Filomat 36:7 (2022), 2427–2438
https://doi.org/10.2298/FIL2207427C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Strong Whitney Convergence on Bornologies

Tarun Kumar Chauhan[a], Varun Jindal[a]

[a] *Department of Mathematics, Malaviya National Institute of Technology Jaipur, Jaipur-302017, Rajasthan, India*

Abstract. The strong Whitney convergence on bornology introduced by Caserta in [9] is a generalization of the strong uniform convergence on bornology introduced by Beer-Levi in [5]. This paper aims to study some important topological properties of the space of all real valued continuous functions on a metric space endowed with the topologies of Whitney and strong Whitney convergence on bornology. More precisely, we investigate metrizability, various countability properties, countable tightness, and Fréchet property of these spaces. In the process, we also present a new characterization for a bornology to be shielded from closed sets.

1. Introduction

Let $C(X,Y)$ denote the set of all continuous functions from a metric space (X,d) to a metric space (Y,ρ). For $Y = \mathbb{R}$ with the usual metric, the space $C(X,\mathbb{R})$ is simply denoted by $C(X)$. A family $\mathcal{B}$ of nonempty subsets of X is called a *bornology* on X if $\mathcal{B}$ forms an ideal and covers X (see [21]). A subfamily $\mathcal{B}_0$ of $\mathcal{B}$ satisfying that for every $B \in \mathcal{B}$ there is an element $B' \in \mathcal{B}_0$ such that $B \subseteq B'$ is called *base* for the bornology $\mathcal{B}$. If every member of $\mathcal{B}_0$ is closed (respectively, compact) in (X,d), then $\mathcal{B}$ is said to have a *closed* (respectively, *compact*) *base*.

The smallest (respectively, largest) bornology on X is the collection $\mathcal{F}$ of all finite (respectively, $\mathcal{P}_0(X)$ of all nonempty) subsets of X. Another important bornology on X is $\mathcal{K}$, the family of all nonempty *relatively compact* subsets (that is, subsets with compact closure) of X.

In the literature, several topologies have been defined and studied on the set $C(X,Y)$. Some of these topologies, such as the topology of pointwise convergence τ_p, the topology of uniform convergence τ^u, and the topology of uniform convergence on compacta τ_k, are extensively studied (see [14, 29]).

Another well-known topology on $C(X,Y)$ is the Whitney topology τ^w, introduced by H. Whitney in [31]. This topology is also sometimes called the fine topology or m-topology and is finer than the topology of uniform convergence τ^u. It plays an important role in the study of rings of continuous functions (see [16, 19]), approximation theory [1], and differential topology [20]. The Whitney topology has been studied in [13, 17, 23, 25, 27]. For more details on Whitney topology, we refer readers to the research monograph [28].

For any two metric spaces (X,d), (Y,ρ) and a bornology $\mathcal{B}$ on X, the topology $\tau^w_{\mathcal{B}}$ of Whitney convergence on $\mathcal{B}$ is a generalization of the Whitney topology τ^w on $C(X,Y)$. The topology $\tau^w_{\mathcal{B}}$ reduces to τ^w when $\mathcal{B} = \mathcal{P}_0(X)$. In [9], Caserta introduced a stronger version of the topology $\tau^w_{\mathcal{B}}$ known as the topology of

2020 *Mathematics Subject Classification*. Primary 54C35; Secondary 54A10, 54C05, 54C30, 54D70
Keywords. Bornology, continuous real functions, countability properties, Fréchet property, shield, strong Whitney convergence
Received: 03 December 2021; Revised: 15 February 2022; Accepted: 18 February 2022
Communicated by Ljubiša D.R. Kočinac
Email addresses: rajputtarun.chauhan@gmail.com (Tarun Kumar Chauhan), vjindal.maths@mnit.ac.in (Varun Jindal)

strong Whitney convergence on $\mathcal{B}$, which is denoted by $\tau^{sw}_{\mathcal{B}}$. The topology $\tau^{sw}_{\mathcal{B}}$ is a stronger form of the topology $\tau^{w}_{\mathcal{B}}$ in the same vein as the topology $\tau^{s}_{\mathcal{B}}$ of strong uniform convergence on $\mathcal{B}$ is of the classical topology of uniform convergence on $\mathcal{B}$, denoted by $\tau_{\mathcal{B}}$. The topology $\tau^{s}_{\mathcal{B}}$ was introduced by Beer and Levi in [5]. However, for $\mathcal{B} = \mathcal{F}$, a similar topology was considered by Bouleau in [7, 8]. The topological space $(C(X), \tau^{s}_{\mathcal{B}})$ is well studied in [3, 5, 10, 11, 22, 24].

The properties of the spaces $(C(X), \tau^{w}_{\mathcal{B}})$ and $(C(X), \tau^{sw}_{\mathcal{B}})$ are yet to be explored in detail. Recently, $\tau^{w}_{\mathcal{B}}$-convergence and $\tau^{sw}_{\mathcal{B}}$-convergence have been studied in [12]. The main objective of this paper is to characterize the metrizability and various countability properties of $C(X)$ equipped with the topologies $\tau^{w}_{\mathcal{B}}$ and $\tau^{sw}_{\mathcal{B}}$. It comes out that the first countability, second countability and metrizability all are equivalent for these function spaces. Note that these three properties of $(C(X), \tau^{sw}_{\mathcal{B}})$ are also considered in [9] (Corollary 2 and Corollary 3). Our results (Theorem 3.2 and Theorem 3.11) show that these corollaries are not true in the form given in [9].

The notion of a bornology being shielded from closed sets is pivotal to studying the topology of strong Whitney (strong uniform) convergence. In [6], many applications of the concept of shielded from closed sets are given. In this paper, we present a new characterization for a bornology to be shielded from closed sets in terms of a covering property associated with the bornology which is useful to relate the tightness and Fréchet property of spaces $(C(X), \tau^{w}_{\mathcal{B}})$ and $(C(X), \tau^{sw}_{\mathcal{B}})$.

2. Preliminaries

All metric spaces are assumed to have at least two points. For any $x \in X$ and $\delta > 0$, $S_\delta(x)$ denotes the open ball with center x and radius δ. For any nonempty subset A of X, A^δ represents the *δ-enlargement* of A defined as $A^\delta = \cup_{x\in A} S_\delta(x) = \{x \in X : d(x, A) < \delta\}$. Note that $A^\delta = \overline{A}^{\delta}$, where $\overline{A}$ denotes the closure of A. We denote by f_0 the constant function zero on X, that is, $f_0 : X \to \mathbb{R}$ such that $f_0(X) = 0$ for all $x \in X$. For other terms and notations, we refer to [14, 29, 32].

