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Stancu Type Operators Including Generalized Brenke Polynomials

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Abstract. Our objective in this paper is to present the sequence of Stancu type operators including generalized Brenke polynomials. We answer the problem of uniform approximation of continuous functions on closed bounded interval and the problem of the order of this convergence estimate by known tools in approximation theory. Furthermore, we apply some of the results obtained in this work to Miller-Lee polynomials and Gould-Hopper polynomials.

1. Introduction

Approximation of functions by polynomials is not only an important topic of the theory of mathematical analysis but also provides powerful mathematical tools to application areas. Theorem of Weierstrass declares that every function f which is continuous on a finite closed interval $[a,b]$ can be developed according to polynomials in a series which is uniformly convergent on $[a,b]$. According to Weierstrass' theorem, the family of all polynomials is dense in $C[a,b]$. Taking advantage of a probabilistic construction, Bernstein presented a definitively proof of the Weierstrass approximation theory.

For $\phi \in C[0,\infty)$ one of the most important sequence of operators is defined by Szasz [18]

$$S_n\left(\phi;x\right) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\phi\left(\frac{k}{n}\right), \tag{1}$$

where the above sum converges under suitable conditions.

Assuming that

$$h(z) = \sum_{k=0}^{\infty} a_k z^k, \quad (a_0 \neq 0)$$

is an analytic function on the following set

$$\{z : |z| < R,\ R > 1\}$$

and $h(1) \neq 0$. If polynomials π_k satisfy the following relation

$$h(u)e^{ux} = \sum_{k=0}^{\infty}\pi_k(x)u^k, \tag{2}$$

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then it is called Appell polynomials [4]. These type of polynomials mentioned above are among the most important special functions and have various applications to engineering and mathematical analysis.

Jakimovski and Leviatan [12] gave sequence of operators $\{J_n\}_{n\geq 1}$ as follows

$$J_n\left(\phi;x\right) := \frac{e^{-nx}}{h(1)}\sum_{k=0}^{\infty}\pi_k(nx)\,\phi\left(\frac{k}{n}\right), \tag{3}$$

where $\pi_k(x) \geq 0$ for $x \in [0,\infty)$. The proof of convergence follows in the same way as the proof of Szasz. This type of method plays an important role in the problem of approximation theory.

Much of the developments of approximation of functions by the sequence of operators involving special functions was performed by many mathematicians ([1],[3],[5],[9],[11],[13],[14],[15],[17],[19],[20]).

The construction of our operators is based on the following relation

$$A_1(h(t))\,A_2(xh(t)) = \sum_{k=0}^{\infty}\pi_k(x)\,t^k, \tag{4}$$

where A_1, A_2 and h are analytic functions on the following set

$$\{t : |t| < R,\ R > 1\}$$

such that

$$A_1(t) = \sum_{k=0}^{\infty}a_{1,k}t^k\ \ (a_{1,0}\neq 0),\ A_2(t) = \sum_{k=0}^{\infty}a_{2,k}t^k\ \ (a_{2,k}\neq 0),\ h(t) = \sum_{k=1}^{\infty}h_kt^k\ \ (h_1\neq 0). \tag{5}$$

If polynomials π_k satisfy the relation (4), then it is called generalized Brenke polynomials [16].

We are mainly concerned with the study of Varma et al. [10]. Therefore, for ν_1, $\nu_2 \geq 0$ we define the sequence of operators $\left\{\mathcal{L}_n^{(\nu_1,\nu_2)}\right\}_{n\geq 1}$ as follows

$$\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) = \frac{1}{A_1(h(1))\,A_2(nxh(1))}\sum_{k=0}^{\infty}\pi_k(nx)\,f\left(\frac{k+\nu_1}{n+\nu_2}\right), \tag{6}$$

where π_k polynomials are defined by (4), the function h and the polynomials π_k have the following properties $h'(1) = 1$ and $\pi_k(x) \geq 0$. In the all section of our study, we assume that

$$\begin{array}{ll} (i) & A_1,\ A_2 : \mathbb{R} \longrightarrow (0,\infty), \\ (ii) & (4)\ \text{and}\ (5)\ \text{converge for}\ |t| < R\ \ (R > 1). \end{array} \tag{7}$$

In this paper we are interested in defining the sequence of Stancu type operators including generalized Brenke polynomials. In the second section, we use the theorem of Korovkin on the sequence of positive linear operators and answer the problem of the order of this convergence estimate by known tools in approximation theory. Furthermore, we apply some of the results obtained in this work to Miller-Lee polynomials and Gould-Hopper polynomials.

