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Generalizations and Refinements of Niezgoda Inequality for Similarly Separable Vectors with Applications

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Abstract. In this article we provide generalizations of Niezgoda inequality for similarly separable vectors followed by refinements. We also highlight its importance by giving applications. Our results will be generalizing many previously established results.

1. Introduction and Preliminaries

Jensen's inequality for convex functions is amongst the most celebrated inequalities in mathematics and statistics. It has a significant role in various branches of sciences. Various renowned inequalities are a consequence of Jensen's inequality: for example, the Arithmetic-Geometric inequality is a consequence of Jensen's Inequality for convex functions. Also, the general inequality between means of orders p and q, such as Hölder's and Minkowski's inequalities, are also consequences of Jensen's inequality. There are numerous variants, generalizations and refinements of Jensen's inequalities (for reference see [2–4, 7–10, 14–22, 42, 43, 45, 46]). We also adduce to [6] and [38] for detailed discussion on Jensen's inequality and for some remarks on literature and history of the topic.

Throughout the article we assume that J is an interval in $\mathbb{R}$ and for real weights $w_1, \ldots, w_n$, we define the notation

$$W_i = \sum_{\gamma=1}^{i} w_\gamma,\ i \in I_n \quad \text{and clearly} \quad W_n = \sum_{\gamma=1}^{n} w_\gamma.$$

Also in our article, we denote $I_m = \{1, 2, \ldots, m\}$.

Here we state some results from [38] (see also [30, 31, 41]). Let us start with Jensen's inequality.

Proposition 1.1. *Assume Ψ is a convex function on J. Take $\mathbf{x}$ to be an n-tuple such that $x_i \in J$, for $i \in I_n$. Let $\mathbf{w}$ be a nonnegative n-tuple such that $W_n > 0$. Then the following inequality holds*

$$\Psi\left(\frac{1}{W_n}\sum_{i\in I_n} w_i x_i\right) \leq \frac{1}{W_n}\sum_{i\in I_n} w_i \Psi(x_i). \tag{1}$$

Steffensen in 1919 [38, p. 57] presented a more general form of Jensen's inequality which we usually refer to as Jensen-Steffensen inequality. This may be stated as:

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Proposition 1.2. *Assume Ψ is a convex function on J. Take $\mathbf{x}$ to be a monotonic n-tuple such that $x_i \in J$, $i \in I_n$. Let $\mathbf{w}$ be a real n-tuple such that*

$$0 \le W_i \le W_n \quad \text{for} \quad i \in I_n, \quad W_n > 0. \tag{2}$$

Then (1) *holds.*

In mathematical literature, the following inequality is referred to as Reverse-Jensen inequality [38, p. 83].

Proposition 1.3. *If Ψ is a convex function on J. Take $\mathbf{x}$ to be an n-tuple such that $x_i \in J$ for $i \in I_n$. Let $\mathbf{w}$ be a real n-tuple with $\frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \in J$, where $w_1 > 0$, $w_i \le 0$ for $i \in \{2, \dots, n\}$ and $W_n > 0$. Then reverse inequality in* (1) *holds.*

Mercer [29] proved a variant of Jensen's inequality as follows. We will refer to it as Jensen-Mercer inequality.

Proposition 1.4. *Under the assumptions of Proposition 1.1, the following inequality holds*

$$\Psi\left(L + M - \frac{1}{W_n}\sum_{i\in I_n} w_i x_i\right) \le \Psi(L) + \Psi(M) - \frac{1}{W_n}\sum_{i\in I_n} w_i \Psi(x_i) \tag{3}$$

where

$$L = \min_{x_i\in J}\{x_i\} \quad \text{and} \quad M = \max_{x_i\in J}\{x_i\}.$$

By imposing different conditions on weights w_i for $i \in I_n$, as we observed in aforementioned propositions, we get different variants of Proposition 1.4.

In [1] (see also [33]), we can find the following variant of Jensen-Mercer inequality.

Proposition 1.5. *Assume Ψ is a convex function on J. Take $\mathbf{x}$ to be a monotonic nondecreasing n-tuple such that $x_i \in J$, $i \in I_n$. Let $\mathbf{w}$ be a real n-tuple such that conditions on weights given in* (2) *be valid. Then inequality* (3) *holds.*

The following result has been proved in [28]:

Proposition 1.6. *Under the assumptions of Proposition* 1.3, *inequality* (3) *holds.*

Now we state the definition of majorization from [25] as follows: Let two m-tuples $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$ be such that $x_{[1]} \ge \dots \ge x_{[m]}$, $y_{[1]} \ge \dots \ge y_{[m]}$ be their ordered components.

Definition 1.7. *For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$,*

$$\mathbf{x} \prec \mathbf{y} \quad \text{if} \quad \begin{cases} \sum_{i\in I_\kappa} x_{[i]} \le \sum_{i\in I_\kappa} y_{[i]} \quad , \quad \kappa \in I_{m-1}, \\ \sum_{i\in I_m} x_{[i]} = \sum_{i\in I_m} y_{[i]} \quad . \end{cases}$$

When $\mathbf{x} \prec \mathbf{y}$, we say "$\mathbf{y}$ majorizes $\mathbf{x}$*" or "*$\mathbf{x}$ is majorized by $\mathbf{y}$*".*

This concept of majorization was first introduced by Hardy et al. in 1934. In their book "Inequalities" [13], we can find the well-known majorization theorem. Using the defintion of majorization stated above, we are ready to state an extension of inequality (3) presented by Niezgoda in [33]. We refer to it as Niezgoda's inequality (see [23, 34, 36] for recent extensions of inequality (3)).

Proposition 1.8. *Assume Ψ is a continuous convex function on J. Suppose $\boldsymbol{a} = (a_1, \dots, a_m)$ is an m-tuple such that $a_i \in J$ and $\boldsymbol{X} = (\mathbf{x}_\gamma) = (x_{i\gamma})$ is an $n \times m$ matrix such that $x_{i\gamma} \in J$ for all $i \in I_n$ and $\gamma \in I_m$.*

If $\boldsymbol{a}$ majorizes each row of $\boldsymbol{X}$, i.e.,

$$\mathbf{x}_{i.} = (x_{i1}, \dots, x_{im}) \prec (a_1, \dots, a_m) = \boldsymbol{a} \text{ for each } i \in I_n,$$

then the following inequality holds:

$$\Psi\left(\sum_{\gamma\in I_m} a_\gamma - \sum_{\gamma\in I_{m-1}}\sum_{i\in I_n} w_i x_{i\gamma}\right) \le \sum_{\gamma\in I_m}\Psi(a_\gamma) - \sum_{\gamma\in I_{m-1}}\sum_{i\in I_n} w_i \Psi(x_{i\gamma}), \tag{4}$$

where $\sum_{i=1}^{n} w_i = 1$ with $w_i \ge 0$.

2. Generalized Niezgoda inequality for similarly separable vectors

In current section we provide generalization of Niezgoda's inequality (4). For this purpose, some notations are required here. We also quote some relevant definitions from [33] (see also [35] and [37]).

Throughout the paper, we assume that $\mathbf{p} = (p_1, \ldots, p_m)$ is a positive m-tuple.
Here we introduce inner product on $\mathbb{R}^m$ for $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ by

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{\gamma \in I_m} p_\gamma a_\gamma b_\gamma. \tag{5}$$

For $\lambda \in I_m$, we denote

$$P_\lambda = \sum_{\gamma \in I_\lambda} p_\gamma, \quad \hat{P}_\lambda = \sum_{\gamma \in I_\lambda} \gamma p_\gamma, \quad \tilde{P}_\lambda = \sum_{\gamma \in I_\lambda} \gamma^2 p_\gamma.$$

Unless specified elsewise, we take $\varepsilon = \{\mathbf{e_1}, \ldots, \mathbf{e_m}\}$ as an ordered basis in $\mathbb{R}^m$ and $D = \{\mathbf{d_1}, \ldots, \mathbf{d_m}\}$ as the dual basis of ε , that is, $\langle \mathbf{e_i}, \mathbf{d}_\gamma \rangle = \delta_{i\gamma}$ (Kronecker delta) for $i, \gamma \in I_m$.

Definition 2.1. *We define a vector $\mathbf{v} \in \mathbb{R}^m$ to be ε-positive if $\langle \mathbf{e_i}, \mathbf{v} \rangle > 0$ for all $i \in I_m$. Let J_1 and J_2 be two sets of indices such that $J_1 \cup J_2 = J$. Given $\mu \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^m$, a vector $\mathbf{z} \in \mathbb{R}^m$ is known as $\mu, \mathbf{v}$-separable on J_1 and J_2 (with respect to basis ε), if*

$$\langle \mathbf{e_i}, \mathbf{z} - \mu \mathbf{v} \rangle \geq 0 \quad for \quad i \in J_1 \quad and \quad \langle \mathbf{e}_\gamma, \mathbf{z} - \mu \mathbf{v} \rangle \leq 0 \quad for \quad \gamma \in J_2. \tag{6}$$

We say $\mathbf{z}$ is $\mu, \mathbf{v}$-separable on J_1 and J_2 with respect to the basis ε if and only if

$$\max_{i \in J_2} \frac{\langle \mathbf{e_i}, \mathbf{z} \rangle}{\langle \mathbf{e_i}, \mathbf{v} \rangle} \leq \mu \leq \min_{\gamma \in J_1} \frac{\langle \mathbf{e}_\gamma, \mathbf{z} \rangle}{\langle \mathbf{e}_\gamma, \mathbf{v} \rangle} \tag{7}$$

where $\mathbf{v}$ is ε-positive.

Definition 2.2. *A vector $\mathbf{z} \in \mathbb{R}^m$ is $\mathbf{v}$-separable on J_1 and J_2 (with respect to the basis ε), if for some $\mu \in \mathbb{R}$, $\mathbf{z}$ is $\mu, \mathbf{v}$-separable on J_1 and J_2.*

Definition 2.3. *A map $\phi : J \to \mathbb{R}$ is said to preserve $\mathbf{v}$-separability on J_1 and J_2 with respect to ε, if $\phi(\mathbf{z}) = (\phi(z_1), \ldots, \phi(z_m))$ is $\mathbf{v}$-separable on J_1 and J_2 with respect to ε, whenever $\mathbf{z} = (z_1, \ldots, z_m) \in J^m$ is $\mathbf{v}$-separable on J_1 and J_2 with respect to ε.*

Remark 2.4. *In case where $\mathbf{v}$ is ε-positive, $J_1 = \{\lambda_0\}$ and $J_2 = J \backslash \{\lambda_0\}$, the $\mathbf{v}$-separability of $\mathbf{z}$ is implied by:*

$$\frac{\langle \mathbf{e}_\lambda, \mathbf{z} \rangle}{\langle \mathbf{e}_\lambda, \mathbf{v} \rangle} \leq \frac{\langle \mathbf{e}_{\lambda_0}, \mathbf{z} \rangle}{\langle \mathbf{e}_{\lambda_0}, \mathbf{v} \rangle} \tag{8}$$

for $\lambda \in I_m$.

Definition 2.5. *[41, pp. 32, 110] Given a real convex function Ψ on J, $\partial\Psi$ denotes the subdifferential of Ψ. It is the set of all functions $\phi : J \to [-\infty, \infty]$ such that $\phi(J^o) \subseteq \mathbb{R}$ and*

$$\Psi(x) \geq \Psi(a) + (x - a)\phi(a) \quad for\ any\ x, a \in J.$$

Using the notations defined until now, we now present our main result:

Theorem 2.6. *Define $\Psi : J \to \mathbb{R}$ to be a convex function on an open interval $J \subseteq \mathbb{R}$. Suppose $\mathbf{a} = (a_1, \ldots, a_m) \in J^m$ and $\mathbf{X} = (\mathbf{x}_\gamma) = (x_{i\gamma})$ is an $n \times m$ matrix such that $x_{i\gamma} \in J$ and $(\mathbf{x}_\gamma)$ is a monotonic m-tuple for all $i \in I_n$, $\gamma \in I_m$. Also assume that the weight w_i for $i \in I_n$ satisfying the conditions as in* (2) *and $\frac{1}{W_n} \sum_{i \in I_n} w_i \Psi(x_{i\gamma}) \leq \Psi(a_\gamma)$ for $\gamma \in I_m$. We further let that $\langle \mathbf{a} - \mathbf{x}_{i.}, \mathbf{v} \rangle = 0$ for $i \in I_n$. Then the following inequality holds:*

$$\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma-\frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma\sum_{i\in I_n}w_i x_{i\gamma}-\frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma\sum_{i\in I_n}w_i x_{i\gamma}\right)$$
$$\leq\frac{1}{p_\kappa}\sum_{\gamma\in I_m}p_\gamma\Psi(a_\gamma)-\frac{1}{p_\kappa}\frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1}p_\gamma\sum_{i\in I_n}w_i\Psi(x_{i\gamma})-\frac{1}{p_\kappa}\frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m}p_\gamma\sum_{i\in I_n}w_i\Psi(x_{i\gamma}),\quad(9)$$

where $\epsilon=\frac{1}{p_\kappa v_\kappa}$ *with* $v_\kappa\neq 0$ *for* $\kappa\in I_m$.

