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Complex Symmetry and Normality of Toeplitz Composition Operators on the Hardy Space

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Abstract. In this paper, we investigate the conditions under which the Toeplitz composition operator on the Hardy space $\mathcal{H}^2$ becomes complex symmetric with respect to a certain conjugation. We also study various normality conditions for the Toeplitz composition operator on $\mathcal{H}^2$.

1. Introduction and Preliminaries

Let $\mathbb{D}$ denote the open unit disc and $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi)\}$ denote the unit circle in the complex plane $\mathbb{C}$. Recall that the *Hardy space* $\mathcal{H}^2$ is a Hilbert space which consists of all those analytic functions f on $\mathbb{D}$ having power series representation with square summable complex coefficients. That is,

$$\mathcal{H}^2 = \{f : \mathbb{D} \to \mathbb{C} \mid f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \text{ and } \|f\|^2_{\mathcal{H}^2} := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}$$

or equivalently,

$$\mathcal{H}^2 = \{f : \mathbb{D} \to \mathbb{C} \text{ analytic } \mid \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty\}.$$

The evaluation of functions in $\mathcal{H}^2$ at each $w \in \mathbb{D}$ is a bounded linear functional and for all $f \in \mathcal{H}^2$, $f(w) = \langle f, K_w \rangle$ where $K_w(z) = 1/(1 - \overline{w}z)$. The function $K_w(z)$ is called the *reproducing kernel* for the Hardy space $\mathcal{H}^2$. Consider the Hilbert space

$$\widetilde{\mathcal{H}^2} = \{f^* : \mathbb{T} \to \mathbb{C} \mid f^*(z) = \sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta} \text{ and } \|f^*\|^2_{\mathcal{H}^2} := \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty\}.$$

Let L^2 denote the Lebesgue (Hilbert) space on the unit circle $\mathbb{T}$. It is well known that every function $f \in \mathcal{H}^2$ satisfies the radial limit $f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})$ for almost every $\theta \in [0, 2\pi)$ and it is obvious that the

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correspondence where $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ is mapped to $f^*(e^{i\theta}) = \sum_{n=0}^{\infty} \hat{f}(n) e^{in\theta}$ is an isometric isomorphism from $\mathcal{H}^2$ onto the closed subspace $\widetilde{\mathcal{H}^2}$ of L^2. Since $\{e_n(z) = z^n : n \in \mathbb{Z}\}$ forms an orthonormal basis for L^2, every function $f \in L^2$ can be expressed as $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n$ where $\hat{f}(n)$ denotes the *nth* Fourier coefficient of f. Let L^∞ be the Banach space of all essentially bounded functions on the unit circle $\mathbb{T}$. For any $\phi \in L^\infty$, the *Toeplitz operator* $T_\phi : \mathcal{H}^2 \to \mathcal{H}^2$ is defined by $T_\phi f = P(\phi \cdot f)$ for $f \in \mathcal{H}^2$ where $P : L^2 \to \mathcal{H}^2$ is the orthogonal projection. It can be easily verified that for $m, n \in \mathbb{Z}$,

$$P(z^m \overline{z}^n) = \begin{cases} z^{m-n} & \text{if } m \geq n, \\ 0 & \text{otherwise.} \end{cases}$$

For a non-zero bounded analytic function u on $\mathbb{D}$ and a self-analytic map ϕ on $\mathbb{D}$, the *weighted composition operator* $W_{u,\phi}$ is defined by $W_{u,\phi} f = u \cdot f \circ \phi$ for every $f \in \mathcal{H}^2$. Over the past several decades, there has been tremendous development in the study of composition operators and weighted composition operators over the Hardy space $\mathcal{H}^2$ and various other spaces of analytic functions. Readers may refer [1, 10] for general study and background of the composition operators on the Hardy space $\mathcal{H}^2$. In this paper, we introduce the notion of the Toeplitz composition operator on the Hardy space $\mathcal{H}^2$ where the symbol u in $W_{u,\phi}$ need not necessarily be analytic. For a function $\psi \in L^\infty$ and a self-analytic map ϕ on $\mathbb{D}$, the *Toeplitz composition operator* $T_\psi C_\phi : \mathcal{H}^2 \to \mathcal{H}^2$ is defined by $T_\psi C_\phi f = P(\psi \cdot f \circ \phi)$ for every $f \in \mathcal{H}^2$ where $C_\phi f := f \circ \phi$ is the composition operator on $\mathcal{H}^2$. The authors in [5] introduced the concept of the Toeplitz composition operators on the Fock space and also studied its various properties.

Let $\mathcal{H}$ be a separable Hilbert space. Then a mapping S on $\mathcal{H}$ is said to be *anti-linear (also conjugate-linear)* if $S(\alpha x_1 + \beta x_2) = \overline{\alpha} S(x_1) + \overline{\beta} S(x_2)$ for all scalars $\alpha, \beta \in \mathbb{C}$ and for all $x_1, x_2 \in \mathcal{H}$.

