} \rangle \right|^2 \leq \left( \frac{\|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \|}{2} \right)^2
\]

\[
 \leq \frac{1}{2} \left( \|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \|^2 \right) + \frac{1}{2} \left( \|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \|^2 \right)
\]

\[
 = \frac{1}{2} \left( \|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \|^2 \right) + \left( \|A \|^2 \|B \|^2 \right) \left( \|B' \|^2 \right) k_{\xi} k_{\xi}
\]

\[
 = \frac{1}{2} \left( \|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \|^2 \right) + \left( \|A \|^2 \|B \|^2 \right) \left( \|B' \|^2 \right) k_{\xi} k_{\xi}
\]

\[
 \leq \frac{1}{2} \left( \|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \|^2 \right) + \left( \|A \|^2 \|B \|^2 \right) \left( \|B' \|^2 \right) k_{\xi} k_{\xi}
\]

(by the inequality (4))

\[
 \leq \frac{1}{2} \left( \|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \|^2 \right) + \lambda \left( \|A \|^2 \|B \|^2 \right) \left( \|B' \|^2 \right) k_{\xi} k_{\xi}
\]

(by the inequality (5)).

Hence,

\[
 \left| \langle A \tilde{k}_{\xi}, \tilde{k}_{\xi} \rangle \langle B \tilde{k}_{\xi}, \tilde{k}_{\xi} \rangle \right|^2 \leq \frac{1}{2} \left( \|A \tilde{k}_{\xi} \| \|B \tilde{k}_{\xi} \|^2 \right) + \lambda \left( \|A \|^2 \|B \|^2 \right) \left( \|B' \|^2 \right) k_{\xi} k_{\xi}
\]

\[
 \tag{14}
\]
Now since $f$ is increasing and convex, (14) implies
\[
\begin{align*}
&f\left(\left\langle \tilde{A}_k, \tilde{k}_\ell \right\rangle \left\langle \tilde{B}_k, \tilde{k}_\ell \right\rangle^2\right) \\
&\leq f\left(\frac{1}{2} \left(\left\langle \tilde{B}\tilde{A}, \tilde{k}_\ell \right\rangle^2 + \lambda \left(\left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle + (1 - \lambda) \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle\right)\right) \\
&\leq f\left(\frac{\left\langle \left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle + \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle}{2}\right) \\
&\leq f\left(\left\langle \left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle + \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle\right) \\
&\leq \frac{f\left(\left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle + \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle\right)}{2} \\
&\leq \frac{f\left(\left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle + \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle\right)}{2} \\
\end{align*}
\]
(by inequality (6))
\[
\begin{align*}
&f\left(\left\langle \left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle + \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle\right) \\
&\leq \frac{f\left(\left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle + \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle\right)}{2} \\
&\end{align*}
\]
which is equivalent to
\[
\begin{align*}
&f\left(\left\langle A (\tilde{\xi}) B (\tilde{\xi}) \right\rangle^2\right) \\
&\leq \frac{1}{2} f\left(\left\langle B\tilde{A} (\tilde{\xi}) \right\rangle^2\right) + \frac{1}{2} \left(\left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle + \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle\right) \\
\end{align*}
\]
On the other hand, from (13), we get
\[
\begin{align*}
\left\langle A_k, \tilde{k}_\ell \right\rangle \left\langle B_k, \tilde{k}_\ell \right\rangle &\leq \frac{1}{2} \left(\left\langle \tilde{B}\tilde{A}, \tilde{k}_\ell \right\rangle + \left\langle A_k, \tilde{k}_\ell \right\rangle \left\langle B_k^2, \tilde{k}_\ell \right\rangle \right) \\
&= \frac{1}{2} \left(\left\langle \tilde{B}\tilde{A}, \tilde{k}_\ell \right\rangle + \left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle \right) \\
&\leq \frac{1}{2} \left(\left\langle \tilde{B}\tilde{A}, \tilde{k}_\ell \right\rangle + \left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle \right) \\
\end{align*}
\]
(by the AM-GM inequality).

Again, since $f$ is increasing and convex, we obtain
\[
\begin{align*}
&f\left(\left\langle A_k, \tilde{k}_\ell \right\rangle \left\langle B_k, \tilde{k}_\ell \right\rangle\right) \\
&\leq \frac{1}{2} \left(f\left(\left\langle \tilde{A}_k, \tilde{k}_\ell \right\rangle + \left\langle \tilde{B}_k, \tilde{k}_\ell \right\rangle\right)\right) \\
&\leq \frac{1}{2} \left(\frac{f\left(\left\langle \tilde{A}_k, \tilde{k}_\ell \right\rangle + \left\langle \tilde{B}_k, \tilde{k}_\ell \right\rangle\right)}{2}\right) \\
&= \frac{1}{2} f\left(\left\langle \tilde{A}_k, \tilde{k}_\ell \right\rangle + \frac{1}{4} \left\langle \left\langle |A|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle + \left\langle |B|^2 \tilde{k}_\ell, \tilde{k}_\ell \right\rangle\right)\right) \\
\end{align*}
\]
Consider the function $\xi$. By taking supremum over $\lambda$, we obtain the desired inequality.