Let $\mathcal{B}$ be a bornology on (X, d) and let (Y, ρ) be any other metric space. Then the classical *topology $\tau_{\mathcal{B}}$ of uniform convergence on $\mathcal{B}$* for the space $C(X, Y)$ is determined by the uniformity which has a base for its entourages all sets of the form

$$[B, \epsilon] = \{(f, g) : \rho(f(x), g(x)) < \epsilon \text{ for all } x \in B\} \quad (B \in \mathcal{B}, \epsilon > 0).$$

Note that the topology $\tau_{\mathcal{F}}$ is the topology of pointwise convergence τ_p; if $\mathcal{B} = \mathcal{P}_0(X)$ (respectively, $\mathcal{K}$), then the topology $\tau_{\mathcal{P}_0(X)}$ (respectively, $\tau_{\mathcal{K}}$) is the topology of uniform convergence τ^u (respectively, the topology of uniform convergence on compact subsets τ_k).

Definition 2.1. ([5]) Let (X, d) and (Y, ρ) be metric spaces and let $\mathcal{B}$ be a bornology on X. Then the *topology $\tau^{s}_{\mathcal{B}}$ of strong uniform convergence on $\mathcal{B}$* is determined by a uniformity on $C(X, Y)$ having as a base all sets of the form

$$[B, \epsilon]^s = \{(f, g) : \exists\, \delta > 0\ \forall x \in B^\delta, \rho(f(x), g(x)) < \epsilon\} \ (B \in \mathcal{B}, \epsilon > 0).$$

In [6], the authors found a necessary and sufficient condition for topologies $\tau^{s}_{\mathcal{B}}$ and $\tau_{\mathcal{B}}$ to agree on $C(X, Y)$ using the notion of a shield. The concept of a shield was introduced by Beer et al. in [4] while studying bornological convergence. Recall that for a nonempty subset A of X, a superset A_1 of A is called a *shield* for A provided that for every closed subset C of X with $C \cap A_1 = \emptyset$, we have C cannot be near to A, that is, there exists a $\delta > 0$ such that $C \cap A^\delta = \emptyset$. Equivalently, a superset A_1 of A is a shield for A if every open neighborhood of A_1 contains A^δ for some $\delta > 0$. Since $A^\delta = \overline{A}^{\delta}$, a set A_1 is a shield for A if and only if A_1 is a shield for $\overline{A}$. Evidently, X is a shield for all $A \in \mathcal{P}_0(X)$.

A bornology $\mathcal{B}$ on X is called *shielded from closed sets* if $\mathcal{B}$ contains a shield for each of its members. It is known that every bornology with a compact base is shielded from closed sets, in particular, the bornologies $\mathcal{F}$ and $\mathcal{K}$ are shielded from closed sets.

Let $\mathcal{B}$ be a bornology on a metric space (X, d). The classical uniformity for the *topology* $\tau^w_{\mathcal{B}}$ *of Whitney convergence on* $\mathcal{B}$ for $C(X, Y)$ has as a base for its entourages all sets of the form

$$[B, \epsilon]^w = \{(f, g) : \rho(f(x), g(x)) < \epsilon(x) \text{ for all } x \in B\} \quad (B \in \mathcal{B}, \epsilon \in C^+(X)).$$

Here $C^+(X)$ represents the set of all positive real-valued continuous functions defined on X. Note that if $\mathcal{B} = \mathcal{P}_0(X)$, then the above uniformity generates the topology τ^w of Whitney convergence on $C(X, Y)$.

Definition 2.2. ([9]) Let (X, d) and (Y, ρ) be metric spaces and let $\mathcal{B}$ be a bornology on X. Then the *topology* $\tau^{sw}_{\mathcal{B}}$ *of strong Whitney convergence on* $\mathcal{B}$ is determined by a uniformity on $C(X, Y)$ having as a base all sets of the form

$$[B, \epsilon]^{sw} = \{(f, g) : \exists\, \delta > 0\ \forall x \in B^\delta, \rho(f(x), g(x)) < \epsilon(x)\}$$

with $B \in \mathcal{B}, \epsilon \in C^+(X)$.

Remark 2.3. For an arbitrary bornology $\mathcal{B}$ on X, we have $[B, \epsilon]^s = [\overline{B}, \epsilon]^s$ and $[B, \epsilon]^{sw} = [\overline{B}, \epsilon]^{sw}$ for every $B \in \mathcal{B}$. Moreover, $\overline{\mathcal{B}} = \{\overline{B} : B \in \mathcal{B}\}$ is a bornology on X. Hence there is no loss of generality, if in the definition of $\tau^s_{\mathcal{B}}$ as well as $\tau^{sw}_{\mathcal{B}}$, we assume $\mathcal{B}$ has a closed base.

In [9], the $\tau^{sw}_{\mathcal{F}}$-convergence is shown to be equivalent to Arzelà-Whitney convergence on compact sets. The Arzelà-Whitney convergence was introduced in [15].

For any bornology $\mathcal{B}$, we have $\mathcal{F} \subseteq \mathcal{B}$. So each of the above defined topologies is finer than the topology of pointwise convergence, and hence is Hausdorff. Since all of them are induced by a uniformity, they are Tychonoff.

In general, on $C(X, Y)$ the above defined topologies are related as follows:

$$\tau_{\mathcal{B}} \subseteq \tau^s_{\mathcal{B}} \subseteq \tau^{sw}_{\mathcal{B}} \text{ and } \tau_{\mathcal{B}} \subseteq \tau^w_{\mathcal{B}} \subseteq \tau^{sw}_{\mathcal{B}} \subseteq \tau^w.$$

The relationships between the above mentioned topologies are thoroughly studied in [12]. The following result proved in [12], we need in the sequel.

Theorem 2.4. ([12, Corollary 3.11] *Let* $\mathcal{B}$ *be a bornology on a metric space* (X, d) *with a closed base. Then the following conditions are equivalent:*

(a) for every metric space (Y, ρ), $\tau^{sw}_{\mathcal{B}} = \tau^s_{\mathcal{B}} = \tau^w_{\mathcal{B}} = \tau_{\mathcal{B}}$ *on* $C(X, Y)$*;*

(b) $\mathcal{B} \subseteq \mathcal{K}$.

Note that for a bornology $\mathcal{B}$ on a metric space (X, d) with a closed base, $\mathcal{B} \subseteq \mathcal{K}$ if and only if $\mathcal{B}$ has a compact base. We can easily observe that if $\mathcal{B}$ is a bornology on (X, d) with a compact base, then $\tau_{\mathcal{F}} \subseteq \tau^w_{\mathcal{B}} = \tau^s_{\mathcal{B}} = \tau^{sw}_{\mathcal{B}} \subseteq \tau^{sw}_{\mathcal{K}} = \tau_{\mathcal{K}}$ on $C(X, Y)$ for every metric space (Y, ρ).

3. Metrizability and Countability Properties

In this section, we characterize metrizability and various countability properties such as countable chain condition, separability, second countability, being a cosmic space, (hereditary) Lindelöf property and ω-narrowness of spaces $(C(X), \tau^{sw}_{\mathcal{B}})$ and $(C(X), \tau^w_{\mathcal{B}})$. Since these spaces are topological groups, it follows from the Birkhoff-Kakutani theorem that they are metrizable if and only if they are first countable.