Proof. First, we use the assumption that for each $\gamma\in I_m$

$$\frac{1}{W_n}\sum_{i\in I_n}w_i\Psi(x_{i\gamma})\leq\Psi(a_\gamma)$$

multiplying it by p_γ and taking sum over $\gamma\in I_m$, we get

$$\frac{1}{W_n}\sum_{i\in I_n}\sum_{\gamma\in I_m}p_\gamma w_i\Psi(x_{i\gamma})\leq\sum_{\gamma\in I_m}p_\gamma\Psi(a_\gamma).\quad(10)$$

Since $\langle\mathbf{a}-\mathbf{x_{i.}},\mathbf{v}\rangle=0$ for all $i\in I_n$ by (5) we have for each $i\in I_n$.

$$\epsilon\left(\sum_{\gamma\in I_m}p_\gamma v_\gamma a_\gamma-\sum_{\gamma=1}^{\kappa-1}p_\gamma v_\gamma x_{i\gamma}-\sum_{\gamma=\kappa+1}^{m}p_\gamma v_\gamma x_{i\gamma}\right)=x_{i\kappa}.\quad(11)$$

Now, using Jensen-Steffensen inequality for weights w_i and then using (11) and inequality (10) for weights p_γ we obtain our required result as follows

$$\begin{aligned}
&p_\kappa\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma-\frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma\sum_{i\in I_n}w_i x_{i\gamma}-\frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma\sum_{i\in I_n}w_i x_{i\gamma}\right)\\
&=p_\kappa\Psi\left(\frac{1}{W_n}\sum_{i\in I_n}w_i\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma-\sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma x_{i\gamma}-\sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma x_{i\gamma}\right)\right)\\
&\leq\frac{1}{W_n}\sum_{i\in I_n}w_i p_\kappa\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma-\sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma x_{i\gamma}-\sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma x_{i\gamma}\right)\\
&=\frac{1}{W_n}\sum_{i\in I_n}w_i p_\kappa\Psi\left(x_{i\kappa}\right)\\
&\leq\frac{1}{W_n}\sum_{i\in I_n}w_i\left(\sum_{\gamma\in I_m}p_\gamma\Psi(a_\gamma)-\sum_{\gamma=1}^{\kappa-1}p_\gamma\Psi(x_{i\gamma})-\sum_{\gamma=\kappa+1}^{m}p_\gamma\Psi(x_{i\gamma})\right)\\
&=\sum_{\gamma\in I_m}p_\gamma\Psi(a_\gamma)-\frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1}p_\gamma\sum_{i\in I_n}w_i\Psi(x_{i\gamma})-\frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m}p_\gamma\sum_{i\in I_n}w_i\Psi(x_{i\gamma}).
\end{aligned}$$

□

Remark 2.7. *Here we observe that the proof of Theorem 2.6 is much more simpler than proof of Theorem 3.1 of [33]. It should be noted that in Theorem 3.1 of [33] the author has used positive weights while we have used real weights satisfying assumptions as stated in (2) with monotonic* $m-$*tuples* $(\mathbf{x}_\gamma)$. *It is also worth mentioning that we did not use* *bifractional inequality [35] or concept of similarly separable vectors, we removed all these assumptions at the expense of the assumption that* $\frac{1}{W_n}\sum_{i\in I_n} w_i\Psi(x_{i\gamma}) \le \Psi(a_\gamma)$ *for* $\gamma \in I_m$.

Now we state a corollary and other special case of Theorem 2.6 as under:

Corollary 2.8. *Define* $\Psi : J \to \mathbb{R}$ *to be a convex function on an open interval* $J \subseteq \mathbb{R}$. *Let* $\partial\Psi : J \to \mathbb{R}$ *be the subdifferential of* Ψ *and* $\phi \in \partial\Psi$. *Suppose* $\mathbf{a} = (a_1, \dots, a_m) \in J^m$ *and* $\boldsymbol{X} = (\mathbf{x}_\gamma) = (x_{i\gamma})$ *is an* $n \times m$ *matrix such that* $x_{i\gamma} \in J$ *for all* $i \in I_n$, $\gamma \in I_m$. *Further suppose that the weight* w_i *are positive real weights for* $i \in I_n$. *Let* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ *such that* $\langle \mathbf{u}, \mathbf{v}\rangle > 0$. *Let there exist index sets* J_1 *and* J_2 *with* $J_1 \cup J_2 = J$ *such that for each* $i \in I_n$ *we have:*

(i) $\mathbf{x}_{\mathbf{i}.}$ *is* $\mathbf{v}$*-separable on* J_1 *and* J_2 *with respect to* ε,
(ii) $\mathbf{a} - \mathbf{x}_{\mathbf{i}.}$ *is* $0, \mathbf{u}$*-separable on* J_1 *and* J_2 *with respect to* D,
(iii) $\langle \mathbf{a} - \mathbf{x}_{\mathbf{i}.}, \mathbf{v}\rangle = 0$,
(iv) ϕ *preserves* $\mathbf{v}$*-separability on* J_1 *and* J_2 *with respect to* ε.

Then the following inequality holds:

$$\Psi\left(\sum_{\gamma\in I_m} \epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1} \epsilon p_\gamma v_\gamma \sum_{i\in I_n} w_i x_{i\gamma} - \frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m} \epsilon p_\gamma v_\gamma \sum_{i\in I_n} w_i x_{i\gamma}\right)$$
$$\le \frac{1}{p_\kappa}\sum_{\gamma\in I_m} p_\gamma \Psi(a_\gamma) - \frac{1}{p_\kappa}\frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1} p_\gamma \sum_{i\in I_n} w_i\Psi(x_{i\gamma}) - \frac{1}{p_\kappa}\frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m} p_\gamma \sum_{i\in I_n} w_i\Psi(x_{i\gamma}), \quad (12)$$

where $\epsilon = \frac{1}{p_\kappa v_\kappa}$ *with* $v_\kappa \neq 0$ *for* $\kappa \in I_m$.

Remark 2.9. *If we simply put* $\kappa = m$ *and* $W_n = 1$ *in Corollary 2.8 we will get Theorem 3.1 of [33] (for further remarks see [19]) and consequently we capture all its corollaries and special cases. Some similar results are stated as under as well.*

Corollary 2.10. *Using the assumptions from Corollary* 2.8, *suppose* $\mathbf{v} = \mathbf{d}_{\lambda_0}$ *for some* $\lambda_0 \in J = I_m$. *Take* $J_1 = \{\lambda_0\}$ *and* $J_2 = J\backslash\{\lambda_0\}$. *Replace conditions* (*i*) *and* (*ii*) *in Corollary* 2.8 *by the following*

(i) $\mathbf{x}_{i.}$ *is* $\mathbf{v}$*-separable on* J_1 *and* J_2 *with respect to* ε, *i.e.,* $\mathbf{v}$ *is* ε*-positive and*

$$\frac{\langle \mathbf{e}_\lambda, \mathbf{x}_{i.}\rangle}{\langle \mathbf{e}_\lambda, \mathbf{v}\rangle} \le \frac{\langle \mathbf{e}_{\lambda_0}, \mathbf{x}_{i.}\rangle}{\langle \mathbf{e}_{\lambda_0}, \mathbf{v}\rangle}$$

for $\lambda \in I_m$,

(ii) $\mathbf{a} - \mathbf{x}_{i.}$ *is* $0, \mathbf{u}$*-separable on* J_1 *and* J_2 *with respect to* D, *i. e.,*

$$\langle \mathbf{d}_\lambda, \mathbf{a} - \mathbf{x}_{i.}\rangle \le \langle \mathbf{d}_{\lambda_0}, \mathbf{a} - \mathbf{x}_{i.}\rangle$$

for $\lambda \in I_m$.

Then inequality (12) *holds.*

Proof. We get condition (*ii*) *of Corollary 2.10 from* (*ii*) *of Corollary* 2.8 *(see inequality (8)). By* (*iii*) *we have* $\langle \mathbf{d}_{\lambda_0}, \mathbf{a} - \mathbf{x}_{i.}\rangle = \langle \mathbf{v}, \mathbf{a} - \mathbf{x}_{i.}\rangle = 0$. *Therefore* (*ii*) *gives*

$$\langle \mathbf{d}_\lambda, \mathbf{a} - \mathbf{x}_{i.}\rangle \le 0 = \langle \mathbf{d}_{\lambda_0}, \mathbf{a} - \mathbf{x}_{i.}\rangle$$

for $\lambda \in I_m$, *implying* $\mathbf{a} - \mathbf{x}_{i.}$ *is* $0, \mathbf{u}$*-separable on* J_1 *and* J_2 *with respect to* D.

The assertion now follows from Corollary 2.8. □

In the remaining part of this section, we interpret our result from Corollary 2.10 for various vectors **u** and **v**. For this end we make use of [37, Corollaries 2.3, 2.6, 2.10 and 2.11]. In what follows we consider the following two pairs of dual bases $\varepsilon = \{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ and $D = \{\mathbf{d}_1, \dots, \mathbf{d}_m\}$:

$$\mathbf{e}_\kappa = \mathbf{d}_\kappa = \frac{1}{\sqrt{p_\kappa}}(\underbrace{0, \dots, 0}_{(\kappa-1)\text{ times}}, 1, 0, \dots, 0) \tag{13}$$

for $\kappa \in I_{m-1}$ (for Corollaries 2.11 and 2.14) and

$$\mathbf{e}_\kappa = (\underbrace{0, \dots, 0}_{(\kappa-1)\text{ times}}, \frac{1}{p_\kappa}, -\frac{1}{p_{\kappa+1}}, 0, \dots, 0) \quad \text{for } \kappa \in I_{m-1} \text{ and} \tag{14}$$

$$\mathbf{e}_m = (0, \dots, 0, \frac{1}{p_m}), \tag{15}$$

$$\mathbf{d}_\kappa = (\underbrace{1, \dots, 1}_{\kappa\text{ times}}, 0, \dots, 0) \quad \text{for } \kappa \in I_{m-1} \tag{16}$$

(for Corollaries 2.12 and 2.15). Inequality (13) gives an orthonormal basis in $\mathbb{R}^m$ with respect to the inner product defined in (5), whereas inequalities (14) – (16) corresponds to weak majorization ordering [27, p. 10], whenever $p_1 = \dots = p_m = 1$ [26, 14,p. 426].