In particular, if $p \geq 1$, then the inequality for the product of two operators.

Corollary 2.7. Let $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$. Then for any $p \geq 1$ and $0 \leq \lambda \leq 1$,

$$
\|A(\xi)B(\xi)\|^{2p} \leq \frac{1}{2} \|\overline{BA}(\xi)\|^{2p} + \frac{1}{4} \left(\|A\|^{2p} + \|B\|^{2p}\right)\|k_\xi, k_\xi\|,
$$

and

$$
\|A(\xi)B(\xi)\|^{p} \leq \frac{1}{2} \|\overline{BA}(\xi)\|^{p} + \frac{1}{4} \left(\|A\|^{p} + \|B\|^{p}\right)\|k_\xi, k_\xi\|.
$$

The first application of Theorem 2.6 and Corollary 2.7 is the following ber-norm and Berezin number inequality for the product of two operators.

Corollary 2.8. If $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ and $f : [0, \infty) \to \mathbb{R}$ is an increasing convex function, then

$$
f(\text{ber}^2(BA)) \leq \frac{1}{2} f(\text{ber}(\|A\|^2)) + \frac{1}{4} \left\| \|A\|^4 + f(\|B\|^4) \right\|_{\text{ber}}.
$$

In particular, if $p \geq 1$, then

$$
\text{ber}^{2p}(BA) \leq \frac{1}{2} \text{ber}^{2p}(\|A\|^2) + \frac{1}{4} \left\| \|A\|^{4p} + |B|^{4p} \right\|_{\text{ber}}. \tag{15}
$$

Proof. Replacing $A$ and $B$ by $|A|^2$ and $|B|^2$ respectively Theorem 2.6, then the inequality (11) reduces to

$$
f\left(\left(\|A\|^2 k_\xi, k_\xi, k_\xi, k_\xi\right) \right) \leq \frac{1}{2} f\left(\left(\|A\|^2 k_\xi, k_\xi, k_\xi, k_\xi\right) \right) + \frac{1}{4} \left(\|A\|^4 + f(\|B\|^4)\right)\|k_\xi, k_\xi\|.
$$

On the other hand,

$$
\left\|B^* A k_\xi, k_\xi\right\|^2 = \left|\langle A k_\xi, B k_\xi \rangle\right|^2 \\
\leq \|A k_\xi\|^2 \|B k_\xi\|^2
\quad \text{(by the Cauchy-Schwarz inequality)} \\
\quad = \left|\langle A k_\xi, k_\xi \rangle \langle B^* k_\xi, k_\xi \rangle\right|.
$$

Since $f$ is increasing, we get

$$
f\left(\left|\langle A k_\xi, k_\xi \rangle \langle B^* k_\xi, k_\xi \rangle\right|^2\right) \leq f\left(\left|\langle A k_\xi, k_\xi \rangle \langle B^* k_\xi, k_\xi \rangle\right|\right)
$$

and this together with (16) imply

$$
f\left(\left|\langle A(\xi)B(\xi)\rangle\right|^2\right) \leq \frac{1}{2} f\left(\left|\langle B^* |A|^2(\xi)\rangle\right|\right) + \frac{1}{4} \left(\|A\|^4 + f(\|B\|^4)\right)\|k_\xi, k_\xi\|.
$$

By taking supremum over $\xi \in \Omega$, we have

$$
f(\text{ber}^2(BA)) \leq \frac{1}{2} f(\text{ber}(\|A\|^2)) + \frac{1}{4} \left\| \|A\|^4 + f(\|B\|^4) \right\|_{\text{ber}}.
$$

Consider the function $f(t) = t^p$, $p \geq 1$, then we get the second inequality. This completes the proof. \qed
Remark 2.9. Since for $p = 1$ and $A = B$, we get on both sides of (15) the same quantity $\|A\|_{\text{ber}}^4$.