We show that metrizability of $(C(X), \tau^{sw}_{\mathcal{B}})$ is also equivalent to some weaker properties. We first recall their definitions. A topological space X is called *pointwise countable type* if every point of X is contained in a compact set having a countable character. A subset A of a topological space X is said to have *countable character* in X if there exists a countable collection $\{U_n : n \in \mathbb{N}\}$ of open subsets of X such that each U_n contains A and for every open set U in X with $A \subseteq U$, there exists a U_n such that $U_n \subseteq U$. A topological space X is called a *q-space* if for each point $x \in X$, there exists a sequence $\{U_n : n \in \mathbb{N}\}$ of neighborhoods of x

such that if $x_n \in U_n$ for each n, then $\{x_n : n \in \mathbb{N}\}$ has a cluster point. It is known that a first countable space is of pointwise countable type and every pointwise countable type space is a q-space (see [30]).

In a number of subsequent results, we use the following lemma given in [28]. We include its statement here for the readers' convenience.

Lemma 3.1. (Lemma 1.1 on page 7, [28]) *If A is a C-embedded subset of a Tychonoff space X, then any continuous function $f : A \to (0,\infty)$ can be extended to a continuous function $F : X \to (0,\infty)$.*

Theorem 3.2. *Let (X,d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) $\mathcal{B}$ has a countable base consisting of compact sets;

(b) for every metric space (Y,ρ), $(C(X,Y),\tau^{sw}_{\mathcal{B}})$ is metrizable;

(c) for every metric space (Y,ρ), $(C(X,Y),\tau^{sw}_{\mathcal{B}})$ is first countable;

(d) $(C(X),\tau^{sw}_{\mathcal{B}})$ is metrizable;

(e) $(C(X),\tau^{sw}_{\mathcal{B}})$ is first countable;

(f) $(C(X),\tau^{sw}_{\mathcal{B}})$ is of pointwise countable type;

(g) $(C(X),\tau^{sw}_{\mathcal{B}})$ is a q-space.

Proof. $(a) \Rightarrow (b)$ Since $\mathcal{B}$ has a countable base consisting of compact sets, by Theorem 2.4, $\tau^{s}_{\mathcal{B}} = \tau^{sw}_{\mathcal{B}}$ on $C(X,Y)$. But for such a bornology, $(C(X,Y),\tau^{s}_{\mathcal{B}})$ is metrizable by Theorem 7.1 of [5].

The implications $(b) \Rightarrow (c) \Rightarrow (e)$ and $(b) \Rightarrow (d) \Rightarrow (e)$ are immediate.

$(e) \Rightarrow (a)$ Let $\mathcal{B}(f_0) = \{[B_n,\phi_n]^{sw}(f_0) : n \in \mathbb{N}\}$ be a countable base at f_0, where $B_n \in \mathcal{B}$ and $\phi_n \in C^+(X)$ for each $n \in \mathbb{N}$.

Suppose $B_0 \in \mathcal{B}$ is closed but not compact. So there exists a countably infinite subset $D = \{x_n : n \in \mathbb{N}\}$ of B_0 which is closed and discrete in X. By Lemma 3.1, the continuous function $\epsilon' : D \to \mathbb{R}$ such that $\epsilon'(x_n) = \frac{\phi_n(x_n)}{2}$ for all $x_n \in D$ can be extended to a function $\epsilon \in C^+(X)$. Thus for every $n \in \mathbb{N}$, we have $[B_n,\phi_n]^{sw}(f_0) \not\subseteq [B_0,\epsilon]^{sw}(f_0)$. We arrive at a contradiction. Hence $\mathcal{B}$ has a base consisting of compact sets. Consequently, by Theorem 2.4, $\tau^{s}_{\mathcal{B}} = \tau^{sw}_{\mathcal{B}}$ on $C(X)$, and by Theorem 7.1 of [5], $\mathcal{B}$ has a countable base.

The implications $(e) \Rightarrow (f) \Rightarrow (g)$ follow from the above discussion.

$(g) \Rightarrow (e)$ We can prove that $\mathcal{B}$ has a countable base in a manner similar to the proof of implication (vi)⇒(i) of Theorem 3.1 in [10]. Hence $(C(X),\tau^{s}_{\mathcal{B}})$ is metrizable. Consequently, every point of $C(X)$ is G_δ in $(C(X),\tau^{sw}_{\mathcal{B}})$. But by Lemma 3.2 in [18] a regular q-space in which singleton sets are G_δ, is first countable. □

Remark 3.3. We now give an example showing that Proposition 4 and Corollary 2 of [9] do not hold in general.

Example 3.4. Let $(X,d) = \mathbb{R}$ with the usual metric and $\mathcal{B} = \mathcal{K}$. Then $\tau^{sw}_{\mathcal{K}} = \tau_{\mathcal{K}}$ on $C(\mathbb{R})$. Hence $(C(\mathbb{R}),\tau^{sw}_{\mathcal{K}})$ is first countable but $\mathbb{R}$ is not compact.

Similarly, we can prove the following result for the topology $\tau^{w}_{\mathcal{B}}$.

Theorem 3.5. *Let (X,d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) $\mathcal{B}$ has a countable base consisting of compact sets;

(b) for every metric space (Y,ρ), $(C(X,Y),\tau^{w}_{\mathcal{B}})$ is metrizable;

(c) for every metric space (Y,ρ), $(C(X,Y),\tau^{w}_{\mathcal{B}})$ is first countable;

(d) $(C(X), \tau^w_{\mathcal{B}})$ *is metrizable;*

(e) $(C(X), \tau^w_{\mathcal{B}})$ *is first countable;*

(f) $(C(X), \tau^w_{\mathcal{B}})$ *is of pointwise countable type;*

(g) $(C(X), \tau^w_{\mathcal{B}})$ *is a q-space.*

A topological space X is said to have *countable chain condition* (ccc) if any family of mutually disjoint open subsets of X is atmost countable.

Theorem 3.6. *Let $\mathcal{B}$ be a bornology on a metric space (X, d) with a closed base. Then the following conditions are equivalent:*

(a) $(C(X), \tau^{sw}_{\mathcal{B}})$ *has countable chain condition;*

(b) $(C(X), \tau^w_{\mathcal{B}})$ *has countable chain condition;*

(c) $\mathcal{B} \subseteq \mathcal{K}$.

Proof. $(a) \Rightarrow (b)$ Since the topology $\tau^{sw}_{\mathcal{B}}$ is finer than $\tau^w_{\mathcal{B}}$, $(C(X), \tau^w_{\mathcal{B}})$ has countable chain condition.