2. Approximation to functions using $\mathcal{L}_n^{(\nu_1,\nu_2)}$ operators

Let us denote set of functions of form

$$\frac{f(x)}{1+x^2}\ \text{is convergent as}\ x \to \infty$$

by E.

The following lemmas give rise to important observations.

Lemma 2.1. *If π_k are polynomials satisfying the relation* (4), *then*

$$\begin{aligned}\sum_{k=0}^{\infty}\pi_k(nx) &= A_1(h(1))A_2(nxh(1)),\\ \sum_{k=0}^{\infty}k\pi_k(nx) &= A_1^{'}(h(1))A_2(nxh(1))+nxA_1(h(1))A_2^{'}(nxh(1)),\\ \sum_{k=0}^{\infty}k^2\pi_k(nx) &= \left[\left(h^{''}(1)+1\right)A_1^{'}(h(1))+A_1^{''}(h(1))\right]A_2(nxh(1))\\ &\quad+\left[2A_1^{'}(h(1))+\left(h^{''}(1)+1\right)A_1(h(1))\right]A_2^{'}(nxh(1))\,nx+A_1(h(1))A_2^{''}(nxh(1))(nx)^2.\end{aligned}$$

As a direct consequence of Lemma 2.1, we have the following:

Lemma 2.2. *For $n\geq 1$, one has the following identities*

$$\begin{aligned}\mathcal{L}_n^{(\nu_1,\nu_2)}(1;x) &= 1,\\ \mathcal{L}_n^{(\nu_1,\nu_2)}(s;x) &= \frac{A_2^{'}(nxh(1))\,n}{A_2(nxh(1))(n+\nu_2)}x+\left(\frac{A_1^{'}(h(1))}{A_1(h(1))}+\nu_1\right)\frac{1}{n+\nu_2},\\ \mathcal{L}_n^{(\nu_1,\nu_2)}\left(s^2;x\right) &= \frac{A_2^{''}(nxh(1))\,n^2}{A_2(nxh(1))(n+\nu_2)^2}x^2+\frac{\left[\left(1+2\nu_1+h^{''}(1)\right)A_1(h(1))+2A_1^{'}(h(1))\right]A_2^{'}(nxh(1))\,n}{A_1(h(1))A_2(nxh(1))(n+\nu_2)^2}x\\ &\quad+\frac{\nu_1^2A_1(h(1))+\left(1+2\nu_1+h^{''}(1)\right)A_1^{'}(h(1))+A_1^{''}(h(1))}{A_1(h(1))(n+\nu_2)^2}.\end{aligned}$$

Proof. *The validity of these identities follows from the definition of operators $\mathcal{L}_n^{(\nu_1,\nu_2)}$ and Lemma 2.1. Together,* (4) *and* (6) *imply*

$$\mathcal{L}_n^{(\nu_1,\nu_2)}(1;x) = \frac{1}{A_1(h(1))A_2(nxh(1))}\sum_{k=0}^{\infty}\pi_k(nx)=1.$$

Comparing (6) *and Lemma 2.1, we see*

$$\begin{aligned}\mathcal{L}_n^{(\nu_1,\nu_2)}(s;x) &= \frac{1}{A_1(h(1))A_2(nxh(1))}\sum_{k=0}^{\infty}\pi_k(nx)\frac{k+\nu_1}{n+\nu_2}\\ &= \frac{1}{A_1(h(1))A_2(nxh(1))(n+\nu_2)}\sum_{k=0}^{\infty}\pi_k(nx)k+\frac{\nu_1}{A_1(h(1))A_2(nxh(1))(n+\nu_2)}\sum_{k=0}^{\infty}\pi_k(nx)\\ &= \frac{A_2^{'}(nxh(1))\,n}{A_2(nxh(1))(n+\nu_2)}x+\left(\frac{A_1^{'}(h(1))}{A_1(h(1))}+\nu_1\right)\frac{1}{n+\nu_2}.\end{aligned}$$