Corollary 2.10. If $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ then

$$\text{ber}(B^*A) \leq \frac{1}{2}\|\|A\|^2 + |B|^2\|$$

and

$$\text{ber}^p(B^*A) \leq \frac{1}{2}\|\|A\|^p + |B|^p\|, \quad p \geq 1.$$  \hfill (17)

Proof. We recall the following arithmetic-geometric mean inequality obtained in [10]

$$\|B^*A\| \leq \left(\|A\| + |B|\right)^2 (\text{by (18)})$$

Hence, by the inequality (1),

$$\text{ber}(B^*A) \leq \|B^*A\| \leq \frac{1}{4}\|\|A\|^2 + |B|^2\|$$

(by the inequality (8)).

Notice that

$$\text{ber}^p(\|A\|^2 |B|^2) \leq \frac{1}{2}\|\|A\|^p + |B|^p\||.$$

Also Corollary 2.8 implies that

$$\text{ber}^p(B^*A) \leq \frac{1}{2}\text{ber}^p(\|A^2 |B|^2\|) + \frac{1}{4}\|\|A|^p + |B|^p\||,$$

explaining why Corollary 2.8 provide a refinement of the inequality (17). Further, the first inequality in Corollary 2.8 provides a generalization of (17). \hfill \Box

Now Theorem 2.6 is utilized to obtain the following one-operator Berezin number inequality.

Corollary 2.11. If $E \in \mathcal{B}(\mathcal{H}(\Omega))$ and $f: [0, \infty) \to \mathbb{R}$ is an increasing convex function, then for $0 \leq \lambda \leq 1$,

$$f(\text{ber}^4(E)) \leq \frac{1}{2}f(\text{ber}^2(|E| |E'|)) + \frac{1}{2}\left\|\left(1 - \lambda\right)f(\left|E\right|^2) + \lambda f\left(\left|E'\right|^2\right)\right\|_{\text{ber}},$$

and

$$f(\text{ber}^2(E)) \leq \frac{1}{2}\left(f(\text{ber}(|E| |E'|)) + \frac{1}{2}\left\|f(\left|E\right|^2) + f\left(\left|E'\right|^2\right)\right\|_{\text{ber}}\right).$$

In particular, if $p \geq 1$, then

$$\text{ber}^{4p}(E) \leq \frac{1}{2}\text{ber}^{2p}(\|E| |E'|\|) + \frac{1}{2}\left\|(1 - \lambda)\|E\|^{2p} + \lambda \|E'|^{2p}\right\|_{\text{ber}},$$

and

$$\text{ber}^{2p}(E) \leq \frac{1}{2}\text{ber}^{p}(\|E| |E'|\|) + \frac{1}{4}\left\|E|^{2p} + \|E'|^{2p}\right\|_{\text{ber}},$$

(19)
Proof. Replacing $A = |E|$ and $B = |E|$ in the inequality (10), we get

$$f\left(\left|\langle E|\kappa_{\xi}\rangle\langle E^*|\kappa_{\xi}\rangle\right|^p\right) \leq f\left(\left|\langle E|E^*|\kappa_{\xi}\rangle\kappa_{\xi}\rangle\right|^p + \left|\left(1 - \lambda\right)f\left(|E|^{2}t^*\right) + \lambda f\left(|E^*|t^*\right)\right|\kappa_{\xi}\rangle\kappa_{\xi}\rangle\right).$$

Since $f$ is increasing, it follows from inequality (7) that

$$f\left(\left|\langle E|\kappa_{\xi}\rangle\langle E^*|\kappa_{\xi}\rangle\right|^2\right) \leq f\left(\left|\langle E^*|\kappa_{\xi}\rangle\kappa_{\xi}\rangle\right|^{2} + \left|\left(1 - \lambda\right)f\left(|E|^{2}t^*\right) + \lambda f\left(|E^*|t^*\right)\right|\kappa_{\xi}\rangle\kappa_{\xi}\rangle\right).$$

and

$$\sup_{\xi \in \Omega} f\left(\left|\langle E|\kappa_{\xi}\rangle\langle E^*|\kappa_{\xi}\rangle\right|^2\right) \leq \sup_{\xi \in \Omega} f\left(\left|\langle E^*|\kappa_{\xi}\rangle\kappa_{\xi}\rangle\right|^{2} + \left|\left(1 - \lambda\right)f\left(|E|^{2}t^*\right) + \lambda f\left(|E^*|t^*\right)\right|\kappa_{\xi}\rangle\kappa_{\xi}\rangle\right).$$

which is equivalent to

$$f\left(\text{ber}^2\left(|E|\right)\right) \leq \frac{1}{2}f\left(\text{ber}^2\left(|E|^{2}\right)\right) + \frac{1}{2} \left|\left(1 - \lambda\right)f\left(|E|^{2}t^*\right) + \lambda f\left(|E^*|t^*\right)\right|_{\text{ber}},$$

and completes the proof of the first inequality of the theorem. By using (11) inequality, the second inequality follows similarly way. The other two inequalities follow by letting $f\left(t\right) = t^p, \, p \geq 1$. □

The following result will be needed for further investigation.