Corollary 2.11. *Using the assumptions from Corollary* 2.8, *let* $\varepsilon = D$ *be the basis in* $\mathbb{R}^m$ *given by* (13) *and let* $\mathbf{u} = \mathbf{v} = (1, \dots, 1)$. *For each* $i \in I_n$, *suppose there exist index sets* J_1 *and* J_2 *with* $J_1 \bigcup J_2 = J$ *such that*

(i) $\mathbf{x}_{i.}$ *is* $\mathbf{v}$*-separable on* J_1 *and* J_2 *with respect to* ε, *i.e.,*
$x_{i\lambda} \le x_{i\gamma}$ *for* $\gamma \in J_1$ *and* $\lambda \in J_2$,

(ii) $\mathbf{a} - \mathbf{x}_{i.}$ *is* $0, \mathbf{u}$*-separable on* J_1 *and* J_2 *with respect to* $D = \varepsilon$, *i.e.,*
$a_\lambda - x_{i\lambda} \le 0 \le a_\gamma - x_{i\gamma}$ *for* $\gamma \in J_1$ *and* $\lambda \in J_2$,

(iii) $\sum_{\kappa \in I_m}(a_\kappa - x_{i\kappa})p_\kappa = 0.$

Then the following inequality holds:

$$\Psi\left(\sum_{\gamma \in I_m} \tilde{p}_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1} \tilde{p}_\gamma \sum_{i \in I_n} w_i x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m} \tilde{p}_\gamma \sum_{i \in I_n} w_i x_{i\gamma}\right)$$
$$\le \sum_{\gamma \in I_m} \tilde{p}_\gamma \Psi(a_\gamma) - \sum_{\gamma=1}^{\kappa-1} \tilde{p}_\gamma \sum_{i \in I_n} w_i \Psi(x_{i\gamma}) - \sum_{\gamma=\kappa+1}^{m} \tilde{p}_\gamma \sum_{i \in I_n} w_i \Psi(x_{i\gamma}), \tag{17}$$

where $\tilde{p}_\gamma = \frac{p_\gamma}{p_\kappa}$ *for* $\kappa \in I_m$.

For instance, if $\tilde{p}_\gamma = 1$ (*i.e.*, $p_1 = \dots = p_m$), *then* (17) *reduces to*

$$\Psi\left(\sum_{\gamma \in I_m} a_\gamma - \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I_n} w_i x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I_n} w_i x_{i\gamma}\right) \le \sum_{\gamma \in I_m} \Psi(a_\gamma) - \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I_n} w_i \Psi(x_{i\gamma}) - \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I_n} w_i \Psi(x_{i\gamma}). \tag{18}$$

Proof. By (7) and (13), it can be seen that a vector $\mathbf{z} = (z_1, \dots, z_n)$ is $\mathbf{v}$-separable on J_1 and J_2 with respect to ε if and only if

$$z_\lambda \le z_\gamma \tag{19}$$

for $\gamma \in J_1$ and $\lambda \in J_2$.

Therefore (*i*) − (*ii*) of Corollary 2.11 imply (*i*) − (*ii*) of Theorem 2.8. Since ϕ is nondecreasing (see [11, p.209]), from (19) we get $\phi(z_\lambda) \le \phi(z_\gamma)$ for $\gamma \in J_1$ and $\lambda \in J_2$, implying $\phi(z)$ is **v**-separable on J_1 and J_2 with respect to ε. Now, condition (*iv*) of Corollary 2.8 is fulfilled. Also, $\langle \mathbf{a} - \mathbf{x}_{i.}, \mathbf{v}\rangle = \sum_{\kappa=1}^{m}(a_\kappa - x_{i\kappa})p_\kappa = 0$ which implies (*iii*) of Corollary 2.8.

To verify inequality (17), we use inequality (12) from Corollary 2.8. Also, inequality (17) can be reduced to obtain inequality (18). □

Note that if $\mathbf{x}_{i.}$ and $\mathbf{a} - \mathbf{x}_{i.}$ are both nondecreasing, i.e., $x_{i1} \le \cdots \le x_{im}$ and $a_1 - x_{i1} \le \cdots \le a_m - x_{im}$, then conditions (*i*) and (*ii*) of Corollary 2.11 are satisfied for the index sets $J_1 = \{\kappa + 1, \ldots, m\}$ and $J_2 = I_\kappa$ for some κ.

Corollary 2.12. *Using the assumptions from Corollary 2.8, take* $\mathbf{u} = \mathbf{v} = (1, \ldots, 1)$. *Let* ε *and* D *be the bases in* $\mathbb{R}^m$ *defined by inequalities* (14) − (16). *For each* $i \in I_n$, *suppose that there exist index sets* J_1 *and* J_2 *with* $J_1 \bigcup J_2 = J$ *such that*

- *(i)* $\mathbf{x}_{i.}$ *is* **v***-separable on* J_1 *and* J_2 *with respect to* ε, *i.e., there exist* $\mu \in \mathbb{R}$ *satisfying* $x_{i,\lambda} - x_{i,\lambda+1} \le 0 \le x_{i,\gamma} - x_{i,\gamma+1}$ *for* $\gamma \in J_1$ *and* $\lambda \in J_2$ *with convention* $x_{i,m+1} = \mu$,
- *(ii)* $\mathbf{a} - \mathbf{x}_{i.}$ *is* $0, \mathbf{u}$*-separable on* J_1 *and* J_2 *with respect to* D, *i.e.,* $\sum_{\kappa\in I_\lambda}(a_\kappa - x_{i\kappa})p_\kappa \le 0 \le \sum_{\kappa\in I_\gamma}(a_\kappa - x_{i\kappa})p_\kappa$ *for* $\gamma \in J_1$ *and* $\lambda \in J_2$,
- *(iii)* $\sum_{\kappa\in I_m}(a_\kappa - x_{i\kappa})p_\kappa = 0$.

Then inequalities (17) − (18) *hold.*

Proof. From inequality (6) and inequality (14) − (16), a vector $\mathbf{z} = (z_1, \ldots, z_n)$ is **v**-separable on J_1 and J_2 with respect to D if and only if there exist $\mu \in \mathbb{R}$ such that

$$z_\lambda - z_{\lambda+1} \le 0 \le z_\gamma - z_{\gamma+1} \tag{20}$$

for $\gamma \in J_1$ and $\lambda \in J_2$ with the convention $z_{m+1} = \mu$.

Also it follows from (7) that a vector $\mathbf{z} = (z_1, \ldots, z_n)$ is $0, \mathbf{u}$- separable on J_1 and J_2 with respect to D if and only if $\sum_{\kappa\in I_\lambda} z_\kappa p_\kappa \le 0 \le \sum_{\kappa\in I_\gamma} z_\kappa p_\kappa$ for $\gamma \in J_1$ and $\lambda \in J_2$.
Therefore (*i*) − (*ii*) of Corollary 2.12 imply statements (*i*) − (*ii*) of Corollary 2.8.
Since ϕ is nondecreasing (see [11, p.209]), from (20) we get $\phi(z_\lambda) - \phi(z_{\lambda+1}) \le 0 \le \phi(z_\gamma) - \phi(z_{\gamma+1})$ for $\gamma \in J_1$ and $\lambda \in J_2$.
In consequence, ϕ preserves **v**-separability on J_1 and J_2 with respect to ε and (*iv*) of Corollary 2.8 is satisfied. From the assumption (*iii*) of Corollary 2.12 we get condition (*iii*) of Corollary 2.8. Lastly, in order to derive inequalities (17) − (18), we use inequality (12) (Corollary 2.8). □

We note that under the assumption (*iii*) of Corollary 2.12, conditions (*i*) − (*ii*) of Corollary 2.12 are satisfied for $J_1 = \{m\}$ and $J_2 = I_{m-1}$ provided $\mathbf{x}_{i.}$ is nondecreasing, i.e., $x_{i1} \le x_{i2} \le \cdots \le x_{im}$ and $\mathbf{a} - \mathbf{x}_{i.}$ is nondecreasing in *P*-mean [44, p.318] , i.e.,

$$\frac{1}{P_\lambda}\sum_{\kappa\in I_\lambda}(a_\kappa - x_{i\kappa})p_\kappa \le \frac{1}{p_{\lambda+1}}\sum_{\kappa\in I_{\lambda+1}}(a_\kappa - x_{i\kappa})p_\kappa \tag{21}$$

for $\lambda \in I_{m-1}$.

Definition 2.13. *[44, p.318] An m-tuple* $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{R}^m$ *is said to be star-shaped if*

$$\frac{z_\lambda}{\lambda} \le \frac{z_{\lambda+1}}{\lambda+1} \tag{22}$$

for $\lambda \in I_{m-1}$.

A function $\phi : I \to \mathbb{R}$, *where* $I \subset \mathbb{R}^+$, *is said to be star-shaped, if the function* $x \mapsto \dfrac{\phi(x)}{x}$ *is nondecreasing [30].*

Here we take $\phi : I \to \mathbb{R}$ to be a convex function which is differentiable positive nondecreasing and convex function on a positive open interval $I \subset \mathbb{R}^+$. We know [37, Lemma 2.8] that if ϕ is star-shaped, then it preserves star-shapeness of m-tuples in the sense that (22) implies
$\frac{\phi(z_\lambda)}{\lambda} \leq \frac{\phi(z_{\lambda+1})}{\lambda+1}$ for $\lambda \in I_{m-1}$.

Corollary 2.14. *Using the assumptions from Corollary 2.8 and let $\varepsilon = D$ be the basis in $\mathbb{R}^m$ given by (13) and $\mathbf{u} = \mathbf{v} = (1, 2, \ldots, m)$. For all $i \in I_n$, suppose there exist index sets J_1 and J_2 with $J_1 \bigcup J_2 = J$ such that*

(i) $\mathbf{x}_{i.}$ *is* $\mathbf{v}$*-separable on J_1 and J_2 with respect to ε, i.e.,*
$\frac{x_{i\lambda}}{\lambda} \leq \frac{x_{i\gamma}}{\gamma}$ *for $\gamma \in J_1$ and $\lambda \in J_2$,*

(ii) $\mathbf{a} - \mathbf{x}_{i.}$ *is* $0, \mathbf{u}$*-separable on J_1 and J_2 with respect to $D = \varepsilon$, i.e.,*
$\frac{a_\lambda - x_{i\lambda}}{\lambda} \leq 0 \leq \frac{a_\gamma - x_{i\gamma}}{\gamma}$ *for $\gamma \in J_1$ and $\lambda \in J_2$,*

(iii) $\sum_{\kappa \in I_m} (a_\kappa - x_{i\kappa}) p_\kappa = 0$,

(iv) ϕ *preserves* $\mathbf{v}$*-separability on J_1 and J_2 with respect to ε, i.e., (i) of Corollary 2.14 implies*
$\frac{\phi(x_{i\lambda})}{\lambda} \leq \frac{\phi(x_{i\gamma})}{\gamma}$ *for $\gamma \in J_1$ and $\lambda \in J_2$.*

Then the following inequality holds:

$$\Psi\left(\sum_{\gamma \in I_m} \tilde{p}_\gamma \tilde{v}_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1} \tilde{p}_\gamma \tilde{v}_\gamma \sum_{i \in I_n} w_i x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m} \tilde{p}_\gamma \tilde{v}_\gamma \sum_{i \in I_n} w_i x_{i\gamma}\right) \leq \sum_{\gamma \in I_m} \tilde{p}_\gamma \Psi(a_\gamma) - \sum_{\gamma=1}^{\kappa-1} \tilde{p}_\gamma \sum_{i \in I_n} w_i \Psi(x_{i\gamma}) - \sum_{\gamma=\kappa+1}^{m} \tilde{p}_\gamma \sum_{i \in I_n} w_i \Psi(x_{i\gamma}), \tag{23}$$

where $\tilde{p}_\gamma = \frac{p_\gamma}{p_\kappa}$, $\tilde{v}_\gamma = \frac{\gamma}{m}$ for $\kappa \in I_m$.

For instance, if $\tilde{p}_\gamma = 1$ (i.e., $p_1 = \cdots = p_m$), then (23) becomes:

$$\Psi\left(\sum_{\gamma \in I_m} \frac{\gamma}{m} a_\gamma - \sum_{\gamma=1}^{\kappa-1} \frac{\gamma}{m} \sum_{i \in I_n} w_i x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m} \frac{\gamma}{m} \sum_{i \in I_n} w_i x_{i\gamma}\right) \leq \sum_{\gamma \in I_m} \Psi(a_\gamma) - \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I_n} w_i \Psi(x_{i\gamma}) - \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I_n} w_i \Psi(x_{i\gamma}). \tag{24}$$

If $\mathbf{x}_i.$ and $\mathbf{a} - \mathbf{x}_i.$ are star-shaped tuples, and the map ϕ preserves star-shaped tuples, then $(i) - (ii)$ of Corollary 2.14 are satisfied for the index set $J_1 = \{\kappa + 1, \ldots, m\}$ and $J_2 = I_\kappa$ for some κ.