The following remaining relation

$$
\begin{aligned}
\mathcal{L}_n^{(\nu_1,\nu_2)}\left(s^2;x\right) &= \frac{1}{A_1\left(h\left(1\right)\right)A_2\left(nxh\left(1\right)\right)}\sum_{k=0}^{\infty}\pi_k\left(nx\right)\left(\frac{k+\nu_1}{n+\nu_2}\right)^2 \\
&= \frac{1}{A_1\left(h\left(1\right)\right)A_2\left(nxh\left(1\right)\right)\left(n+\nu_2\right)^2}\sum_{k=0}^{\infty}\pi_k\left(nx\right)\left(k^2+2k\nu_1+\nu_1^2\right) \\
&= \frac{A_2''\left(nxh\left(1\right)\right)n^2}{A_2\left(nxh\left(1\right)\right)\left(n+\nu_2\right)^2}x^2+\frac{\left[\left(1+2\nu_1+h''\left(1\right)\right)A_1\left(h\left(1\right)\right)+2A_1'\left(h\left(1\right)\right)\right]A_2'\left(nxh\left(1\right)\right)n}{A_1\left(h\left(1\right)\right)A_2\left(nxh\left(1\right)\right)\left(n+\nu_2\right)^2}x \\
&\quad+\frac{\nu_1^2A_1\left(h\left(1\right)\right)+\left(1+2\nu_1+h''\left(1\right)\right)A_1'\left(h\left(1\right)\right)+A_1''\left(h\left(1\right)\right)}{A_1\left(h\left(1\right)\right)\left(n+\nu_2\right)^2}
\end{aligned}
$$

can be obtained from (6) *and Lemma 2.1.* □

Lemma 2.3. *Let $\mathcal{L}_n^{(\nu_1,\nu_2)}$ be operators defined by* (6). *Then it follows that*

$$
\begin{aligned}
\mathcal{L}_n^{(\nu_1,\nu_2)}\left(s-x;x\right) &= \left(\frac{A_2'\left(nxh\left(1\right)\right)n}{A_2\left(nxh\left(1\right)\right)\left(n+\nu_2\right)}-1\right)x+\left(\frac{A_1'\left(h\left(1\right)\right)}{A_1\left(h\left(1\right)\right)}+\nu_1\right)\frac{1}{n+\nu_2}, \\
\mathcal{L}_n^{(\nu_1,\nu_2)}\left(\left(s-x\right)^2;x\right) &= \left[\frac{A_2''\left(nxh\left(1\right)\right)n^2}{A_2\left(nxh\left(1\right)\right)\left(n+\nu_2\right)^2}-\frac{2A_2'\left(nxh\left(1\right)\right)n}{A_2\left(nxh\left(1\right)\right)\left(n+\nu_2\right)}+1\right]x^2 \\
&\quad+\left\{\frac{\left(1+2\nu_1+h''\left(1\right)\right)A_2'\left(nxh\left(1\right)\right)n}{A_2\left(nxh\left(1\right)\right)\left(n+\nu_2\right)^2}\right. \\
&\quad\left.+\frac{2A_1'\left(h\left(1\right)\right)A_2'\left(nxh\left(1\right)\right)}{A_1\left(h\left(1\right)\right)A_2\left(nxh\left(1\right)\right)\left(n+\nu_2\right)^2}n-\left(\frac{A_1'\left(h\left(1\right)\right)}{A_1\left(h\left(1\right)\right)}+\nu_1\right)\frac{2}{n+\nu_2}\right\}x \\
&\quad+\frac{\nu_1^2}{\left(n+\nu_2\right)^2}+\frac{\left(1+2\nu_1+h''\left(1\right)\right)A_1'\left(h\left(1\right)\right)+A_1''\left(h\left(1\right)\right)}{A_1\left(h\left(1\right)\right)\left(n+\nu_2\right)^2}.
\end{aligned}
$$