$(b) \Rightarrow (c)$ Suppose there exists a non-compact closed member $B_0 \in \mathcal{B}$. So we can find a countably infinite subset $D = \{x_n : n \in \mathbb{N}\}$ of B_0 which is closed and discrete in X. If $\mathcal{P}(D)$ denotes the power set of D, then $|\mathcal{P}(D)| > \aleph_0$. For every $A \in \mathcal{P}(D)$, by Tietze's extension theorem define a function $f_A \in C(X)$ such that $f_A(x) = 0$ for all $x \in A$ and $f_A(x) = 1$ for all $x \in D \setminus A$. Also by Lemma 3.1, the continuous function $\epsilon' : D \to \mathbb{R}$ defined by $\epsilon'(x_n) = \frac{1}{4n}$ for all $x_n \in D$ can be extended to a function $\epsilon \in C^+(X)$. Then the collection $\mathcal{G} = \{[B_0, \epsilon]^w(f_A) : A \in \mathcal{P}(D)\}$ of basic open sets in $(C(X), \tau^w_{\mathcal{B}})$ is a pairwise disjoint family with $|\mathcal{G}| > \aleph_0$. We arrive at a contradiction.

$(c) \Rightarrow (a)$ Suppose $\mathcal{B} \subseteq \mathcal{K}$. By Theorem 2.4, $\tau^{sw}_{\mathcal{B}} = \tau_{\mathcal{B}}$ and $\tau_{\mathcal{B}}$ is coarser than $\tau_{\mathcal{K}}$ on $C(X)$. Since X is a metric space, $(C(X), \tau_{\mathcal{K}})$ has countable chain condition (see Exercise 2(a), page 68 in [29]). Hence $(C(X), \tau^{sw}_{\mathcal{B}})$ has countable chain condition. □

Theorem 3.7. *Let $\mathcal{B}$ be a bornology on a metric space (X, d) with a closed base. Then the following conditions are equivalent:*

(a) $(C(X), \tau^{sw}_{\mathcal{B}})$ *is separable;*

(b) $(C(X), \tau^w_{\mathcal{B}})$ *is separable;*

(c) $\mathcal{B} \subseteq \mathcal{K}$ *and X has a weaker separable metrizable topology.*

Proof. $(a) \Rightarrow (b)$ Since $\tau^w_{\mathcal{B}} \subseteq \tau^{sw}_{\mathcal{B}}$, $(C(X), \tau^w_{\mathcal{B}})$ is separable.

$(b) \Rightarrow (c)$ If $(C(X), \tau^w_{\mathcal{B}})$ is separable, then $(C(X), \tau^w_{\mathcal{B}})$ has countable chain condition. Consequently, by Theorem 3.6, $\mathcal{B} \subseteq \mathcal{K}$. Since $\tau_{\mathcal{F}}$ is coarser than $\tau^w_{\mathcal{B}}$, $(C(X), \tau_{\mathcal{F}})$ is also separable. Hence by Corollary 4.2.2 in [29], X has a weaker separable metrizable topology.

$(c) \Rightarrow (a)$ Since $\mathcal{B} \subseteq \mathcal{K}$, $\tau^{sw}_{\mathcal{B}}$ is weaker than $\tau^{sw}_{\mathcal{K}} = \tau_{\mathcal{K}}$ (by Theorem 2.4). But by Corollary 4.2.2 in [29], $(C(X), \tau_{\mathcal{K}})$ is separable. □

For a topological space X, a family $\mathcal{N}$ of nonempty subsets of X is called a *network* (respectively, *k-network*) provided that for every $x \in X$ (respectively, compact subset K of X) and every open set U containig x (respectively, K), there exists a member $N \in \mathcal{N}$ such that $x \in N \subseteq U$ (respectively, $K \subseteq N \subseteq U$). A topological space having a countable network is called a *cosmic* space. It is easy to see that every cosmic space is Lindelöf as well as separable, and a separable metric space has a countable k-network.

Theorem 3.8. *Let (X, d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) $(C(X), \tau^{sw}_{\mathcal{B}})$ *is a cosmic space;*

(b) $(C(X), \tau^{w}_{\mathcal{B}})$ *is a cosmic space;*

(c) $\mathcal{B} \subseteq \mathcal{K}$ *and X is separable.*

Proof. $(a) \Rightarrow (b)$ is immediate.

$(b) \Rightarrow (c)$ $(C(X), \tau^{w}_{\mathcal{B}})$ being a cosmic space implies that $(C(X), \tau^{w}_{\mathcal{B}})$ is separable and $(C(X), \tau_{\mathcal{F}})$ is cosmic. Consequently, by Theorem 3.7, $\mathcal{B} \subseteq \mathcal{K}$ and by Corollary 4.1.3 in [29] X has a countable network and therefore X is separable.

$(c) \Rightarrow (a)$ Since X is separable, X has a countable k-network. By Corollary 4.1.3 in [29], $(C(X), \tau_{\mathcal{K}})$ has a countable network. Since $\mathcal{B} \subseteq \mathcal{K}$, $\tau^{sw}_{\mathcal{B}} = \tau_{\mathcal{B}}$ and $\tau_{\mathcal{B}}$ is coarser than $\tau_{\mathcal{K}}$ on $C(X)$. Hence $(C(X), \tau^{sw}_{\mathcal{B}})$ has a countable network. □

Theorem 3.9. *Let* (X, d) *be a metric space and let* $\mathcal{B}$ *be a bornology with a closed base. Then the following conditions are equivalent:*

(a) $(C(X), \tau^{sw}_{\mathcal{B}})$ *is (hereditary) Lindelöf;*

(b) $(C(X), \tau^{w}_{\mathcal{B}})$ *is (hereditary) Lindelöf;*

(c) $\mathcal{B} \subseteq \mathcal{K}$ *and X is separable.*

Proof. $(a) \Rightarrow (b)$ is immediate.

$(b) \Rightarrow (c)$ Suppose $\mathcal{B} \nsubseteq \mathcal{K}$. Then there exists a closed member $B_0 \in \mathcal{B}$ which is not compact. So there is a countably infinite subset $D = \{x_n : n \in \mathbb{N}\}$ of B_0 which is closed and discrete in X. By Lemma 3.1, the continuous function $\epsilon' : D \to \mathbb{R}$ such that $\epsilon'(x_n) = \frac{1}{n}$ for all $x_n \in D$ can be extended to a function $\epsilon \in C^+(X)$. Then the collection $\mathcal{G} = \{[B_0, \epsilon]^w(f) : f \in C(X)\}$ of basic open sets in $(C(X), \tau^{w}_{\mathcal{B}})$ forms an open cover of $C(X)$. We claim that there is no countable subcover of $\mathcal{G}$. Let $\mathcal{G}_0 = \{[B_0, \epsilon]^w(f_n) : n \in \mathbb{N}\}$ be any countable subfamily of $\mathcal{G}$. Define a continuous function $h : D \to \mathbb{R}$ such that $h(x_n) = f_n(x_n) + \frac{1}{n}$ for all $x_n \in D$. By Tietze's extension theorem, there is a function $H \in C(X)$ such that $H|_D = h$. Clearly, no member of $\mathcal{G}_0$ contains H.

Since $\tau^{w}_{\mathcal{B}}$ is finer than $\tau_{\mathcal{F}}$ on $C(X)$, the space $(C(X), \tau_{\mathcal{F}})$ is Lindelöf. By Corollary 2 of [26], X is Lindelöf. Since X is a metric space, it is separable.