Corollary 2.15. *Using the assumptions from Corollary 2.8, suppose that ε and D are the bases in $\mathbb{R}^m$ defined by (14) – (16) and $\mathbf{u} = \mathbf{v} = (1, 2, \ldots, m)$. For each $i \in \{1, \ldots, n\}$, suppose there exist index sets J_1 and J_2 with $J_1 \bigcup J_2 = J$ such that*

(i) $\mathbf{x}_{i.}$ *is* $\mathbf{v}$*-separable on J_1 and J_2 with respect to ε, i.e., there exist $\mu \in \mathbb{R}$ satisfying*

$$x_{i,\lambda+1} - x_{i,\lambda} \geq \mu \geq x_{i,\gamma+1} - x_{i\gamma}, \quad for \quad \gamma \in J_1 \quad and \quad \lambda \in J_2 \tag{25}$$

with convention $x_{i,m+1} = \mu(m+1)$,

(ii) $\mathbf{a} - \mathbf{x}_{i.}$ *is* $0, \mathbf{u}$*-separable on J_1 and J_2 with respect to D, i.e.,*

$$\sum_{\kappa \in I_\lambda} (a_\kappa - x_{i\kappa}) p_\kappa \leq 0 \leq \sum_{\kappa \in I_\gamma} (a_\kappa - x_{i\kappa}) p_\kappa \quad where \quad \gamma \in J_1, \quad \lambda \in J_2 \tag{26}$$

(iii) $\sum_{\kappa\in I_m}(a_\kappa - x_{i\kappa})\kappa p_\kappa = 0$,

(iv) ϕ *preserves* $\mathbf{v}$*-separability on* J_1 *and* J_2 *with respect to* ε*, i.e.,* (*i*) *of Corollary* 2.15 *implies there exist* $\nu \in \mathbb{R}$ *satisfying*

$$\phi(x_{i,\lambda+1}) - \phi(x_{i\lambda}) \geq \nu \geq \phi(x_{i,\gamma+1}) - \phi(x_{i\gamma}) \quad for \quad \gamma \in J_1 \quad and \quad \lambda \in J_2 \tag{27}$$

with the convention $\phi(x_{i,m+1}) = \nu(m+1)$*. Hence inequalities* (23) *and* (24) *hold.*

Definition 2.16. *An m-tuple* $\mathbf{z} = (z_1, \ldots, z_m)$ *is said to be convex [44, p.318] if*

$$z_\lambda \leq \frac{z_{\lambda-1} + z_{\lambda+1}}{2} \tag{28}$$

for $\lambda \in \{2, \ldots, m-1\}$.

Remark 2.17. *Say* $\phi : I \to \mathbb{R}$ *is a nonincreasing convex map such that* $\phi(0) = 0$*. Then conditions* (25) − (27) *are satisfied for the index sets* $J_1 = I_\kappa$ *and* $J_2 = \{\kappa+1, \ldots, m\}$ *for some* κ*, whenever* $\mathbf{a} - \mathbf{x}_{i.}$ *in nondecreasing in* $\hat{P}$ − *mean, i.e.,*

$$\frac{1}{\hat{P}} \sum_{\kappa=1}^{\lambda} (a_\kappa - x_{i\kappa}) p_\kappa \geq \frac{1}{\hat{P}_{\lambda+1}} \sum_{\kappa=1}^{\lambda+1} (a_\kappa - x_{i\kappa}) p_\kappa$$

for $\lambda \in I_{m-1}$.

Also, $\mathbf{x}_{i.} = (x_{i1}, \ldots, x_{im})$ *is a decreasing convex m-tuple such that* $x_{i1} \leq m(x_{i2} - x_{i1})$.

Remark 2.18. *By putting special conditions,* $\kappa = m$ *and* $w_i > 0$ *for all i with* $W_n = 1$ *we obtain Theorem 3.1 of [33]. Consequently all the corollaries of Theorem 3.1 of [33] become special cases of our article.*

Theorem 2.19. *If in Corollary* 2.8*,* Ψ *is a differential function with* $\mathbf{a} = (a, a, \ldots, a) \in J^m$*,* $\mathbf{p} = (p_1, \ldots, p_m)$ *is a real m-tuple satisfying the conditions given in* (2) *and* (5)*, then inequality* (12) *can be written as:*

$$p_\kappa \Psi\left(a \sum_{\gamma\in I_m} \epsilon p_\gamma v_\gamma - \frac{1}{W_n} \sum_{\gamma=1}^{m-1} \epsilon p_\gamma v_\gamma \sum_{i\in I_n} w_i x_{i\gamma}\right) \leq P_m \Psi(a) - \frac{1}{W_n} \sum_{\gamma=1}^{m-1} p_\gamma \sum_{i\in I_n} w_i \Psi(x_{i\gamma}), \tag{29}$$

Proof. For an $n \times m$ matrix $\mathbf{X} = (\mathbf{x}_\gamma) = (x_{i\gamma})$ such that $x_{i\gamma} \in J$ and $(\mathbf{x}_\gamma)$ is a monotonic m-tuple for all $i \in I_n$, $\gamma \in I_m$, then for $a_i \equiv a \in J$, if Ψ is a differential function then, for m-tuple $\mathbf{p} = (p_1, \ldots, p_m)$ satisfying conditions (2) and (5), [24, Theorem 2.1] can be written as

$$\frac{1}{P_m}\left[\sum_{\gamma\in I_m} p_\gamma \Psi(x_{i\gamma}) - \sum_{\gamma\in I_m} p_\gamma \Psi(d)\right] \leq \frac{1}{P_m} \sum_{\gamma\in I_m} p_\gamma \Psi'(x_{i\gamma})(x_{i\gamma} - d)$$

which implies that

$$\sum_{\gamma\in I_m} p_\gamma \left(\Psi(x_{i\gamma}) - \Psi(d)\right) \leq \sum_{\gamma\in I_m} p_\gamma \Psi'(x_{i\gamma})(x_{i\gamma} - d)$$

multiplying both sides by (−1)

$$\sum_{\gamma\in I_m} p_\gamma \left(\Psi(d) - \Psi(x_{i\gamma})\right) \geq \sum_{\gamma\in I_m} p_\gamma \Psi'(x_{i\gamma})(d - x_{i\gamma})$$

that follows to inequality (12) in the following form:

$$p_\kappa \Psi\left(a \sum_{\gamma\in I_m} \epsilon p_\gamma v_\gamma - \frac{1}{W_n} \sum_{\gamma=1}^{m-1} \epsilon p_\gamma v_\gamma \sum_{i\in I_n} w_i x_{i\gamma}\right) \leq P_m \Psi(a) - \frac{1}{W_n} \sum_{\gamma=1}^{m-1} p_\gamma \sum_{i\in I_n} w_i \Psi(x_{i\gamma}). \tag{30}$$

□

Remark 2.20. *Similar results can be produced for concave functions by making use of the definition of concave functions, i.e,* Ψ *is concave if and only if* $-\Psi$ *is convex.*

3. Index Set Functions and Refinements of Generalized Niezgoda's Inequality For Similarly Separable Vectors

In start of this section we give some construction which we will use throughout this section: Let I be a finite nonempty set of positive integers. Let $\mathbf{w} = (w_i)$, $i \in I$ be a real sequence and let $(\mathbf{x}_\gamma) = (x_{i\gamma})$ be a sequence of vectors such that $x_{i\gamma} \in J$ for all $i \in I$, $\gamma \in I_m$. Moreover we define $A_I(\mathbf{x}_\gamma, \mathbf{w}) = \frac{1}{W_I}\sum_{i\in I} w_i x_{i\gamma}$ where $W_I = \sum_{i\in I} w_i$. For a convex function $\Psi : J \to \mathbb{R}$. Also if assumptions of Theorem 2.6 are valid we define the index set function F as

$$F(I) = W_I\Bigg[\sum_{\gamma\in I_m}\tilde{p}_\gamma\Psi(a_\gamma) - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\tilde{p}_\gamma w_i\Psi(x_{i\gamma}) - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\tilde{p}_\gamma w_i\Psi(x_{i\gamma}) \\ -\Psi\Bigg(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}\Bigg)\Bigg] \tag{31}$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ is an m-tuple such that $a_\gamma \in J$ for $\gamma \in I_m$ and $\tilde{p}_\gamma = \frac{p_\gamma}{p_\kappa}$ and $\kappa \in I_m$.

Now, by using techniques of article [28] we prove some results here.

Theorem 3.1. *Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)$ be an m-tuple such that $a_\gamma \in J$ for $\gamma \in I_m$, I and $\bar{I}$ be nonempty sets such that $I \cup \bar{I} = I_n$ and $I \cap \bar{I} = \emptyset$. Let $(\mathbf{x}_\gamma) = (x_{i\gamma})$ be a sequence of vectors such that $x_{i\gamma} \in J$ for all $i \in I$, $\gamma \in I_m$ and $\mathbf{w} = (w_i)$, $i \in I_n$ such that $W_{I_n} > 0$. Let Ψ be a convex function on J and $A_S(\mathbf{x}_\gamma, \mathbf{w}) \in J$ ($S \in \{I, \bar{I}, I \cup \bar{I}\}$). If $W_I > 0$ and $W_{\bar{I}} > 0$, then*

$$F(I \cup \bar{I}) \geq F(I) + F(\bar{I}). \tag{32}$$

If $W_I \cdot W_{\bar{I}} < 0$, then inequality (32) *is reversed.*

Proof. *Fix $\kappa \in I_m$. Since Ψ is continuous convex and composition with an affine function, we get convex function g which we define as:*

$$g(\mathbf{t}_\alpha) = \Psi\Big(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\varepsilon p_\gamma v_\gamma t_\gamma^{(\alpha)} - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma t_\gamma^{(\alpha)}\Big)$$

where $\mathbf{t}_\alpha = (t_1^\alpha, \ldots, t_m^\alpha) \in J^m$. We use the definition of convex function, for all $\mathbf{t}_1$, $\mathbf{t}_2 \in J^m$ and $\lambda_1, \lambda_2 > 0$ to get

$$g\Big(\frac{\lambda_1 t_1 + \lambda_2 t_2}{\lambda_1 + \lambda_2}\Big) \leq \frac{\lambda_1 g(t_1) + \lambda_2 g(t_2)}{\lambda_1 + \lambda_2}, \tag{33}$$

which gives

$$(\lambda_1 + \lambda_2)\Psi\Bigg(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\varepsilon p_\gamma v_\gamma\frac{\lambda_1 t_\gamma^{(1)} + \lambda_2 t_\gamma^{(2)}}{\lambda_1 + \lambda_2} - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma\frac{\lambda_1 t_\gamma^{(1)} + \lambda_2 t_\gamma^{(2)}}{\lambda_1 + \lambda_2}\Bigg) \\ \leq \lambda_1\Psi\Big(\sum_{\gamma\in I_m}a_\gamma\varepsilon p_\gamma v_\gamma - \sum_{\gamma=1}^{\kappa-1}t_\gamma^{(1)} - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma t_\gamma^{(1)}\Big) + \lambda_2\Psi\Big(\sum_{\gamma\in I_m}a_\gamma - \sum_{\gamma=1}^{\kappa-1}\varepsilon p_\gamma v_\gamma t_\gamma^{(2)} - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma t_\gamma^{(2)}\Big). \tag{34}$$