**Proposition 2.12.** If $A \in \mathcal{B}\left(\mathcal{H}\left(\Omega\right)\right)$, then for any $p \geq 1$ and $0 \leq \lambda \leq 1$,

$$\text{ber}^2\left(|A|\right) \leq \left|\left(1 - \lambda\right)|A|^2 + \lambda |A^*|^2\right|_{\text{ber}},$$

and

$$\text{ber}^p\left(|A|\right) \leq \frac{1}{2} \left|\left|A\right|\right|^p + \left|\left|A^*\right|\right|^p, \quad (20)$$

**Proof.** Let $\kappa_{\xi} \in \mathcal{H}$ be a normalized reproducing kernel. We have

$$\left|\left|A|A^*|\kappa_{\xi}\rangle\kappa_{\xi}\rangle\right|^2\right| = \left|\left|A^*\langle \kappa_{\xi}\rangle\langle A|\kappa_{\xi}\rangle\right|^2\right| \leq \left|\left|A^*\kappa_{\xi}\rangle\kappa_{\xi}\rangle\right|^2\left|\left|A\kappa_{\xi}\rangle\kappa_{\xi}\rangle\right|^2\right|$$

(by the Cauchy-Schwarz inequality)
By the inequality (19) and by the AM-GM inequality, we have

\[ \sup_{\xi \in \Omega} |A| |A^\ast (\xi)|^{2p} \leq \sup_{\xi \in \Omega} \left( (1 - \lambda) |A|^{2p} + \lambda |A^\ast|^{2p} \right) \]

which clearly implies that

\[ \text{ber}^{2p} (|A| |A^\ast|) \leq \left\| (1 - \lambda) |A|^{2p} + \lambda |A^\ast|^{2p} \right\|_{\text{ber}}. \]  

(21)

Similar arguments implies

\[ \| A | A^\ast (\xi) \|^p \leq \frac{1}{2} \left( \| A^{2p} + |A^\ast|^{2p} \|_{\text{ber}} \right), \]

for any \( \xi \in \Omega \). By taking supremum over \( \lambda \in \Omega \), we have

\[ \text{ber}^{2p} (|A| |A^\ast|) \leq \frac{1}{2} \left\| A^{2p} + |A^\ast|^{2p} \right\|_{\text{ber}}. \]

Hence, we get the desired inequality (20).

\[ \Box \]

**Remark 2.13.** By combining inequalities (19) and (20), we infer that

\[ \text{ber}^{2p} (A) \leq \frac{1}{2} \text{ber}^{p} (|A| |A^\ast|) + \frac{1}{4} \left\| A^{2p} + |A^\ast|^{2p} \right\|_{\text{ber}} \leq \frac{1}{2} \left\| A^{2p} + |A^\ast|^{2p} \right\|_{\text{ber}}. \]  

(22)

The inequalities (22) provide a refinement of the inequality (3) (also, [24, Theorem 1]).

Now we are in a position to present our refined Berezin number inequality.

**Theorem 2.14.** If \( A \in B(H(\Omega)) \), then

\[ \text{ber} (A) \leq \frac{1}{2} \left( \| A^{2} \|^{1/2}_{\text{ber}} + \| A \|_{\text{ber}} \right). \]

(23)

**Proof.** By the inequality (19) and by the AM-GM inequality, we have

\[ \left\| A \kappa_{\xi}, \kappa_{\bar{\xi}} \right\| \leq \left\| A \kappa_{\xi}, \kappa_{\bar{\xi}} \right\|^{1/2} \left\| A^\ast \kappa_{\bar{\xi}}, \kappa_{\bar{\xi}} \right\|^{1/2} \]

\[ \leq \frac{1}{2} \left( \left\| A \kappa_{\xi}, \kappa_{\bar{\xi}} \right\| + \left\| A^\ast \kappa_{\bar{\xi}}, \kappa_{\bar{\xi}} \right\| \right) \]

\[ \leq \frac{1}{2} \left( \left\| A + |A^\ast| \right\| \kappa_{\bar{\xi}}, \kappa_{\bar{\xi}} \right). \]
for every $\xi \in \Omega$. Thus
\begin{equation}
\text{ber} (A) = \sup_{\xi \in \Omega} \left| A \left( \xi, \xi \right) \right| = \sup_{\xi \in \Omega} \left| \left( A \tilde{k}_\xi, \tilde{k}_\xi \right) \right| \leq \frac{1}{2} \sup_{\xi \in \Omega} \left( \left| A \right| + \left| A^* \right| \right) \| \tilde{k}_\xi, \tilde{k}_\xi \| \leq \frac{1}{2} \| A \| + \| A^* \| \| \text{ber} \|.
\end{equation}