Proof. According to the rule of linearity of $\mathcal{L}_n^{(\nu_1,\nu_2)}$ operators and applying Lemma 2.2, this establishes the result. □

Now we define

$$
\begin{aligned}
\Delta_{1,n}^{(\nu_1,\nu_2)}\left(x\right) &:= \mathcal{L}_n^{(\nu_1,\nu_2)}\left(s-x;x\right), \\
\Delta_{2,n}^{(\nu_1,\nu_2)}\left(x\right) &:= \mathcal{L}_n^{(\nu_1,\nu_2)}\left(\left(s-x\right)^2;x\right).
\end{aligned}
$$

For $i=0,1,2$, let us determine under what conditions the sequence of operators

$$\mathcal{L}_n^{(\nu_1,\nu_2)}\left(s^i;x\right)$$

will approach x^i respectively.

Theorem 2.4. *Let $f:[0,\infty)\to\mathbb{R}$ be a continuous function belonging to class E. Set*

$$\lim_{y\to\infty}\frac{A_2'\left(y\right)}{A_2\left(y\right)}=1 \text{ and } \lim_{y\to\infty}\frac{A_2''\left(y\right)}{A_2\left(y\right)}=1. \tag{8}$$

Then,

$$\mathcal{L}_n^{(\nu_1,\nu_2)}\left(f;x\right)\to f\left(x\right)$$

uniformly as $n\to\infty$ on each compact subset of $[0,\infty)$.

Proof. Under the assumptions of the (8) and by using Lemma 2.2, for $i = 0, 1, 2$ we obtain the following

$$\mathcal{L}_n^{(\nu_1,\nu_2)}\left(s^i; x\right) \to x^i$$

which converge uniformly in each compact subset of $[0, \infty)$. The universal Korovkin-type property [2] enables us to obtain the assertion of theorem. □

Now define $\tilde{C}[0,\infty)$ and $C_B[0,\infty)$ to be the set of all uniformly continuous functions and to be the set of all bounded and continuous functions on $[0,\infty)$, respectively.

We are now in a position to obtain the order of approximation for the functions which belongs to the space $\tilde{C}[0,\infty) \cap E$.

Theorem 2.5. *Let f be a function belonging to the class $f \in \tilde{C}[0,\infty) \cap E$, then*

$$\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| \le 2\omega\left(f;\delta_n(x)\right),$$

where $\delta_n(x) := \sqrt{\Delta_{2,n}^{(\nu_1,\nu_2)}(x)}$ and $\omega(f;.)$ is modulus of continuity [7] of the function f.

Proof. It follows from Lemma 2.2 and monotonicity property of operators $\mathcal{L}_n^{(\nu_1,\nu_2)}$ that

$$\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| \le \mathcal{L}_n^{(\nu_1,\nu_2)}\left(\left|f(s) - f(x)\right|; x\right).$$

With the aid of the property of $\omega(f;.)$ we can write

$$\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| \le \omega(f;\delta)\left(1 + \frac{1}{\delta}\mathcal{L}_n^{(\nu_1,\nu_2)}(|s-x|;x)\right).$$

After using Cauchy-Schwarz inequality, the above inequality leads to

$$\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| \le \omega(f;\delta)\left(1 + \frac{1}{\delta}\sqrt{\Delta_{2,n}^{(\nu_1,\nu_2)}(x)}\right).$$

With $\delta := \delta_n(x) = \sqrt{\Delta_{2,n}^{(\nu_1,\nu_2)}(x)}$, the proof is completed. □

Definition 2.6. *For α and M with $0 < \alpha \le 1$, $M > 0$, a function f is said to be Lipschitz property of order α if it satisfies*

$$\left|f(t_1) - f(t_2)\right| \le M\left|t_1 - t_2\right|^{\alpha}, \qquad t_1, t_2 \in [0,\infty). \tag{9}$$

Theorem 2.7. *Let f be a function satisfying the condition (9). Then for $x \ge 0$*

$$\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| \le M\delta_n^{\alpha}(x),$$

where $\delta_n(x)$ is defined as Theorem 2.5.