An anti-linear mapping $\mathcal{C} : \mathcal{H} \to \mathcal{H}$ is said to be a *conjugation* if it is involutive (*i.e.* $\mathcal{C}^2 = I$) and isometric (*i.e.* $\|\mathcal{C}x\| = \|x\|$ for every $x \in \mathcal{H}$). A *complex symmetric operator* S on $\mathcal{H}$ is a bounded linear operator such that $S = \mathcal{C} S^* \mathcal{C}$ for some conjugation $\mathcal{C}$ on $\mathcal{H}$. We call such an operator S to be a *$\mathcal{C}-$symmetric operator*.

Garcia and Putinar [3, 4] began the general study of complex symmetric operators on Hilbert spaces which are the natural generalizations of complex symmetric matrices. There exist a wide variety of complex symmetric operators which include normal operators, compressed Toeplitz operators, Volterra integration operators *etc.* Jung *et al.* [7] studied the complex symmetry of the weighted composition operators on the Hardy space in the unit disc $\mathbb{D}$. Garcia and Hammond [2] undertook the study of complex symmetry of weighted composition operators on the weighted Hardy spaces. Ko and Lee [8] gave a characterization of the complex symmetric Toeplitz operators on the Hardy space $\mathcal{H}^2$ of the unit disc $\mathbb{D}$. Motivated by this, we study the complex symmetry of the Toeplitz composition operators on the Hardy space $\mathcal{H}^2$. In this paper we give a characterization of such types of operators. We also investigate certain conditions under which a complex symmetric operator turns out to be a normal operator. In the concluding section of this article, we discuss the normality of the Toeplitz composition operators on $\mathcal{H}^2$.

2. Complex Symmetric Toeplitz Composition Operators

In this section we aim to find the conditions under which a Toeplitz composition operator becomes complex symmetric with respect to a certain fixed conjugation. In order to determine these conditions, we need an explicit formula for the adjoint C_ϕ^* of a composition operator C_ϕ where ϕ is a self-analytic map on the unit disc $\mathbb{D}$. But there exists no general formula and there are only a few special cases where it is possible to find a formula for C_ϕ^* explicitly. C. Cowen was the first to find the representation for the adjoint of a composition operator C_ϕ on $\mathcal{H}^2$, famously known as the Cowen's Adjoint Formula, where the symbol ϕ is a linear fractional self-map of the unit disc $\mathbb{D}$. The Cowen's Adjoint Formula was extended to the Bergman space $\mathcal{A}^2$ by P. Hurst [6] and it is stated as follows:

Theorem 2.1 ([1]). *(Cowen's Adjoint Formula) Let $\phi(z) = \frac{az+b}{cz+d}$ be a linear fractional self-map of the unit disc where $ad - bc \neq 0$. Then $\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}$ maps disc into itself, $g(z) = (-\overline{b}z + \overline{d})^{-p}$ and $h(z) = (cz + d)^p$ are bounded analytic* *functions on the disc and on $\mathcal{H}^2$ or $\mathcal{A}^2$, $C_\phi^* = M_g C_\sigma M_h^*$ where $p = 1$ on $\mathcal{H}^2$ and $p = 2$ on $\mathcal{A}^2$.(Note that the operator M_g is the multiplication operator defined by $M_g f = g \cdot f$.)*

Next we have the following lemmas which would be instrumental in proving certain results throughout this article :

Lemma 2.2 ([9]). *A linear fractional map ϕ, written in the form $\phi(z) = \frac{az+b}{cz+d}$; $ad - bc \neq 0$, maps $\mathbb{D}$ into itself if and only if:*

$$|b\overline{d} - a\overline{c}| + |ad - bc| \leq |d|^2 - |c|^2. \tag{1}$$

Lemma 2.3 ([1]). *Let $\phi(z) = \frac{az+b}{cz+d}$ be a linear fractional map and define the associated linear fractional transformation ϕ^* by*

$$\phi^*(z) = \frac{1}{\overline{\phi^{-1}(\frac{1}{\overline{z}})}} = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}.$$

Then ϕ is a self-map of the disc if and only if ϕ^* is also a self-map of the disc.

Lemma 2.4 ([1]). *If $\phi(z) = \frac{az+b}{cz+d}$ is a linear fractional transformation mapping $\mathbb{D}$ into itself where $ad - bc = 1$, then $\sigma(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}$ maps $\mathbb{D}$ into itself.*

In the following lemma, a conjugation on the Hardy space $\mathcal{H}^2$ has been defined with respect to which we will find the complex symmetry of the operator $T_\psi C_\phi$.