Applying Lemmas 2.4 and 2.5 to the positive operators $|A|$ and $|A^*|$, and using the facts that $\|A\| = \|A^*\| = \|A\|$ and $\|A\| |A^*\| = \|A^2\|$, we have
\begin{equation}
\| A \| + \| A^* \| \| \text{ber} \| \leq \| A^2 \|^{1/2} + \| A \| \| \text{ber} \|.
\end{equation}
The desired inequality (23) now follows from (24) and (25).  

The following result is a consequence of the inequality (23).

**Lemma 2.15.** If $A \in \mathcal{B}(\mathcal{H}(\Omega))$ is such that $\text{ber} (A) = \| A \| \| \text{ber} \|$, then $\| A^2 \| \| \text{ber} \| = \| A \| \| \text{ber} \|^2$.

**Proof.** It follows from the inequality (23) that
\[ 2 \text{ber} (A) \leq \| A^2 \|^{1/2} \| \text{ber} \| \| A \| \| \text{ber} \|. \]

The following another result shows that the inequality (19) provides an improvement of the inequality (23).

**Corollary 2.16.** If $A \in \mathcal{B}(\mathcal{H}(\Omega))$, then
\[ \text{ber} (A) \leq \frac{1}{2} \sqrt{2 \text{ber} (|A| \left| A^* \right|) + \| |A| \left| A^* \right| \| \text{ber} + \| A \| \| A^* \| \| \text{ber} + \| A \| \| \text{ber} \|^2}. \]

**Proof.** Let $\tilde{k}_\xi \in \mathcal{H}$ be a normalized reproducing kernel. We get
\begin{align*}
\text{ber} (A) &\leq \frac{1}{2} \sqrt{2 \text{ber} (|A| \left| A^* \right|) + \| |A| \left| A^* \right| \| \text{ber} + \| A \| \| A^* \| \| \text{ber} + \| A \| \| \text{ber} \|^2} \\
&\leq \frac{1}{2} \sqrt{2 \| |A| \left| A^* \| \| \text{ber} + \| A \| \| A^* \| \| \text{ber} + \| A \| \| \text{ber} \|^2} \\
&\leq \frac{1}{2} \sqrt{2 \| A^2 \| \| \text{ber} + \| A \| \| A^* \| \| \text{ber} + \| A \| \| \text{ber} \|^2} \\
&\leq \frac{1}{2} \sqrt{2 \| A \| \| A^* \| \| \text{ber} + \| A \| \| A^* \| \| \text{ber} + \| A \| \| \text{ber} \|^2} \\
&\leq \frac{1}{2} \left( \| A^2 \|^{1/2} + \| A \| \| \text{ber} \|^2 \right) \\
&\leq \frac{1}{2} \left( \| A^2 \|^{1/2} + \| A \| \| \text{ber} \|^2 \right).
\end{align*}

This completes the proof.  

We give the following example which show that $\text{ber}(A) = \max_{1 \leq j \leq n} |a_{jj}|$ for any complex $n \times n$ matrix $A = (a_{jk})_{j,k=1}^n$.

**Example 2.17.** Let us consider the finite dimensional setting. $A = (a_{jk})_{j,k=1}^n$ be a $n \times n$ matrix. Let $\nu = (\nu_1, ..., \nu_n) \in \mathbb{C}^n$ and $X = \{1, ..., n\}$. We can consider $\mathbb{C}^n$ as the set of all functions mapping $X \to \mathbb{C}$ by $\nu(j) = \nu_j$. Letting $e_j$ be the $j$th standard basis vector for $\mathbb{C}^n$ under the standard inner product, we can view $\mathbb{C}^n$ as an RKHS with kernel

$$k(i, j) = \langle e_j, e_i \rangle.$$ 

Note that $k_j = k_i$ for each $j = 1, ..., n$. We have $a_{jj} = \langle Ae_j, e_j \rangle$. Thus, the Berezin set of $A$ is simply

$$\text{Ber}(A) = \{a_{jj} : j = 1, ..., n\},$$

which is just the collection of diagonal elements of $A$. Therefore $\text{ber}(A) = \max_{1 \leq j \leq n} |a_{jj}|$.

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**References**