Now using $\lambda_1 = W_I,\ \lambda_2 = W_{\bar{I}},\ t_\gamma^{(1)} = A_I(\mathbf{x}_\gamma, \mathbf{w})$ *and* $t_\gamma^{(2)} = A_{\bar{I}}(\mathbf{x}_\gamma, \mathbf{w})$ *we have*

$$
\begin{aligned}
& W_{I\cup\bar{I}}\Psi\left(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\varepsilon p_\gamma v_\gamma \frac{W_I A_I(\mathbf{x}_\gamma,\mathbf{w}) + W_{\bar{I}} A_{\bar{I}}(\mathbf{x}_\gamma,\mathbf{w})}{W_{I\cup\bar{I}}}\right.\\
& \left. - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma \frac{W_I A_I(\mathbf{x}_\gamma,\mathbf{w}) + W_{\bar{I}} A_{\bar{I}}(\mathbf{x}_\gamma,\mathbf{w})}{W_{I\cup\bar{I}}}\right)\\
& \leq W_I\Psi\left(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\varepsilon p_\gamma v_\gamma A_I(\mathbf{x}_\gamma,\mathbf{w}) - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma A_I(\mathbf{x}_\gamma,\mathbf{w})\right)\\
& + W_{\bar{I}}\Psi\left(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\varepsilon p_\gamma v_\gamma A_{\bar{I}}(\mathbf{x}_\gamma,\mathbf{w}) - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma A_{\bar{I}}(\mathbf{x}_\gamma,\mathbf{w})\right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
& W_{I\cup\bar{I}}\Psi\left(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\varepsilon p_\gamma v_\gamma A_{I\cup\bar{I}}(\mathbf{x}_\gamma,\mathbf{w}) - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma A_{I\cup\bar{I}}(\mathbf{x}_\gamma,\mathbf{w})\right)\\
& \leq W_I\Psi\Big(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\varepsilon p_\gamma v_\gamma A_I(\mathbf{x}_\gamma,\mathbf{w}) - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma A_I(\mathbf{x}_\gamma,\mathbf{w})\Big)\\
& + W_{\bar{I}}\Psi\Big(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\varepsilon p_\gamma v_\gamma A_{\bar{I}}(\mathbf{x}_\gamma,\mathbf{w}) - \sum_{\gamma=\kappa+1}^{m}\varepsilon p_\gamma v_\gamma A_{\bar{I}}(\mathbf{x}_\gamma,\mathbf{w})\Big).
\end{aligned}
$$

Multiplying both sides of the above inequality by (-1)*, putting values of* A_S *and adding to both sides of the inequality the term shown below*

$$
W_{I\cup\bar{I}}\left[\sum_{\gamma\in I_m}\tilde{p}_\gamma\Psi(a_\gamma) - \frac{1}{W_{I\cup\bar{I}}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I\cup\bar{I}}\tilde{p}_\gamma w_i\Psi(x_{i\gamma}) - \frac{1}{W_{I\cup\bar{I}}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I\cup\bar{I}}\tilde{p}_\gamma w_i\Psi(x_{i\gamma})\right]
$$

we get

$$
\begin{aligned}
& W_{I\cup\bar{I}}\left[\sum_{\gamma\in I_m}\tilde{p}_\gamma\Psi(a_\gamma) - \frac{1}{W_{I\cup\bar{I}}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I\cup\bar{I}}\tilde{p}_\gamma w_i\Psi(x_{i\gamma}) - \frac{1}{W_{I\cup\bar{I}}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I\cup\bar{I}}\tilde{p}_\gamma w_i\Psi(x_{i\gamma})\right.\\
& \left. -\Psi\left(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_{I\cup\bar{I}}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I\cup\bar{I}}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_{I\cup\bar{I}}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I\cup\bar{I}}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\right]
\end{aligned}
$$

$$\geq W_I\left[\sum_{\gamma\in I_m}\tilde{p}_\gamma\Psi(a_\gamma)-\frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\tilde{p}_\gamma w_i\Psi(x_{i\gamma})-\frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\tilde{p}_\gamma w_i\Psi(x_{i\gamma})\right.$$
$$\left.-\Psi\left(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma-\frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}-\frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\right]$$
$$+W_{\bar{I}}\left[\sum_{\gamma\in I_m}\tilde{p}_\gamma\Psi(a_\gamma)-\frac{1}{W_{\bar{I}}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in J}\tilde{p}_\gamma w_i\Psi(x_{i\gamma})-\frac{1}{W_{\bar{I}}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in J}\tilde{p}_\gamma w_i\Psi(x_{i\gamma})\right.$$
$$\left.-\Psi\left(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma a_\gamma-\frac{1}{W_{\bar{I}}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in J}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}-\frac{1}{W_{\bar{I}}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in J}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\right].$$

In index set function notation we finally get

$$F(I\cup\bar{I})\geq F(I)+F(\bar{I}).$$

In case when $W_I.W_{\bar{I}}<0$, for instance $W_I>0$ and $W_{\bar{I}}<0$, we again let $\lambda_1=W_I$, $\lambda_2=W_{\bar{I}}$, $t_\gamma^{(1)}=A_I(\mathbf{x}_\gamma,\mathbf{w})$ and $t_\gamma^{(2)}=A_{\bar{I}}(\mathbf{x}_\gamma,\mathbf{w})$ and reversed inequality in (32) follows by using reverse Jensen's inequality for two variable case. □

Corollary 3.2. *Let $\boldsymbol{a}=(a_1,a_2,\ldots,a_m)$ be an m-tuple such that $a_\gamma\in J$ for $\gamma\in I_m$. Let I_t, $t\in I_\lambda$ be finite nonempty sets of positive integers such that $I_s\cap I_t=\emptyset$ for all $s\neq t\in I_\lambda$. We further suppose that $(\mathbf{x}_\gamma)=(x_{i\gamma})$ is a real sequence of vectors such that $x_{i\gamma}\in J$ for all $i\in\bigcup_{t=1}^{\lambda}I_t$, $\gamma\in I_m$ and let $\mathbf{w}=(w_i)$, $i\in\bigcup_{t=1}^{\lambda}I_t$ such that $W_{i\in\bigcup_{t=1}^{\lambda}I_t}>0$ and $A_S(\mathbf{x}_\gamma,\mathbf{w})\in J$ $(S\in\{I_1,\ldots,I_t,\bigcup_{t=1}^{r}I_t\})$ $(r\in\{2,\ldots,\lambda\})$. Take Ψ to be a continuous convex function on J. Then:*

(a) If $W_{I_t}>0$ for $t\in I_\lambda$, we have

$$F\left(\bigcup_{t=1}^{\lambda}I_t\right)\geq\sum_{t=1}^{\lambda}F(I_t). \tag{35}$$

(b) If $W_{I_1}>0$ and $W_{I_t}<0$ for $t\in\{2,\ldots,\lambda\}$, then inequality (35) *is reversed.*

Proof. Proof follows directly from Theorem 3.1 by using induction. □

The following results give us refinements of Niezgoda's Inequality. For the remaining part of this section we assume $x_{i\gamma}\in[a,b]\subseteq J$ for all i and γ.

Corollary 3.3. *Let $\boldsymbol{a}=(a_1,a_2,\ldots,a_m)$ be an m-tuple such that $a_\gamma\in J$ for $\gamma\in I_m$. We further suppose that $(\mathbf{x}_\gamma)=(x_{i\gamma})$ is a real sequence of vectors such that $x_{i\gamma}\in J$ for all $i\in I_n$, $\gamma\in I_m$ and let Ψ be a continuous convex function on J. If $w_1>0$ and $w_i\geq 0$ for $i\in\{2,\ldots,n\}$, then under the assumptions in Corollary* 2.8 *we have*

$$F(I_n)\geq F(I_{n-1})\geq\cdots\geq F(I_2)\geq F(I_1)\geq 0. \tag{36}$$

If $w_i\leq 0$ for $i\in\{2,\ldots,n\}$, $W_{I_n}>0$ and $A_{I_n}(\mathbf{x}_\gamma,\mathbf{w})\in[a,b]\subseteq J$, then

$$0\leq F(I_n)\leq F(I_{n-1})\leq\cdots\leq F(I_2)\leq F(I_1). \tag{37}$$

Proof. Fix $\kappa\in I_m$. Suppose that $w_i\geq 0$ for $i\in\{2,\ldots,n\}$. From generalized Niezgoda's inequality (12) it follows that

$$F(\{t\})=w_t\left[\sum_{\gamma\in I_m}\Psi(a_\gamma)-\sum_{\gamma=1}^{\kappa-1}\Psi(x_{t\gamma})-\sum_{\gamma=\kappa+1}^{m}\Psi(x_{t\gamma})-\Psi\left(\sum_{\gamma\in I_m}a_\gamma-\sum_{\gamma=1}^{\kappa-1}x_{t\gamma}-\sum_{\gamma=\kappa+1}^{m}x_{t\gamma}\right)\right]\geq 0$$

for $t \in I_n$. Now, by Theorem 3.1 we have

$$F(I_t) = F(I_{t-1} \cup \{t\}) \geq F(I_{t-1}) + F(\{t\}) \geq F(I_{t-1})$$

for all $t \in \{2, \dots, n\}$.

For second part, we suppose that $w_i \leq 0$ for $i \in \{2, \dots, n\}$ with $W_{I_n} > 0$ and $A_{I_n}(\mathbf{x}_\gamma, \mathbf{w}) \in [a, b]$. Now we show that $A_{I_{n-1}}(\mathbf{x}_\gamma, \mathbf{w}) \in [a, b]$ as follows.

Given that

$$a \leq A_{I_n}(\mathbf{x}_\gamma, \mathbf{w}) \leq b$$

multiplying both sides by $W_{I_n} > 0$ and adding $-w_n x_{n\gamma}$ we obtain

$$W_{I_n} a - w_n x_{n\gamma} \leq \sum_{i \in I_n} w_i x_{i\gamma} - w_n x_{n\gamma} \leq W_{I_n} b - w_n x_{n\gamma}$$

or we may write

$$W_{I_n} a - w_n x_{n\gamma} \leq \sum_{i \in I_{n-1}} w_i x_{i\gamma} \leq W_{I_n} b - w_n x_{n\gamma}.$$

Now multiplying both sides by $\frac{1}{W_{I_{n-1}}} > 0$ we get

$$\frac{1}{W_{I_{n-1}}}(W_{I_n} a - w_n x_{n\gamma}) \leq A_{I_{n-1}}(\mathbf{x}_\gamma, \mathbf{w}) \leq \frac{1}{W_{I_{n-1}}}(W_{I_n} b - w_n x_{n\gamma}),$$

or we may write

$$a + \frac{w_n}{W_{I_{n-1}}}(a - x_{n\gamma}) \leq A_{I_{n-1}}(\mathbf{x}_\gamma, \mathbf{w}) \leq b + \frac{w_n}{W_{I_{n-1}}}(b - x_{n\gamma}),$$

clearly

$$\frac{w_n}{W_{I_{n-1}}}(a - x_{n\gamma}) \geq 0 \quad \text{and} \quad \frac{w_n}{W_{I_{n-1}}}(b - x_{n\gamma}) \leq 0,$$

and hence we conclude that

$$a \leq A_{I_{n-1}}(\mathbf{x}_\gamma, \mathbf{w}) \leq b.$$

By iteration we obtain $A_{I_t}(\mathbf{x}_\gamma, \mathbf{w}) \in [a, b]$ for all $t \in \{2, \dots, n\}$. Similarly as before we have $F(\{t\}) \leq 0$ for all $t \in \{2, \dots, n\}$. Now by (32) reversed, we have

$$F(I_t) = F(I_{t-1} \cup \{t\}) \leq F(I_{t-1}) + F(\{t\}) \leq F(I_{t-1})$$

for all $t \in \{2, \dots, n\}$ and finally by Theorem 3.1: $F(I_n) \geq 0$. □

For our main result of this section we can also state results analogous to Theorem 3.1 and its corollaries.