$(c) \Rightarrow (a)$ It follows from the Theorem 3.8 and the fact that a space having a countable network is (hereditary) Lindelöf (see, 3.12.7 (e) page 225 of [14]). □

To give a characterization of the second countability of these spaces, we need the following lemma.

Lemma 3.10. *If a bornology* $\mathcal{B}$ *on a metric space* (X, d) *has a countable base consisting of compact sets, then X is separable.*

Theorem 3.11. *Let* (X, d) *be a metric space and let* $\mathcal{B}$ *be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) $\mathcal{B}$ *has a countable base consisting of compact sets;*

(b) $(C(X), \tau^{sw}_{\mathcal{B}})$ *is second countable;*

(c) $(C(X), \tau^{w}_{\mathcal{B}})$ *is second countable.*

Proof. $(a) \Rightarrow (b)$ Since $\mathcal{B}$ has a countable base consisting of compact sets, by Theorems 3.2, $(C(X), \tau^{sw}_{\mathcal{B}})$ is metrizable. Also by Lemma 3.10 and Theorem 3.7, $(C(X), \tau^{sw}_{\mathcal{B}})$ is separable.

$(b) \Rightarrow (a)$. If $(C(X), \tau^{sw}_{\mathcal{B}})$ is second countable, then $(C(X), \tau^{sw}_{\mathcal{B}})$ is first countable. Hence by Theorem 3.2, $\mathcal{B}$ has a countable base consisting of compact sets.

Similarly, $(a) \Leftrightarrow (c)$ follows from Theorems 3.5 and 3.7. □

Example 3.12. For $X = \mathbb{R}$ with the usual metric, we have $(C(\mathbb{R}), \tau_{\mathcal{K}}^{sw})$ is second countable.

Remark 3.13. The above example together with Theorem 3.11 shows that the assumption of X being pseudocompact in Corollary 3 of [9] is redundant.

A topological group G (under addition) is said to be *ω-narrow* if for every neighborhood U of the identity element of G, there exists a countable subset S of G such that $U + S = \{u + s : u \in U \text{ and } s \in S\} = G$.

Theorem 3.14. *Let (X, d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) $(C(X), \tau_{\mathcal{B}}^{sw})$ *is ω-narrow;*

(b) $(C(X), \tau_{\mathcal{B}}^{w})$ *is ω-narrow;*

(c) $\mathcal{B} \subseteq \mathcal{K}$.

Proof. $(a) \Rightarrow (b)$ It follows as $\tau_{\mathcal{B}}^{sw}$ is finer than $\tau_{\mathcal{B}}^{w}$.

$(b) \Rightarrow (c)$ Suppose there is a member $B_0 \in \mathcal{B}$ such that B_0 is closed but not compact. So there exists a countably infinite subset $D = \{x_n : n \in \mathbb{N}\}$ of B_0 which is closed and discrete in X. Consider a basic open neighborhood $U = [B_0, \epsilon]^w(f_0)$ of f_0, the identity element of the topological group $(C(X), \tau_{\mathcal{B}}^{w})$, where $\epsilon \in C^+(X)$ such that $0 < \epsilon(x) < 1$ for all $x \in X$. Since $(C(X), \tau_{\mathcal{B}}^{w})$ is ω-narrow, there is a countable subset $S = \{f_n : n \in \mathbb{N}\}$ of $C(X)$ such that $U + S = \{h + f_n : h \in U \text{ and } n \in \mathbb{N}\} = C(X)$. Define a continuous function $g' : D \to \mathbb{R}$ such that $g'(x_n) = f_n(x_n)$ for all $n \in \mathbb{N}$. So by Tietze's extension theorem there is a function $g \in C(X)$ such that $g(x) = g'(x)$ for all $x \in D$. Since for every $h \in U$, $h(x) < \epsilon(x) < 1$ for all $x \in B_0$ and $g(x_n) = f_n(x_n)$ for every $n \in \mathbb{N}$, we have $|h(x_n) + f_n(x_n)| < 1 + |g(x_n)|$ for all $n \in \mathbb{N}$. Thus $1 + |g| \in C(X)$ but $1 + |g| \notin U + S$. Which gives a contradiction.

$(c) \Rightarrow (a)$ Suppose $\mathcal{B} \subseteq \mathcal{K}$. By Theorem 3.6, the space $(C(X), \tau_{\mathcal{B}}^{sw})$ has countable chain condition. Hence by Proposition 3.4.7 on page 163, [2], it is ω-narrow. □

In a similar manner, we can also prove the following result.

Theorem 3.15. *Let (X, d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) $(C(X), \tau_{\mathcal{B}}^{s})$ *is ω-narrow;*

(b) $(C(X), \tau_{\mathcal{B}})$ *is ω-narrow;*

(c) $\mathcal{B} \subseteq \mathcal{K}$.

We now give some examples to clarify the relations among various properties studied above for the space $(C(X), \tau_{\mathcal{B}}^{sw})$.

Example 3.16. For any bornology $\mathcal{B}$ on a compact metric space (X, d) with a closed base, the space $(C(X), \tau_{\mathcal{B}}^{sw})$ is separable, Lindelöf, cosmic space, ω-narrow, and has countable chain condition but need not be first countable. For example, $(C([0, 1]), \tau_{\mathcal{F}}^{sw})$ is not first countable.

Example 3.17. For $X = \mathbb{R}$ with the usual metric, the space $(C(\mathbb{R}), \tau_{\mathcal{F}}^{sw})$ is separable and Lindelöf but not first countable. However, $(C(\mathbb{R}), \tau_{\mathcal{K}}^{sw})$ is second countable.

Example 3.18. For $X = \mathbb{R}$ with the discrete metric, we have $(C(\mathbb{R}), \tau_{\mathcal{F}}^{sw}) = (C(\mathbb{R}), \tau_{\mathcal{K}}^{sw})$ and by Theorem 3.7, $(C(\mathbb{R}), \tau_{\mathcal{F}}^{sw})$ is separable. But it is neither Lindelöf nor second countable.

Example 3.19. Let X be a countable discrete metric space. Then $(C(X), \tau_{\mathcal{F}}^{sw}) = (C(X), \tau_{\mathcal{K}}^{sw})$ is second countable. So it is first countable, separable, and Lindelöf.

4. Countable Tightness and Fréchet Property

In this section, we characterize the countable tightness and Fréchet property of the spaces $(C(X), \tau^w_{\mathcal{B}})$ and $(C(X), \tau^{sw}_{\mathcal{B}})$. We first recall definitions of a Fréchet and countably tight space. A topological space X is called *Fréchet* if for every nonempty subset A of X and every $x \in \overline{A}$ there exists a sequence (x_n) in A that converges to x; and a topological space X is called *countably tight* if for every $x \in X$ and every subset C of X such that $x \in \overline{C}$, there exists a countable subset C_0 of C such that $x \in \overline{C_0}$. Clearly, every first countable space is Fréchet, and every Fréchet space is countably tight.