Proof. Since the $\mathcal{L}_n^{(\nu_1,\nu_2)}$ are monotone, then we have

$$\begin{aligned} \left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| &= \left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f(s) - f(x);x)\right| \\ &\le \mathcal{L}_n^{(\nu_1,\nu_2)}\left(\left|f(s) - f(x)\right|;x\right) \\ &\le M\mathcal{L}_n^{(\nu_1,\nu_2)}\left(|s-x|^{\alpha};x\right). \end{aligned} \tag{10}$$

On the other hand the following inequality follows from the Hölder inequality

$$\mathcal{L}_n^{(\nu_1,\nu_2)}\left(|s-x|^{\alpha};x\right) \le \left[\mathcal{L}_n^{(\nu_1,\nu_2)}(1;x)\right]^{\frac{2-\alpha}{2}}\left[\Delta_{2,n}^{(\nu_1,\nu_2)}(x)\right]^{\frac{\alpha}{2}}.$$

From this and (10) we obtain

$$\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x)-f(x)\right|\leq M\left[\Delta_{2,n}^{(\nu_1,\nu_2)}(x)\right]^{\alpha/2}.$$

Hence the assertion of theorem holds. □

A norm on a linear space $C_B[0,\infty)$ is defined by

$$\left\|f\right\|_{C_B[0,\infty)}=\sup_{x\in[0,\infty)}\left|f(x)\right|.$$

Furthermore, the following set of functions

$$C_B^2[0,\infty)=\left\{\psi\in C_B[0,\infty):\psi^{'},\ \psi^{''}\in C_B[0,\infty)\right\}$$

is a normed space with

$$\left\|\psi\right\|_{C_B^2[0,\infty)}=\left\|\psi\right\|_{C_B[0,\infty)}+\left\|\psi^{'}\right\|_{C_B[0,\infty)}+\left\|\psi^{''}\right\|_{C_B[0,\infty)}.$$

An estimate of the convergence rate is obtained in the following theorem by using Peetre's K -functional [7] which is relevant in approximation theory.

Theorem 2.8. *Suppose $f\in C_B[0,\infty)$ and $x\in[0,\infty)$. Then*

$$\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x)-f(x)\right|\leq 2\mathcal{K}(f;\lambda_n(x)),$$

where

$$\lambda_n(x)=\frac{1}{2}\left[\Delta_{1,n}^{(\nu_1,\nu_2)}(x)+\Delta_{2,n}^{(\nu_1,\nu_2)}(x)\right]$$

and $\mathcal{K}(f;.)$ is the Peetre's K -functional of the function f.

Proof. Expand $\psi\in C_B^2[0,\infty)$ about x using Taylor's theorem we get

$$\psi(s)=\psi(x)+(s-x)\psi^{'}(x)+\frac{\psi^{''}(\eta)}{2}(s-x)^2,\quad \eta\in(x,s).$$

This yields

$$\mathcal{L}_n^{(\nu_1,\nu_2)}(\psi;x)-\psi(x)=\psi^{'}(x)\Delta_{1,n}^{(\nu_1,\nu_2)}(x)+\frac{\psi^{''}(\eta)}{2}\Delta_{2,n}^{(\nu_1,\nu_2)}(x).$$

From this it follows easily that

$$\begin{aligned}\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(\psi;x)-\psi(x)\right| &\leq \left|\psi^{'}(x)\right|\Delta_{1,n}^{(\nu_1,\nu_2)}(x)+\frac{\left|\psi^{''}(\eta)\right|}{2}\Delta_{2,n}^{(\nu_1,\nu_2)}(x)\\ &\leq \left[\Delta_{1,n}^{(\nu_1,\nu_2)}(x)+\Delta_{2,n}^{(\nu_1,\nu_2)}(x)\right]\left\|\psi\right\|_{C_B^2[0,\infty)}. \qquad (11)\end{aligned}$$