Theorem 3.4. *Using the assumptions from Corollary* 2.8*, we have the following refinement:*

$$p_\kappa \Psi\left(\sum_{\gamma \in I_m} \epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_n} \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I_n} \epsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_n} \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I_n} \epsilon p_\gamma v_\gamma w_i x_{i\gamma} \right)$$
$$\leq \tilde{D}(\mathbf{p}, \mathbf{w}, \mathbf{X}, \Psi; I) \leq \sum_{\gamma \in I_m} p_\gamma \Psi(a_\gamma) - \frac{1}{W_n} \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I_n} p_\gamma w_i \Psi(x_{i\gamma}) - \frac{1}{W_n} \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I_n} p_\gamma w_i \Psi(x_{i\gamma}), \quad (38)$$

where $W_I = \sum_{i\in I} w_i$, $W_{\bar{I}} = \sum_{i\in\bar{I}} w_i$, $\bar{I} = I_n \backslash I$ *and* $\kappa \in I_m$ *and*

$$\tilde{D}(\mathbf{p},\mathbf{w},\mathbf{X},\Psi;I) =$$

$$p_\kappa \frac{W_I}{W_n}\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma \sum_{i\in I} w_i x_{i\gamma} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma \sum_{i\in I} w_i x_{i\gamma}\right)$$
$$+ p_\kappa \frac{W_{\bar{I}}}{W_n}\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_{\bar{I}}}\sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma \sum_{i\in\bar{I}}^{n} w_i x_{i\gamma} - \frac{1}{W_{\bar{I}}}\sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma \sum_{i\in\bar{I}} w_i x_{i\gamma}\right). \quad (39)$$

Proof. Fixing $\kappa \in I_m$, and suppose that $w_i^* = \frac{w_i}{W_n}$ where $\sum_{i=1}^n w_i^* = 1$. Also $W_I^* = \sum_{i\in I} w_i^*$. By the convexity of function Ψ we have

$$\begin{aligned}
& p_\kappa \Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I_n}\epsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I_n}\epsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\\
= \; & p_\kappa \Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\sum_{i\in I_n}\epsilon p_\gamma v_\gamma w_i^* x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m}\sum_{i\in I_n}\epsilon p_\gamma v_\gamma w_i^* x_{i\gamma}\right)\\
= \; & p_\kappa \Psi\left(\sum_{i\in I_n} w_i^*\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma x_{i\gamma}\right)\right)\\
= \; & p_\kappa \Psi\left(W_I^*\left(\frac{1}{W_I^*}\sum_{i\in I} w_i^*\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma x_{i\gamma}\right)\right)\right.\\
+ \; & \left. W_{\bar{I}}^*\left(\frac{1}{W_{\bar{I}}^*}\sum_{i\in\bar{I}} w_i^*\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma x_{i\gamma}\right)\right)\right)\\
\leq \; & p_\kappa W_I^*\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_I^*}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i^* x_{i\gamma} - \frac{1}{W_I^*}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i^* x_{i\gamma}\right)\\
+ \; & p_\kappa W_{\bar{I}}^*\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_{\bar{I}}^*}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i^* x_{i\gamma} - \frac{1}{W_{\bar{I}}^*}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i^* x_{i\gamma}\right)\\
\leq \; & p_\kappa \frac{W_I}{W_n}\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{\frac{W_I}{W_n}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{\frac{W_I}{W_n}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\\
+ \; & p_\kappa \frac{W_{\bar{I}}}{W_n}\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{\frac{W_{\bar{I}}}{W_n}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{\frac{W_{\bar{I}}}{W_n}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\\
= \; & p_\kappa \frac{W_I}{W_n}\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\\
+ \; & p_\kappa \frac{W_{\bar{I}}}{W_n}\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_{\bar{I}}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in\bar{I}}\epsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_{\bar{I}}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in\bar{I}}\epsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\\
= \; & D(\mathbf{p},\mathbf{w},\mathbf{X},\Psi;I)
\end{aligned}$$

for any I, which proves the first inequality in (38).

By inequality (39) we have

$$
\begin{aligned}
& D(\mathbf{p},\mathbf{w},\mathbf{X},\Psi;I)\\
= & \; p_\kappa \frac{W_I}{W_n}\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\\
+ & \; p_\kappa \frac{W_{\bar I}}{W_n}\Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_{\bar I}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in \bar I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_{\bar I}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in \bar I}\epsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\\
= & \; p_\kappa \frac{W_I}{W_n}\Psi\left(\frac{1}{W_I}\sum_{i\in I} w_i\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma x_{i\gamma}\right)\right)\\
+ & \; p_\kappa \frac{W_{\bar I}}{W_n}\Psi\left(\frac{1}{W_{\bar I}}\sum_{i\in \bar I} w_i\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma x_{i\gamma}\right)\right)\\
\le & \; \frac{W_I}{W_n}\left(\frac{1}{W_I}\sum_{i\in I} w_i p_\kappa \Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma x_{i\gamma}\right)\right)\\
+ & \; \frac{W_{\bar I}}{W_n}\left(\frac{1}{W_{\bar I}}\sum_{i\in \bar I} w_i p_\kappa \Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\epsilon p_\gamma v_\gamma x_{i\gamma} - \sum_{\gamma=\kappa+1}^{m}\epsilon p_\gamma v_\gamma x_{i\gamma}\right)\right)\\
\le & \; \frac{1}{W_n}\sum_{i\in I_n} w_i\left(\sum_{\gamma\in I_m} p_\gamma \Psi(a_\gamma) - \sum_{\gamma=1}^{\kappa-1} p_\gamma\Psi(x_{i\gamma}) - \sum_{\gamma=\kappa+1}^{m} p_\gamma\Psi(x_{i\gamma})\right)\\
= & \; \sum_{\gamma\in I_m} p_\gamma\Psi(a_\gamma) - \frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1} p_\gamma\sum_{i\in I_n} w_i\Psi(x_{i\gamma}) - \frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m} p_\gamma\sum_{i\in I_n} w_i\Psi(x_{i\gamma}).
\end{aligned}
$$

In the last inequality we used the fact that

$$
\frac{W_I}{W_n}\left(\frac{1}{W_I}\sum_{i\in I} w_i p_\kappa \Psi\left(x_{i\gamma}\right)\right) + \frac{W_{\bar I}}{W_n}\left(\frac{1}{W_{\bar I}}\sum_{i\in \bar I} w_i p_\kappa \Psi\left(x_{i\gamma}\right)\right) = \frac{1}{W_n}\sum_{i\in I_n} w_i p_\kappa \Psi\left(x_{i\kappa}\right).
$$

This proves the second inequality in (38), for any I. □

Remark 3.5. *It holds that*

$$
\begin{aligned}
& p_\kappa \Psi\left(\sum_{\gamma\in I_m}\epsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I_n}\epsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I_n}\epsilon p_\gamma v_\gamma w_i x_{i\gamma}\right)\\
& \le \min_I D(\mathbf{p},\mathbf{w},\mathbf{X},\Psi;I)
\end{aligned}
$$

and

$$
\begin{aligned}
& \max_I D(\mathbf{p},\mathbf{w},\mathbf{X},\Psi;I)\\
& \le \sum_{\gamma\in I_m} p_\gamma\Psi(a_\gamma) - \frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I_n} p_\gamma w_i\Psi(x_{i\gamma}) - \frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I_n} p_\gamma w_i\Psi(x_{i\gamma}).
\end{aligned}
$$

We will be needing the following definition for the corollary that follows.

Definition 3.6. *[25, p. 10] An $m \times m$ matrix $\boldsymbol{B} = (b_{\gamma\lambda})$ is known as doubly stochastic, if $b_{\gamma\lambda} \geq 0$ and $\sum_{\gamma=1}^{m} b_{\gamma\lambda} = \sum_{\kappa=1}^{m} b_{\gamma\lambda} = 1$ for all $\gamma, \kappa \in I_m$.*

Also, if $\mathbf{B}$ is an $m \times m$ doubly stochastic matrix, then [25, p. 31] :

$$\boldsymbol{b}\mathbf{B} \prec \boldsymbol{b} \text{ for each real } m\text{-tuple } \boldsymbol{b} = (b_1, \ldots, b_m). \tag{40}$$

By applying Theorem 3.4 and inequality (40), one gets:

Corollary 3.7. *Assume Ψ to be a continuous convex function on J. Suppose that $\boldsymbol{b} = (b_1, \ldots, b_m) \in J^m$ for $\gamma \in I_m$ and $\mathbf{B}_1, \ldots, \mathbf{B}_n$ are $m \times m$ doubly stochastic matrices. Set*

$$\mathbf{X} = (x_{i\gamma}) = \begin{pmatrix} \boldsymbol{b}\ \mathbf{B}_1 \\ \vdots \\ \boldsymbol{b}\ \mathbf{B}_n \end{pmatrix}.$$

Then inequality (38) *holds.*

Remark 3.8. *Analagous assertion can be formulated for concave functions using the fact that Ψ is concave iff $-f$ is convex.*

4. Applications

H: For $\emptyset \neq I \subseteq I_n = \{1, \ldots, n\}$, let A_I, G_I, H_I and $M_I^{[r]}$ be the arithmetic mean, geometric mean, harmonic mean and power mean of order $r \in R$, respectively for $x_{i\gamma} \in [a, b] \subseteq J$ where $\gamma \in I_m$, $i \in I$, and $0 < a < b$, formed with weights w_i, $i \in I$ satisfies the conditions stated in (2). For $I = I_n$ we denote the arithmetic mean, geometric mean, harmonic mean and power mean by A_n, G_n, H_n and $M_n^{[r]}$ respectively. For detailed understanding of these means and their relations with each other, the reader can go through [6] and [22]. For example, it is well known that

$$A_n \geq G_n \geq H_n. \tag{41}$$

$$\left(\frac{A_n}{G_n}\right)^{W_n} \geq \left(\frac{A_{n-1}}{G_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{A_1}{G_1}\right)^{W_1} \geq 1. \tag{42}$$

$$W_n(A_n - G_n) \geq W_{n-1}(A_{n-1} - G_{n-1}) \geq \cdots \geq W_1(A_1 - G_1) \geq 0. \tag{43}$$

Also we have renowned Ky Fan Inequality [5, p. 5] given by

$$\frac{A_n(\mathbf{x})}{A_n(\mathbf{1} - \mathbf{x})} \geq \frac{G_n(\mathbf{x})}{G_n(\mathbf{1} - \mathbf{x})}, \quad 0 < x_\gamma \leq \frac{1}{2} \quad \forall\, \gamma. \tag{44}$$

If we define

$$\begin{aligned} \tilde{A}_I : \ &= \sum_{\gamma \in I_m} \varepsilon p_\gamma v_\gamma a_\gamma - \frac{1}{W_I} \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I} \varepsilon p_\gamma v_\gamma w_i x_{i\gamma} - \frac{1}{W_I} \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I} \varepsilon p_\gamma v_\gamma w_i x_{i\gamma} \\ &= \sum_{\gamma \in I_m} \varepsilon p_\gamma v_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1} \varepsilon p_\gamma v_\gamma A_I(\mathbf{x}_\gamma, \mathbf{w}) - \sum_{\gamma=\kappa+1}^{m} \varepsilon p_\gamma v_\gamma A_I(\mathbf{x}_\gamma, \mathbf{w}) \end{aligned}$$

$$\tilde{G}_I : = \frac{\prod_{\gamma\in I_m} a_\gamma^{\tilde{p}_\gamma}}{\left(\prod_{\gamma=1}^{\kappa-1}\prod_{i\in I} x_{i\gamma}^{w_i\tilde{p}_\gamma}\right)^{\frac{1}{W_I}}\left(\prod_{\gamma=\kappa+1}^{m}\prod_{i\in I} x_{i\gamma}^{w_i\tilde{p}_\gamma}\right)^{\frac{1}{W_I}}}$$

$$\tilde{H}_I : = \left(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma {a_\gamma}^{-1} - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}^{-1} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}^{-1}\right)^{-1}$$

$$\tilde{M}_I^{[r]} : = \begin{cases}\left(\sum_{\gamma\in I_m}\varepsilon p_\gamma v_\gamma {a_\gamma}^{r} - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}^{r} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\varepsilon p_\gamma v_\gamma w_i x_{i\gamma}^{r}\right)^{\frac{1}{r}} & r\neq 0,\\ \tilde{G}_I & r=0.\end{cases}$$