In order to study countable tightness of $(C(X), \tau^w_{\mathcal{B}})$ and $(C(X), \tau^{sw}_{\mathcal{B}})$, we need the following definitions.

Definitions 4.1. Let $\mathcal{A}$ be a family of nonempty subsets of a metric space (X, d). Then a cover $\mathcal{G}$ of X is said to be an *$\mathcal{A}$-cover*, if for every $A \in \mathcal{A}$ there exists $G \in \mathcal{G}$ such that $A \subseteq G$; and a cover $\mathcal{G}$ of X is called a *strong $\mathcal{A}$-cover* (denoted by $\mathcal{A}^s$-cover, see Definition 3.9 of [10]), if for every $A \in \mathcal{A}$ there exist $G \in \mathcal{G}$ and $\delta > 0$ such that $A^\delta \subseteq G$.

Note that every strong $\mathcal{A}$-cover is always an $\mathcal{A}$-cover. The next example shows that converse need not be true.

Example 4.2. Let $X = \mathbb{R}$ with the usual metric and let $\mathcal{B}$ be the bornology on $\mathbb{R}$ with base $\mathcal{B}_0 = \{F \cup \mathbb{N} : F \in \mathcal{F}\}$. Consider the family $\mathcal{G} = \{(\mathbb{R} \setminus A) \cup F : F \in \mathcal{F}\}$ of open sets in $\mathbb{R}$ where $A = \{n + \frac{1}{n+1} : n \in \mathbb{N}\}$. Clearly, $\mathcal{G}$ is an open $\mathcal{B}$-cover. But $\mathcal{G}$ is not an open $\mathcal{B}^s$-cover as $\mathbb{N} \in \mathcal{B}$ and for every $\delta > 0$ and every $F \in \mathcal{F}$, we have $\mathbb{N}^\delta \cap (A \setminus F) \neq \emptyset$.

Theorem 4.3. *For a bornology $\mathcal{B}$ on a metric space (X, d) with a closed base, the following conditions are equivalent:*

- *(a) $\mathcal{B}$ is shielded from closed sets;*
- *(b) every open $\mathcal{B}$-cover is an open $\mathcal{B}^s$-cover.*

Proof. $(a) \Rightarrow (b)$ Let $\mathcal{G}$ be an open $\mathcal{B}$-cover and consider $B \in \mathcal{B}$. Without loss of generality, suppose B is closed. Since $\mathcal{B}$ is shielded from closed sets, there is a member $B_1 \in \mathcal{B}$ which is a shield of B. Since $\mathcal{G}$ is an open $\mathcal{B}$-cover, there exists $G \in \mathcal{G}$ such that $B \subseteq B_1 \subseteq G$. Our claim is that $B^\delta \subseteq G$ for some $\delta > 0$. Suppose no such $\delta > 0$ exists. So for every $n \in \mathbb{N}$ there exists $x_n \in B^{1/n} \setminus G$. If $A = \{x_n : n \in \mathbb{N}\}$, then $A \cap B_1 = \emptyset$. If A is closed, then as B_1 is a shield of B there exists $n_0 \in \mathbb{N}$ such that $A \cap B^{1/n_0} = \emptyset$. Which is not possible as $x_{n_0} \in A \cap B^{1/n_0}$. If A has a cluster point x_0, then $x_0 \in B_1 \subseteq G$. Therefore G contains infinitely many points of A which contradicts that $A \cap G = \emptyset$.

$(b) \Rightarrow (a)$ Suppose $B_0 \in \mathcal{B}$ has no shield in $\mathcal{B}$. So for every $B \in \mathcal{B}$ with $B_0 \subseteq B$, there exists a closed subset C_B of X such that $C_B \cap B = \emptyset$ but for every $\delta > 0$, $C_B \cap B_0^\delta \neq \emptyset$. Let $\mathcal{G} = \{X \setminus C_B : B \in \mathcal{B} \text{ with } B_0 \subseteq B\}$. Then every member of $\mathcal{G}$ is open and $B \subseteq X \setminus C_{B'}$ for every $B \in \mathcal{B}$ where $B' = B_0 \cup B$. So $\mathcal{G}$ is an open $\mathcal{B}$-cover of X. But $\mathcal{G}$ is not an open $\mathcal{B}^s$-cover of X as $B_0^\delta \not\subseteq X \setminus C_B$ for any $B \in \mathcal{B}$ with $B_0 \subseteq B$ and $\delta > 0$. □

Corollary 4.4. *Let $\mathcal{B}$ be a bornology on a metric space (X, d) having a compact base. Then $\mathcal{G}$ is an open $\mathcal{B}$-cover of X if and only if $\mathcal{G}$ is an open $\mathcal{B}^s$-cover of X.*

Theorem 4.5. *Let (X, d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

- *(a) $(C(X), \tau^{sw}_{\mathcal{B}})$ has countable tightness;*
- *(b) every open $\mathcal{B}^s$-cover of X has a countable $\mathcal{B}^s$-subcover and $\mathcal{B} \subseteq \mathcal{K}$.*

Proof. $(a) \Rightarrow (b)$ To show that $\mathcal{B} \subseteq \mathcal{K}$, suppose that there exists a closed set $B_0 \in \mathcal{B}$ which is not compact. Let $D = \{x_n : n \in \mathbb{N}\} \subseteq B_0$ be closed and discrete in X. Define $E \subseteq C(X)$ by $E = \{g \in C(X) : g(x_n) \neq 0 \text{ for all } n \in \mathbb{N}\}$. For every $B \in \mathcal{B}$ and every $\epsilon \in C^+(X)$, we have $\frac{\epsilon}{2} \in [B, \epsilon]^{sw}(f_0) \cap E$. Thus $f_0 \in \overline{E}$ in $(C(X), \tau^{sw}_{\mathcal{B}})$. So there exists a countable subset $E' = \{g_n : n \in \mathbb{N}\}$ of E such that $f_0 \in \overline{E'}$ in $(C(X), \tau_{\mathcal{B}}^{sw})$. By Lemma 3.1, the function $\epsilon_1 : D \to (0, \infty)$ such that $\epsilon_1(x_n) = \frac{|g_n(x_n)|}{2}$ can be extended to a continuous function $\epsilon_0 : X \to (0, \infty)$. Then $f_0 \notin \overline{E'}$ as $[B_0, \epsilon_0]^{sw}(f_0) \cap E' = \emptyset$. This contradiction implies that $\mathcal{B} \subseteq \mathcal{K}$. Hence by Theorem 2.4, $(C(X), \tau_{\mathcal{B}}^{sw}) = (C(X), \tau_{\mathcal{B}}^{s})$. The remaining part of the implication now follows from Theorem 3.12 of [10].

The implication $(b) \Rightarrow (a)$ follows from Theorem 3.12 of [10] and Theorem 2.4. □

The next result can be proved in a manner similar to the previous theorem.