With Lemma 2.2 and expression (11) the estimation can be written as

$$\begin{aligned}\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x)-f(x)\right| &\leq \left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f-\psi;x)\right|+\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(\psi;x)-\psi(x)\right|+\left|f(x)-\psi(x)\right|\\ &\leq 2\left\|f-\psi\right\|_{C_B[0,\infty)}+\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(\psi;x)-\psi(x)\right|\\ &\leq 2\left(\left\|f-\psi\right\|_{C_B[0,\infty)}+\lambda_n(x)\left\|\psi\right\|_{C_B^2[0,\infty)}\right).\end{aligned}$$

Taking the infimum over all $\psi \in C_B^2[0,\infty)$, the last inequality together with the definition of $\mathcal{K}(f;.)$ implies the following desired result

$$\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| \leq 2\mathcal{K}(f;\lambda_n(x)).$$

□

Theorem 2.9. *Suppose $f \in C_B[0,\infty)$. Then*

$$\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| \leq C\omega_2\left(f;\sqrt{\mu_n(x)}\right) + \omega\left(f;\Delta_{1,n}^{(\nu_1,\nu_2)}(x)\right),$$

where C is a positive constant and

$$\mu_n(x) = \frac{1}{8}\left\{\Delta_{2,n}^{(\nu_1,\nu_2)}(x) + \left[\Delta_{1,n}^{(\nu_1,\nu_2)}(x)\right]^2\right\},$$

and $\omega_2(f;.)$ is the second order modulus of smoothness [7] of function f.

Proof. Firstly let us consider the operators $\mathcal{F}_n^{(\nu_1,\nu_2)}$ given by

$$\mathcal{F}_n^{(\nu_1,\nu_2)}(f;x) = \mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f\left(\mathcal{L}_n^{(\nu_1,\nu_2)}(s;x)\right) + f(x).$$

Then one obtains from Lemma 2.2

$$\mathcal{F}_n^{(\nu_1,\nu_2)}(s-x;x) = 0. \tag{12}$$

Moreover, for $\psi \in C_B^2[0,\infty)$ the following equality can be obtained by the Taylor formula

$$\psi(s) = \psi(x) + (s-x)\psi'(x) + \int_x^s (s-u)\psi''(u)\,du.$$

This equation, together with (12), leads immediately to

$$\begin{aligned}
\left|\mathcal{F}_n^{(\nu_1,\nu_2)}(\psi;x) - \psi(x)\right| &= \left|\mathcal{F}_n^{(\nu_1,\nu_2)}\left(\int_x^s (s-u)\psi''(u)\,du;x\right)\right| \\
&\leq \left|\mathcal{L}_n^{(\nu_1,\nu_2)}\left(\int_x^s (s-u)\psi''(u)\,du;x\right)\right| + \left|\int_x^{\mathcal{L}_n^{(\nu_1,\nu_2)}(s;x)} \left(\mathcal{L}_n^{(\nu_1,\nu_2)}(s;x) - u\right)\psi''(u)\,du\right| \\
&\leq \frac{1}{2}\left\{\Delta_{2,n}^{(\nu_1,\nu_2)}(x) + \left[\Delta_{1,n}^{(\nu_1,\nu_2)}(x)\right]^2\right\}\left\|\psi''\right\|_{C_B[0,\infty)} \\
&\leq 4\mu_n(x)\left\|\psi\right\|_{C_B^2[0,\infty)}.
\end{aligned} \tag{13}$$

Combining the definition of $\mathcal{F}_n^{(\nu_1,\nu_2)}$ operator, Lemma 2.2 and (13), we obtain the estimate

$$\begin{aligned}
\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| &\leq \left|\mathcal{F}_n^{(\nu_1,\nu_2)}(f-\psi;x) - (f-\psi)(x)\right| \\
&\quad + \left|\mathcal{F}_n^{(\nu_1,\nu_2)}(\psi;x) - \psi(x)\right| + \left|f\left(\mathcal{L}_n^{(\nu_1,\nu_2)}(s;x)\right) - f(x)\right| \\
&\leq 4\left\|f-\psi\right\|_{C_B[0,\infty)} + 4\mu_n(x)\left\|\psi\right\|_{C_B^2[0,\infty)} + \omega\left(f;\Delta_{1,n}^{(\nu_1,\nu_2)}(x)\right).
\end{aligned}$$