Under the assumption made in Corollary 2.11 we deduce $\varepsilon p_\gamma v_\gamma = \tilde{p}_\gamma$, where $\tilde{p}_\gamma = \frac{p_\gamma}{p_\kappa}$ and hence the arithmetic mean, geometric mean, harmonic mean and power mean defined above can written as:

$$\begin{aligned}\tilde{A}_I : &= \sum_{\gamma\in I_m}\tilde{p}_\gamma a_\gamma - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\tilde{p}_\gamma w_i x_{i\gamma} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\tilde{p}_\gamma w_i x_{i\gamma}\\ &= \sum_{\gamma\in I_m}\tilde{p}_\gamma a_\gamma - \sum_{\gamma=1}^{\kappa-1}\tilde{p}_\gamma A_I(\mathbf{x}_\gamma,\mathbf{w}) - \sum_{\gamma=\kappa+1}^{m}\tilde{p}_\gamma A_I(\mathbf{x}_\gamma,\mathbf{w})\end{aligned}$$

$$\tilde{G}_I : = \frac{\prod_{\gamma\in I_m} a_\gamma^{\tilde{p}_\gamma}}{\left(\prod_{\gamma=1}^{\kappa-1}\prod_{i\in I} x_{i\gamma}^{w_i\tilde{p}_\gamma}\right)^{\frac{1}{W_I}}\left(\prod_{\gamma=\kappa+1}^{m}\prod_{i\in I} x_{i\gamma}^{w_i\tilde{p}_\gamma}\right)^{\frac{1}{W_I}}}$$

$$\tilde{H}_I : = \left(\sum_{\gamma\in I_m}\tilde{p}_\gamma {a_\gamma}^{-1} - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\tilde{p}_\gamma w_i x_{i\gamma}^{-1} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\tilde{p}_\gamma w_i x_{i\gamma}^{-1}\right)^{-1}$$

$$\tilde{M}_I^{[r]} : = \begin{cases}\left(\sum_{\gamma\in I_m}\tilde{p}_\gamma {a_\gamma}^{r} - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\tilde{p}_\gamma w_i x_{i\gamma}^{r} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\tilde{p}_\gamma w_i x_{i\gamma}^{r}\right)^{\frac{1}{r}} & r\neq 0,\\ \tilde{G}_I & r=0.\end{cases}$$

Then under the assumptions given in **H** along with Corollary 2.11, the following results are valid:

Theorem 4.1.

$$\tilde{A} \geq \tilde{G} \tag{45}$$

$$\frac{\tilde{A}(\mathbf{x})}{\tilde{A}(\mathbf{1}-\mathbf{x})} \geq \frac{\tilde{G}(\mathbf{x})}{\tilde{G}(\mathbf{1}-\mathbf{x})} \quad \textit{provided that} \quad 0 < x_{i\gamma} \leq \frac{1}{2} \quad \textit{for all } i,\gamma. \tag{46}$$

Proof. Using the convex function $\phi(x) = -\ln x$ in inequality (12), we obtain inequality (45).

Using the convex function $\phi(x) = \ln\left(\frac{1-x}{x}\right)$ $(0 < x \leq \frac{1}{2})$ in inequality (12), we obtain inequality (46). The reverse inequalities (45) and (46) hold when we take ϕ as a concave function. □

Theorem 4.2.

$$\left(\frac{\tilde{A}_n}{\tilde{G}_n}\right)^{W_n} \geq \left(\frac{\tilde{A}_{n-1}}{\tilde{G}_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{\tilde{A}_1}{\tilde{G}_1}\right)^{W_1} \geq 1. \quad (47)$$

$$W_n(\tilde{A}_n - \tilde{G}_n) \geq W_{n-1}(\tilde{A}_{n-1} - \tilde{G}_{n-1}) \geq \cdots \geq W_1(\tilde{A}_1 - \tilde{G}_1) \geq 0. \quad (48)$$

Proof. Using the convex function $\Psi(x) = -\ln x$ in inequality (36), we obtain:

$$\ln\left(\frac{\tilde{A}_n}{\tilde{G}_n}\right)^{W_n} \geq \ln\left(\frac{\tilde{A}_{n-1}}{\tilde{G}_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \ln\left(\frac{\tilde{A}_1}{\tilde{G}_1}\right)^{W_1} \geq 0. \quad (49)$$

from which inequality (47) follows. Using the convex function $\Psi(x) = \exp x$ and replacing a_γ and $x_{i\gamma}$ with $\ln(a_\gamma)$ and $\ln(x_{i\gamma})$ respectively in inequality (36), we obtain:

$$W_n(\tilde{A}_n - \tilde{G}_n) \geq W_{n-1}(\tilde{A}_{n-1} - \tilde{G}_{n-1}) \geq \cdots \geq W_1(\tilde{A}_1 - \tilde{G}_1) \geq 0,$$

since in this case

$$F(I_t) = W_t\left[\sum_{\gamma\in I_m} \tilde{p}_\gamma a_\gamma - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I} \tilde{p}_\gamma w_i x_{i\gamma} - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I} \tilde{p}_\gamma w_i x_{i\gamma} - \exp\left(\sum_{\gamma\in I_m} \tilde{p_\gamma} ln(a_\gamma) - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I} \tilde{p}_\gamma w_i ln(x_{i\gamma}) - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I} \tilde{p}_\gamma w_i ln(x_{i\gamma})\right)\right] = W_t(\tilde{A}_t - \tilde{G}_t).$$

The reverse inequalities (48) and (47) hold when we take ϕ as a concave function. □

Remark 4.3. *If in Theorem 4.2 we simply put $w_i = 1$ for all $i \in I_n$, we get the following results which are of Popoviciu- [39] and Rado- [40] types respectively (see also [32, p. 13]).*

Corollary 4.4.

$$\left(\frac{\tilde{A}_n}{\tilde{G}_n}\right)^{n} \geq \left(\frac{\tilde{A}_{n-1}}{\tilde{G}_{n-1}}\right)^{n-1} \geq \cdots \geq \left(\frac{\tilde{A}_1}{\tilde{G}_1}\right)^{1} \geq 1.$$

$$n(\tilde{A}_n - \tilde{G}_n) \geq (n-1)(\tilde{A}_{n-1} - \tilde{G}_{n-1}) \geq \cdots \geq 1\cdot(\tilde{A}_1 - \tilde{G}_1) \geq 0.$$

Corollary 4.5.

$$\left(\frac{\tilde{G}_n}{\tilde{H}_n}\right)^{W_n} \geq \left(\frac{\tilde{G}_{n-1}}{\tilde{H}_{n-1}}\right)^{W_{n-1}} \geq \cdots \geq \left(\frac{\tilde{G}_1}{\tilde{H}_1}\right)^{W_1} \geq 1.$$

$$W_n\left(\frac{1}{\tilde{H}_n} - \frac{1}{\tilde{G}_n}\right) \geq W_{n-1}\left(\frac{1}{\tilde{H}_{n-1}} - \frac{1}{\tilde{G}_{n-1}}\right) \geq \cdots \geq W_1\left(\frac{1}{\tilde{H}_1} - \frac{1}{\tilde{G}_1}\right) \geq 0.$$

Proof. Follows directly from Theorem 4.2 by the substitutions $a_\gamma \to \frac{1}{a_\gamma}$ and $x_{i\gamma} \to \frac{1}{x_{i\gamma}}$. □

Theorem 4.6. *For $r \leq 1$, we have the following inequalities*

$$W_n(\tilde{A}_n - \tilde{M}_n^{[r]}) \geq W_{n-1}(\tilde{A}_{n-1} - \tilde{M}_{n-1}^{[r]}) \geq \cdots \geq W_1(\tilde{A}_1 - \tilde{M}_1^{[r]}) \geq 0. \quad (50)$$

For $r \geq 1$, the inequalities in (50) *are reversed.*

Proof. For $r \leq 1$, using the convex function $\Psi(x) = x^{\frac{1}{r}}$ and replacing a_γ and $x_{i\gamma}$ with a_γ^r and $x_{i\gamma}^r$ respectively in inequality (36), we obtain inequality (50), since in this case

$$F(I_t) = W_t \Bigg[\sum_{\gamma \in I_m} \tilde{p}_\gamma a_\gamma - \frac{1}{W_I} \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I} \tilde{p}_\gamma w_i x_{i\gamma} - \frac{1}{W_I} \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I} \tilde{p}_\gamma w_i x_{i\gamma} - \left(\sum_{\gamma \in I_m} \tilde{p}_\gamma a_\gamma^r - \frac{1}{W_I} \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I} \tilde{p}_\gamma w_i x_{i\gamma}^r - \frac{1}{W_I} \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I} \tilde{p}_\gamma w_i x_{i\gamma}^r \right)^{1/r} \Bigg] = W_t(\tilde{A}_t - \tilde{M}_t^{[r]}).$$

If $r \geq 1$, then the function $\Psi(x) = x^{\frac{1}{r}}$ is concave, so the inequalities in (50) are reversed. □

By simply taking $r = -1$ we get the following corollary.

Corollary 4.7.

$$W_n(\tilde{A}_n - \tilde{H}_n) \geq W_{n-1}(\tilde{A}_{n-1} - \tilde{H}_{n-1}) \geq \cdots \geq W_1(\tilde{A}_1 - \tilde{H}_1) \geq 0.$$

Remark 4.8. *It is easy to see that inequality (48) is a direct consequence of Theorem 4.6.*

Theorem 4.9. *Let $r, s \in R$, $r \leq s$. If $s > 0$, then*

$$W_n \left(\left(\tilde{M}_n^{[s]}\right)^s - \left(\tilde{M}_n^{[r]}\right)^s \right) \geq W_{n-1} \left(\left(\tilde{M}_{n-1}^{[s]}\right)^s - \left(\tilde{M}_{n-1}^{[r]}\right)^s \right) \geq \cdots \geq W_1 \left(\left(\tilde{M}_1^{[s]}\right)^s - \left(\tilde{M}_1^{[r]}\right)^s \right) \geq 0. \quad (51)$$

If $s < 0$, then the inequalities in (51) *are reversed.*

Proof. For $s > 0$, using the convex function $\Psi(x) = x^{\frac{s}{r}}$ and replacing a_γ and $x_{i\gamma}$ with a_γ^r and $x_{i\gamma}^r$ respectively in inequality (36), we obtain inequality (51), since in this case

$$F(I_t) = W_t \Bigg[\sum_{\gamma \in I_m} \tilde{p}_\gamma a_\gamma^s - \frac{1}{W_I} \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I} \tilde{p}_\gamma w_i x_{i\gamma}^s - \frac{1}{W_I} \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I} \tilde{p}_\gamma w_i x_{i\gamma}^s - \left(\sum_{\gamma \in I_m} \tilde{p}_\gamma a_\gamma^r - \frac{1}{W_I} \sum_{\gamma=1}^{\kappa-1} \sum_{i \in I} \tilde{p}_\gamma w_i x_{i\gamma}^r - \frac{1}{W_I} \sum_{\gamma=\kappa+1}^{m} \sum_{i \in I} \tilde{p}_\gamma w_i x_{i\gamma}^r \right)^{s/r} \Bigg] = W_t \left(\left(\tilde{M}_t^{[s]}\right)^s - \left(\tilde{M}_t^{[r]}\right)^s \right).$$

If $s < 0$, then the function $\Psi(x) = x^{\frac{s}{r}}$ is concave, so the inequalities in (51) are reversed. □

Theorem 4.10.

$$(i) \quad \tilde{G}_n \leq \min_I \tilde{A}_I^{\frac{W_I}{W_n}} \tilde{A}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}} \quad \text{and } \tilde{A}_n \geq \max_I \tilde{A}_I^{\frac{W_I}{W_n}} \tilde{A}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}}. \quad (52)$$

$$(ii) \quad \tilde{G}_n \leq \min_I \left[\frac{W_I}{W_n} \tilde{G}_I + \frac{W_{\bar{I}}}{W_n} \tilde{G}_{\bar{I}} \right] \text{ and } \tilde{A}_n \geq \max_I \left[\frac{W_I}{W_n} \tilde{G}_I + \frac{W_{\bar{I}}}{W_n} \tilde{G}_{\bar{I}} \right]. \quad (53)$$

Proof. (i) Applying the convex function $\Psi(x) = -\ln x$ in Theorem 3.4, we obtain

$$-\ln \tilde{A}_n \leq -\frac{W_I}{W_n} \ln \tilde{A}_I - \frac{W_{\bar{I}}}{W_n} \ln \tilde{A}_{\bar{I}} \leq -\ln \tilde{G}_n. \quad (54)$$

Now inequality (52) follows from Remark 3.5 and (54).

(ii) Applying the convex function $\Psi(x) = \exp x$ and replacing a_γ and $x_{i\gamma}$ with $\ln a_\gamma$ and $\ln x_{i\gamma}$ respectively in Theorem 3.4 and using Remark 3.5, we obtain inequality (53). □

The following particular case of Theorem 4.10 is of interest.