Theorem 4.6. *Let (X, d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) $(C(X), \tau_{\mathcal{B}}^{w})$ *has countable tightness;*

(b) every open $\mathcal{B}$-cover of X has a countable $\mathcal{B}$-subcover and $\mathcal{B} \subseteq \mathcal{K}$.

By Corollary 4.4, we have the following result.

Theorem 4.7. *Let (X, d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) $(C(X), \tau_{\mathcal{B}}^{sw})$ *has countable tightness;*

(b) $(C(X), \tau_{\mathcal{B}}^{w})$ *has countable tightness;*

(c) every open $\mathcal{B}$-cover of X has a countable $\mathcal{B}$-subcover and $\mathcal{B} \subseteq \mathcal{K}$;

(d) every open $\mathcal{B}^s$-cover of X has a countable $\mathcal{B}^s$-subcover and $\mathcal{B} \subseteq \mathcal{K}$.

To study the Fréchet property of the spaces $(C(X), \tau_{\mathcal{B}}^{sw})$ and $(C(X), \tau_{\mathcal{B}}^{w})$, we need the concepts of $\mathcal{A}$-sequence and strong $\mathcal{A}$-sequence for a family $\mathcal{A}$ of nonempty subsets of X.

Definitions 4.8. Let $\mathcal{A}$ be a family of nonempty subsets of a space X. Then a sequence $\{C_n : n \in \mathbb{N}\}$ of subsets of X is called an *$\mathcal{A}$-sequence* if for each $A \in \mathcal{A}$ there exists $m \in \mathbb{N}$ such that for all $n \geq m$, $A \subseteq C_n$; and a sequence $\{C_n : n \in \mathbb{N}\}$ of subsets of X is called a *strong $\mathcal{A}$-sequence* (denoted by $\mathcal{A}^s$-sequence, see Definition 3.10 of [10]) if for each $A \in \mathcal{A}$ there exist $m \in \mathbb{N}$ and a sequence $\{\delta_n : n \geq m\}$ of positive real numbers such that for all $n \geq m$, $A^{\delta_n} \subseteq C_n$.

Note that every strong $\mathcal{A}$-sequence is always an $\mathcal{A}$-sequence. However, the converse is not true.

Example 4.9. Let $X = \mathbb{R}$ with the usual metric and let $\mathcal{A} = \mathcal{F}$. For every $n \in \mathbb{N}$, define $D_n = \{\frac{1}{m} : m \in \mathbb{N} \text{ and } m \geq n\}$. Consider the sequence $\{C_n = \mathbb{R} \setminus D_n : n \in \mathbb{N}\}$ of subsets of $\mathbb{R}$. Clearly, the sequence $\{C_n : n \in \mathbb{N}\}$ is an $\mathcal{F}$-sequence. Since $\{0\}^{\delta} \nsubseteq C_n$ for every $\delta > 0$ and every $n \in \mathbb{N}$, the sequence $\{C_n : n \in \mathbb{N}\}$ is not an $\mathcal{F}^s$-sequence.

Note that if a bornology $\mathcal{B}$ is stable under small enlargements, then every $\mathcal{B}$-sequence is also a strong $\mathcal{B}$-sequence.

Theorem 4.10. *Let (X, d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) the space $(C(X), \tau_{\mathcal{B}}^{sw})$ *is Fréchet;*

(b) every open $\mathcal{B}^s$-cover of X has a $\mathcal{B}^s$-sequence and $\mathcal{B} \subseteq \mathcal{K}$;

(c) every open $\mathcal{B}$-cover of X has a $\mathcal{B}^s$-sequence and $\mathcal{B} \subseteq \mathcal{K}$.

Proof. $(a) \Rightarrow (b)$ Since $(C(X), \tau_{\mathcal{B}}^{sw})$ is a Fréchet space, it has countable tightness. By Theorem 4.5, $\mathcal{B} \subseteq \mathcal{K}$. Consequently, by Theorem 2.4, $(C(X), \tau_{\mathcal{B}}^{s}) = (C(X), \tau_{\mathcal{B}}^{sw})$ is a Fréchet space. Hence by Theorem 3.14 of [10], every open $\mathcal{B}^s$-cover of X has a $\mathcal{B}^s$-sequence.

$(b) \Rightarrow (a)$ This implication follows from Theorem 3.14 of [10] and Theorem 2.4.

$(b) \Rightarrow (c)$ follows from Corollary 4.4 and $(c) \Rightarrow (b)$ is immediate. □

Lemma 4.11. *If $\mathcal{B}$ is a bornology on a metric space (X, d) with a compact base, then every $\mathcal{B}$-sequence of open subsets is also a $\mathcal{B}^s$-sequence.*

Proof. It follows from the fact that for any compact subset K and open subset U of X with $K \subseteq U$, there exists a $\delta > 0$ such that $K \subseteq K^{\delta} \subseteq U$. □

Theorem 4.12. *Let (X, d) be a metric space and let $\mathcal{B}$ be a bornology on X with a closed base. Then the following conditions are equivalent:*

(a) the space $(C(X), \tau_{\mathcal{B}}^{sw})$ is Fréchet;

(b) the space $(C(X), \tau_{\mathcal{B}}^{w})$ is Fréchet;

(c) every open $\mathcal{B}^s$-cover of X has a $\mathcal{B}^s$-sequence and $\mathcal{B} \subseteq \mathcal{K}$;

(d) every open $\mathcal{B}$-cover of X has a $\mathcal{B}^s$-sequence and $\mathcal{B} \subseteq \mathcal{K}$;

(e) every open $\mathcal{B}$-cover of X has a $\mathcal{B}$-sequence and $\mathcal{B} \subseteq \mathcal{K}$.

Proof. The equivalences $(a) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$ follow from Theroem 4.10, Corollary 4.4, and Lemma 4.11. The equivalence $(b) \Leftrightarrow (c)$ follows from Theorems 2.4, 4.7 and 4.10. □

Proposition 4.13. *Let $\mathcal{B}$ be a bornology on a metric space (X, d) with a closed base. If $\mathcal{B}$ is stable under small enlargements, then the following conditions are equivalent:*

(a) $\mathcal{B}$ has a countable base;

(b) every open $\mathcal{B}$-cover of X has a countable $\mathcal{B}$-subcover.

Proof. $(a) \Rightarrow (b)$ It is immediate.

$(b) \Rightarrow (a)$ Since $\mathcal{B}$ is stable under small enlargements, for every $B \in \mathcal{B}$ there is a $\delta_B > 0$ such that $B^{\delta_B} \in \mathcal{B}$. Thus by the hypothesis, the open $\mathcal{B}$-cover $\{B^{\delta_B} : B \in \mathcal{B}\}$ of X has a countable $\mathcal{B}$-subcover $\{B_n^{\delta_{B_n}} : B_n \in \mathcal{B}$ and $n \in \mathbb{N}\}$. Since $\mathcal{B}$ has a closed base, $\overline{(B_n^{\delta_{B_n}})} \in \mathcal{B}$ for every $n \in \mathbb{N}$. Therefore the collection $\{\overline{(B_n^{\delta_{B_n}})} : B_n \in \mathcal{B}$ and $n \in \mathbb{N}\}$ forms a countable base for $\mathcal{B}$. □

Our next result follows from Theorems 3.2, 3.11, 4.7, and Proposition 4.13.