If we take into account the relation between $\mathcal{K}(f;.)$ and $\omega_2(f;.)$, we have

$$\begin{aligned}
\left|\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) - f(x)\right| &\leq 4\mathcal{K}(f;\mu_n(x)) + \omega\left(f;\Delta_{1,n}^{(\nu_1,\nu_2)}(x)\right) \\
&\leq C\omega_2\left(f;\sqrt{\mu_n(x)}\right) + \omega\left(f;\Delta_{1,n}^{(\nu_1,\nu_2)}(x)\right).
\end{aligned}$$

That is the assertion. □

3. Examples

In this section, we will clarify our analysis. To illustrate our situation, we now consider two examples.

Example 3.1. *The function*

$$e^{bt^{d+1}} \exp(xt)$$

is the generating function of the Gould-Hopper polynomials [8], i.e., the expansion

$$e^{bt^{d+1}} \exp(xt) = \sum_{k=0}^{\infty} g_k^{d+1}(x,b) \frac{t^k}{k!} \tag{14}$$

holds. The general expression for these polynomials is given as

$$g_k^{d+1}(x,b) = \sum_{s=0}^{\left[\frac{k}{d+1}\right]} \frac{k!}{s!\,(k-(d+1)s)!} b^s x^{k-(d+1)s}.$$

The polynomials g_k^{d+1} *defined by* (14) *are the generalized Brenke polynomials with*

$$A_1(t) = e^{bt^{d+1}},\ A_2(t) = e^t \ \text{ and } \ h(t) = t.$$

Under the assumption $b \geq 0$*, these polynomials satisfy the conditions of Theorem 2.4 and assumptions given in introduction part of this work. Hence, the explicit form of* $\mathcal{L}_n^{(\nu_1,\nu_2)}$ *operators is given by*

$$\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) = e^{-nx-b} \sum_{k=0}^{\infty} \frac{g_k^{d+1}(nx,b)}{k!} f\left(\frac{k+\nu_1}{n+\nu_2}\right),$$

where $x \in [0,\infty)$.

Example 3.2. *The function*

$$\frac{1}{(1-t)^{m+1}} \exp(xt)$$

is the generating function of the Miller-Lee polynomials [6], i.e., the expansion

$$\frac{1}{(1-t)^{m+1}} \exp(xt) = \sum_{k=0}^{\infty} G_k^{(m)}(x)\, t^k \tag{15}$$

holds for $|t| < 1$*. The general expression for these polynomials is given as*

$$G_k^{(m)}(x) = \sum_{r=0}^{k} \frac{(m+1)_r}{r!\,(k-r)!} x^{k-r},$$

where $(.)_r$ *is the Pochhammer's symbol. The polynomials* $G_k^{(m)}$ *defined by* (15) *are the generalized Brenke polynomials with*

$$A_1(t) = \frac{1}{(1-t)^{m+1}},\ A_2(t) = e^t \ \text{ and } \ h(t) = t.$$

For $t \to \frac{t}{2}$ *and* $x \to 2x$, *equation* (15) *takes the form*

$$\frac{1}{\left(1-\frac{t}{2}\right)^{m+1}} \exp(xt) = \sum_{k=0}^{\infty} \frac{G_k^{(m)}(2x)}{2^k} t^k, \quad |t| < 2.$$

Hence, the explicit form of $\mathcal{L}_n^{(\nu_1,\nu_2)}$ *operators is given by*

$$\mathcal{L}_n^{(\nu_1,\nu_2)}(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{G_k^{(m)}(2nx)}{2^{m+k+1}} f\left(\frac{k+\nu_1}{n+\nu_2}\right),$$

where $m > -1$ *and* $x \in [0,\infty)$.

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