Corollary 4.11.

$$(i)\ \ \frac{1}{\tilde{G}_n} \le \min_I \frac{1}{\tilde{H}_I^{\frac{W_I}{W_n}} \tilde{H}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}}} \quad and \quad \frac{1}{\tilde{H}_n} \ge \max_I \frac{1}{\tilde{H}_I^{\frac{W_I}{W_n}} \tilde{H}_{\bar{I}}^{\frac{W_{\bar{I}}}{W_n}}}.$$

$$(ii)\ \ \frac{1}{\tilde{G}_n} \le \min_I \left[\frac{W_I}{W_n \tilde{G}_I} + \frac{W_{\bar{I}}}{W_n \tilde{G}_{\bar{I}}}\right] \quad and \quad \frac{1}{\tilde{H}_n} \ge \max_I \left[\frac{W_I}{W_n \tilde{G}_I} + \frac{W_{\bar{I}}}{W_n \tilde{G}_{\bar{I}}}\right].$$

Proof. Follows directly from Theorem 4.10 by the substitutions $a_\gamma \to \frac{1}{a_\gamma}$ and $x_{i\gamma} \to \frac{1}{x_{i\gamma}}$. □

Theorem 4.12. *For $r \le 1$ ($r \ne 0$), the following inequalities hold:*

$$\tilde{M}_n^{[r]} \le \min_I \left[\frac{W_I}{W_n}\tilde{M}_I^{[r]} + \frac{W_{\bar{I}}}{W_n}\tilde{M}_{\bar{I}}^{[r]}\right], \tag{55}$$

$$\tilde{A}_n \ge \max_I \left[\frac{W_I}{W_n}\tilde{M}_I^{[r]} + \frac{W_{\bar{I}}}{W_n}\tilde{M}_{\bar{I}}^{[r]}\right]. \tag{56}$$

In case $r \ge 1$, the above inequalities (55) *are reversed.*

Proof. When $r \le 1$, ($r \ne 0$) use Theorem 3.4 and then Remark 3.5 for the convex function $\Psi(x) = x^{\frac{1}{r}}$ and replacing a_γ and $x_{i\gamma}$ with a_γ^r and $x_{i\gamma}^r$ respectively [for $r = 0$ use Theorem 3.4 for the convex function $\Psi(x) = \exp x$, replacing a_γ and $x_{i\gamma}$ with $\ln a_\gamma$ and $\ln x_{i\gamma}$ respectively, we obtain (53)].

In case $r \ge 1$, the inequalities in (55) are reversed since the function $\Psi(x) = x^{\frac{1}{r}}$ is concave. □

Corollary 4.13.

$$\tilde{H}_n \le \min_I \left[\frac{W_I}{W_n}\tilde{H}_I + \frac{W_{\bar{I}}}{W_n}\tilde{H}_{\bar{I}}\right],$$

$$\tilde{A}_n \ge \max_I \left[\frac{W_I}{W_n}\tilde{H}_I + \frac{W_{\bar{I}}}{W_n}\tilde{H}_{\bar{I}}\right].$$

Remark 4.14. *It is easy to see that Theorem (53) is also direct consequence of Theorem 4.12.*

Theorem 4.15. *Let $r, s \in \mathbb{R}$, $r \le s$.*

(i) For $s \ge 0$, the following inequalities hold:

$$\left(\tilde{M}_n^{[r]}\right)^s \le \min_I \left[\frac{W_I}{W_n}\left(\tilde{M}_I^{[r]}\right)^s + \frac{W_{\bar{I}}}{W_n}\left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right],$$

$$\tilde{A}_n \ge \max_I \left[\frac{W_I}{W_n}\left(\tilde{M}_I^{[r]}\right)^s + \frac{W_{\bar{I}}}{W_n}\left(\tilde{M}_{\bar{I}}^{[r]}\right)^s\right]. \tag{57}$$

(ii) In case $s < 0$, the above inequalities (57) *are reversed.*

Proof. Let $s \ge 0$. Using the convex function $\Psi(x) = x^{\frac{s}{r}}$ and replacing a_γ and $x_{i\gamma}$ with a_γ^r and $x_{i\gamma}^r$ respectively in Theorem 3.4 and making use of Remark 3.5, we obtain inequality (57).
In case $s < 0$, the inequalities in (57) are reversed since the function $\Psi(x) = x^{\frac{s}{r}}$ is concave. □

Definition 4.16. *Let ϕ be a strictly monotonic continuous function on J. Then for a given n-tuple $\mathbf{x} = (x_1, \ldots, x_n) \in J^n$ and real n-tuple $\boldsymbol{w} = (w_1, \ldots, w_n)$ with $W_n \ne 0$, the value*

$$M_\phi^{[n]} = \phi^{-1}\left(\frac{1}{W_n}\sum_{i=1}^{n} w_i \phi(x_i)\right)$$

is well defined and is called quasi – arithmetic mean of $\mathbf{x}$ with weight $\boldsymbol{w}$ (see for example [6, p. 215]).

Under the assumptions of Corollary 2.8 and Corollary 2.11, we define:

$$\tilde{M}_{\phi}^{[n]} = \phi^{-1}\left(\sum_{\gamma\in I_m}\tilde{p}_\gamma\phi(a_\gamma) - \frac{1}{W_n}\sum_{\gamma=1}^{\kappa-1}\sum_{i=1}^{n}\tilde{p}_\gamma w_i\phi(x_{i\gamma}) - \frac{1}{W_n}\sum_{\gamma=\kappa+1}^{m}\sum_{i=1}^{n}\tilde{p}_\gamma w_i\phi(x_{i\gamma})\right). \tag{58}$$

then the following results hold:

Theorem 4.17. *Let ϕ and ψ be two strictly monotonic continuous functions on J. If $\psi\circ\phi^{-1}$ is convex on J, then*

$$W_n\left(\psi\left(\tilde{M}_{\psi}^{[n]}\right) - \psi\left(\tilde{M}_{\phi}^{[n]}\right)\right) \geq W_{n-1}\left(\psi\left(\tilde{M}_{\psi}^{[n-1]}\right) - \psi\left(\tilde{M}_{\phi}^{[n-1]}\right)\right) \geq \cdots \geq W_1\left(\psi\left(\tilde{M}_{\psi}^{[1]}\right) - \psi\left(\tilde{M}_{\phi}^{[1]}\right)\right) \geq 0. \tag{59}$$

If $\psi\circ\phi^{-1}$ is concave on J, then inequalities (59) *are reversed.*

Proof. Applying (36) to the convex function $f = \psi\circ\phi^{-1}$ and replacing a_γ and $x_{i\gamma}$ with $\phi(a_\gamma)$ and $\phi(x_{i\gamma})$ respectively we obtain (59), since in this case

$$\begin{aligned}F(I_t) = W_t&\left[\sum_{\gamma\in I_m}\tilde{p}_\gamma\psi(a_\gamma) - \frac{1}{W_{I_t}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I_t}\tilde{p}_\gamma w_i\psi(x_{i\gamma}) - \frac{1}{W_{I_t}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I_t}\tilde{p}_\gamma w_i\psi(x_{i\gamma})\right]\\ &-(\psi\circ\phi^{-1})\left[\sum_{\gamma\in I_m}\tilde{p}_\gamma\phi(a_\gamma) - \frac{1}{W_{I_t}}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I_t}\tilde{p}_\gamma w_i\phi(x_{i\gamma}) - \frac{1}{W_{I_t}}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I_t}\tilde{p}_\gamma w_i\phi(x_{i\gamma})\right]\\ &= W_t\left(\psi\left(\tilde{M}_{\psi}^{[t]}\right) - \psi\left(\tilde{M}_{\phi}^{[t]}\right)\right).\end{aligned}$$

□

Remark 4.18. *Theorem 4.2, 4.6 and 4.9 follow from Theorem 4.17, if we choose suitable functions ϕ, ψ and make substitutions accordingly.*

Corollary 4.19. *Let $\phi,\psi : J\to\mathbb{R}$ be strictly monotonic and continuous functions. If $\psi\circ\phi^{-1}$ is convex on J, then*

$$\begin{aligned}&W_n\left(\psi\left(\tilde{M}_{\psi}^{[n]}\right) - \psi\left(\tilde{M}_{\phi}^{[n]}\right)\right) \geq\\ &\max_{1\le s\le t\le n}\left[(w_s+w_t)\left[\sum_{\gamma\in I_m}\psi(a_\gamma) - \sum_{\gamma=1}^{\kappa-1}\frac{w_s\psi(x_{sj})+w_t\psi(x_{t\gamma})}{w_s+w_t} - \sum_{\gamma=\kappa+1}^{m}\frac{w_s\psi(x_{sj})+w_t\psi(x_{t\gamma})}{w_s+w_t}\right.\right.\\ &\left.\left.-(\psi\circ\phi^{-1})\left(\sum_{\gamma\in I_m}\phi(a_\gamma) - \sum_{\gamma=1}^{\kappa-1}\frac{w_s\phi(x_{sj})+w_t\phi(x_{t\gamma})}{w_s+w_t} - \sum_{\gamma=\kappa+1}^{m}\frac{w_s\phi(x_{sj})+w_t\phi(x_{t\gamma})}{w_s+w_t}\right)\right]\right]\end{aligned} \tag{60}$$

and

$$\begin{aligned}W_n\left(\psi\left(\tilde{M}_{\psi}^{[n]}\right) - \psi\left(\tilde{M}_{\phi}^{[n]}\right)\right) \geq \max_{1\le t\le n}&\left[w_t\left[\sum_{\gamma\in I_m}\psi(a_\gamma) - \sum_{\gamma=1}^{\kappa-1}\psi(x_{t\gamma}) - \sum_{\gamma=\kappa+1}^{m}\psi(x_{t\gamma})\right.\right.\\ &\left.\left.-(\psi\circ\phi^{-1})\left(\sum_{\gamma\in I_m}\phi(a_\gamma) - \sum_{\gamma=1}^{\kappa-1}\phi(x_{t\gamma}) - \sum_{\gamma=\kappa+1}^{m}\phi(x_{t\gamma})\right)\right]\right].\end{aligned} \tag{61}$$

If $\psi\circ\phi^{-1}$ is concave on J, then inequalities in (60) *and* (61) *are reversed and maximum is replaced with minimum.*

Theorem 4.20. *Let ϕ and ψ be two strictly monotonic continuous functions on J. If $\psi \circ \phi^{-1}$ is convex on J, then*

$$\begin{aligned} \psi\left(\tilde{M}_{\phi}^{[n]}\right) &\leq \min_{I}\left[\frac{W_I}{W_n}\psi\left(\tilde{M}_{\phi}^{[I]}\right) + \frac{W_{\bar{I}}}{W_n}\psi\left(\tilde{M}_{\phi}^{[\bar{I}]}\right)\right], \\ \psi\left(\tilde{M}_{\psi}^{[n]}\right) &\geq \max_{I}\left[\frac{W_I}{W_n}\psi\left(\tilde{M}_{\phi}^{[I]}\right) + \frac{W_{\bar{I}}}{W_n}\psi\left(\tilde{M}_{\phi}^{[\bar{I}]}\right)\right], \end{aligned} \tag{62}$$

where $\tilde{M}_{\phi}^{[I]}$ is defined as

$$\tilde{M}_{\phi}^{[I]} = \phi^{-1}\left(\sum_{\gamma\in I_m}\tilde{p}_{\gamma}\phi(a_{\gamma}) - \frac{1}{W_I}\sum_{\gamma=1}^{\kappa-1}\sum_{i\in I}\tilde{p}_{\gamma}w_i\phi(x_{i\gamma}) - \frac{1}{W_I}\sum_{\gamma=\kappa+1}^{m}\sum_{i\in I}\tilde{p}_{\gamma}w_i\phi(x_{i\gamma})\right).$$

Proof. Using the convex function $f = \psi \circ \phi^{-1}$ and replacing a_{γ} and $x_{i\gamma}$ with $\phi(a_{\gamma})$ and $\phi(x_{i\gamma})$ respectively in Theorem 3.4 and then using Remark 3.5, we obtain inequality (62). □

Remark 4.21. *(a) Theorem 4.10, 4.12 and 4.15 follow from Theorem 4.20, by choosing adequate functions ϕ, ψ and appropriate substitutions.*

(b) In all the theorems, reverse inequalities hold for concave functions.

(c) By imposing different conditions on κ and weights w_i's we can obtain many special cases of our results proved in this section, in articles [19, 21, 28].

Remark 4.22. *Similar results can be proved for concave functions given that Ψ is concave if and only if $-\Psi$ is convex.*

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