Theorem 4.14. *Let $\mathcal{B}$ be a bornology on a metric space (X, d) with a closed base. If $\mathcal{B}$ is stable under small enlargements, then the following conditions are equivalent:*

(a) $\mathcal{B}$ has a countable base consisting of compact sets;

(b) for every metric space (Y, ρ), $(C(X, Y), \tau_{\mathcal{B}}^{sw})$ is metrizable;

(c) for every metric space (Y, ρ), $(C(X, Y), \tau_{\mathcal{B}}^{sw})$ is first countable;

(d) $(C(X), \tau_{\mathcal{B}}^{sw})$ is metrizable;

(e) $(C(X), \tau_{\mathcal{B}}^{sw})$ is first countable;

(f) $(C(X), \tau_{\mathcal{B}}^{sw})$ is second countable;

(g) $(C(X), \tau_{\mathcal{B}}^{sw})$ *is a Fréchet space;*

(h) $(C(X), \tau_{\mathcal{B}}^{sw})$ *has countable tightness.*

Corollary 4.15. *If* (X, d) *is locally compact, then the following conditions are equivalent:*

(a) $\mathcal{K}$ *has a countable base;*

(b) for every metric space (Y, ρ), $(C(X, Y), \tau_{\mathcal{K}})$ *is metrizable;*

(c) for every metric space (Y, ρ), $(C(X, Y), \tau_{\mathcal{K}})$ *is first countable;*

(d) $(C(X), \tau_{\mathcal{K}})$ *is metrizable;*

(e) $(C(X), \tau_{\mathcal{K}})$ *is first countable;*

(f) $(C(X), \tau_{\mathcal{K}})$ *is second countable;*

(g) $(C(X), \tau_{\mathcal{K}})$ *is a Fréchet space;*

(h) $(C(X), \tau_{\mathcal{K}})$ *has countable tightness.*

Acknowledgement: We thank the referee for his/her valuable suggestions.

The first author acknowledges the support of CSIR Grant 09/964(0016)/2018-EMR-I.

The second author acknowledges the support of NBHM Research Grant 02011/6/2020/NBHM(R.P) R&D II/6277.

References

[1] F.W. Anderson, Approximation in systems of real-valued functions, Trans. Amer. Math. Soc. 163 (1962) 249–271.
[2] A.V. Arhangel'skiĭ, M. Tkachenko, Topological Groups and Related Structures, Atlantis Stud. Math. Vol. 1, Atlantis Press, Amsterdam, Paris, 2008.
[3] G. Beer, The Alexandroff property and the preservation of strong uniform continuity, Appl. Gen. Topol. 11 (2010) 117–133.
[4] G. Beer, C. Costantini, S. Levi, Bornological convergence and shields, Mediterr. J. Math. 10 (2013) 529–560.
[5] G. Beer, S. Levi, Strong uniform continuity, J. Math. Anal. Appl. 350 (2009) 568–589.
[6] . Beer, S. Levi, Uniform continuity, uniform convergence, and shields, Set-Valued Anal. 18 (2010) 251–275.
[7] N. Bouleau, Une structure uniforme sur un espace $\mathcal{F}(E, F)$, Cah. Topol. Géom. Différ. Catég. 11 (1969) 207–214.
[8] N. Bouleau, On the coarsest topology preserving continuity, arXiv preprint math/0610373, 2006.
[9] A. Caserta, Strong Whitney convergence, Filomat 26 (2012) 81–91.
[10] A. Caserta, G. Di Maio, L. Holá, Arzelà's Theorem and strong uniform convergence on bornologies, J. Math. Anal. Appl. 371 (2010) 384–392.
[11] A. Caserta, G. Di Maio, Lj.D.R. Kočinac, Bornologies, selection principles and function spaces, Topol. Appl. 159 (2012) 1847–1852.
[12] T.K. Chauhan, V. Jindal, Strong Whitney and strong uniform convergences on a bornology, J. Math. Anal. Appl. 505 (2022) 125634.
[13] G. Di Maio, L. Holá, D. Holý, R.A. McCoy, Topologies on the space of continuous functions, Topol. Appl. 86 (1998) 105–122.
[14] R. Engelking, General Topology, Sigma Series in Pure Mathematics, Vol. 6, Heldermann Verlag, Berlin, 1989.
[15] J. Ewert, J. Jedrzejewski, Between Arzelà and Whitney convergence, Real Anal. Exch. 29 (2004) 257–264.
[16] L. Gillman, M. Jerison, Rings of Continuous Functions, Springer Verlag, New York, 1976.
[17] J. Gómez-Pérez, W. W. McGovern, The m-topology on $C_m(X)$ revisited, Topol. Appl. 153 (2006) 1838–1848.
[18] G. Gruenhage, Generalized metric spaces, In Handbook of Set-Theoretic Topology, North-Holland, Amsterdam. (1984) 423–501.
[19] E. Hewitt, Rings of real-valued continuous functions. I, Trans. Amer. Math. Soc. 64 (1948) 45–99.
[20] M.W. Hirsch, Differential Topology, Graduate Texts in Mathematics, Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.
[21] H. Hogbe-Nlend, Bornologies and Functional Analysis, North-Holland, Amsterdam, 1977.
[22] L. Holá, Complete metrizability of topologies of strong uniform convergence on bornologies, J. Math. Anal. Appl. 387 (2012) 770–775.
[23] L. Holá, V. Jindal, On graph and fine topologies, Topol. Proc. 49 (2017) 65–73.
[24] L. Holá, B. Novotný, Cardinal functions, bornologies and function spaces, Ann. Mat. Pura Appl. 193 (2014) 1319–1327.
[25] L. Holá, B. Novotný, Topology of uniform convergence and m-topology on $C(X)$, Mediterr. J. Math. 14 (2017) doi.org/10.1007/s00009-017-0861-6.
[26] R. A. McCoy, Necessary conditions for function spaces to be Lindelöf, Glasnik Mat. 15 (1980) 163–168.
[27] R.A. McCoy, Fine topology on function spaces, Int. J. Math. Math. Sci. 9 (1986) 417–424.

[28] R.A. McCoy, S. Kundu, V. Jindal, Function Spaces with Uniform, Fine and Graph Topologies, Springer Briefs in Mathematics, Springer, Cham, 2018.
[29] R.A. McCoy, I. Ntantu, Topological Properties of Spaces of Continuous Functions, Lecture Ntes in Mathematics, vol. 1315, Springer-Verlag, Berlin, 1988.
[30] F. Siwiec, Generalizations of the first axiom of countability, Rocky Mt. J. Math. 5 (1975) 1–60.
[31] H. Whitney, Differentiable manifolds, Ann. Math. 37 (1936) 645–680.
[32] S. Willard, General Topology, Addison-Wesley Publishing Company, Reading, MA, 1970.