Lemma 2.5 ([8]). *For every ξ and θ, let $C_{\xi,\theta} : \mathcal{H}^2 \to \mathcal{H}^2$ be defined by*

$$C_{\xi,\theta} f(z) = e^{i\xi}\overline{f(e^{i\theta}\overline{z})}.$$

Then $C_{\xi,\theta}$ is a conjugation on $\mathcal{H}^2$. Moreover, $C_{\xi,\theta}$ and $C_{\tilde{\xi},\tilde{\theta}}$ are unitarily equivalent where $(\tilde{\xi}, \tilde{\theta})$ satisfies the equation $\tilde{\xi} - k\tilde{\theta} = -\xi + k\theta - 2n\pi$ for every $k \in \mathbb{N}$ and $n \in \mathbb{Z}$.

In the next theorem, we determine the conditions under which the Toeplitz composition operator $T_\psi C_\phi$ turns out to be complex symmetric with respect to the conjugation $C_{\xi,\theta}$ on $\mathcal{H}^2$.

Theorem 2.6. *For $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n) z^n \in L^\infty$ and for self-analytic linear transformation $\phi(z) = az + b$ $(a \neq 0)$ mapping $\mathbb{D}$ into itself, let $T_\psi C_\phi$ be a Toeplitz composition operator on $\mathcal{H}^2$. Then $T_\psi C_\phi$ is complex symmetric with the conjugation $C_{\xi,\theta}$ if and only if for each $k, p \in \mathbb{N} \cup \{0\}$ and for every $n \in \mathbb{Z}$, we have :*
(i) $\sum_{n=-k+p}^{p} \binom{k}{p-n} \overline{\hat{\psi}(n)} \overline{a}^{p-n} \overline{b}^{-n+k-p} \lambda^p = \sum_{n=-k}^{-k+p} \binom{p}{p-n-k} \overline{\hat{\psi}(-n)} \overline{a}^{n+k} \overline{b}^{p-n-k} \lambda^k$ *for $b \neq 0$ and,* ***(ii)*** $\overline{\hat{\psi}(n)} \lambda^n = \overline{\hat{\psi}(-n)} \overline{a}^n$ *for $b = 0$.*

Proof. If $T_\psi C_\phi$ is complex symmetric with respect to the conjugation $C_{\xi,\theta}$, then for all $k \in \mathbb{N} \cup \{0\}$ we have

$$C_{\xi,\theta} T_\psi C_\phi z^k = (T_\psi C_\phi)^* C_{\xi,\theta} z^k. \tag{2}$$

We take $\mu = e^{i\xi}$ and $\lambda = e^{-i\theta}$ and consider the following two cases:

Case (i) : Let $b \neq 0$. Then

$$\begin{aligned}
C_{\xi,\theta}T_{\psi}C_{\phi}z^k &= C_{\xi,\theta}T_{\psi}(\phi(z))^k \\
&= C_{\xi,\theta}T_{\psi}(az+b)^k \\
&= C_{\xi,\theta}P(\psi(z)\cdot\sum_{m=0}^{k}\binom{k}{m}a^m b^{k-m}z^m) \\
&= C_{\xi,\theta}P(\sum_{m=0}^{k}(\sum_{n=-\infty}^{\infty}\binom{k}{m}\hat{\psi}(n)a^m b^{k-m}z^{m+n})) \\
&= C_{\xi,\theta}(\sum_{m=0}^{k}P(\sum_{n=-\infty}^{\infty}\binom{k}{m}\hat{\psi}(n)a^m b^{k-m}z^{m+n})) \\
&= C_{\xi,\theta}(\sum_{m=0}^{k}(\sum_{n=-m}^{\infty}\binom{k}{m}\hat{\psi}(n)a^m b^{k-m}z^{m+n})) \\
&= \sum_{m=0}^{k}C_{\xi,\theta}(\sum_{n=-m}^{\infty}\binom{k}{m}\hat{\psi}(n)a^m b^{k-m}z^{m+n}) \\
&= e^{i\xi}\sum_{m=0}^{k}(\sum_{n=-m}^{\infty}\binom{k}{m}\overline{\hat{\psi}(n)}\overline{a}^m\overline{b}^{k-m}e^{-i(m+n)\theta}z^{m+n}) \\
&= \mu\sum_{m=0}^{k}(\sum_{n=-m}^{\infty}\binom{k}{m}\overline{\hat{\psi}(n)}\overline{a}^m\overline{b}^{k-m}\lambda^{m+n}z^{m+n}) \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
(T_{\psi}C_{\phi})^*C_{\xi,\theta}z^k &= C_{\phi}^*T_{\psi}^*C_{\xi,\theta}z^k \\
&= C_{\phi}^*T_{\overline{\psi}}(e^{i\xi}e^{-ik\theta}z^k) \\
&= C_{\phi}^*T_{\overline{\psi}}(\mu\lambda^k z^k) \\
&= C_{\phi}^*P(\mu\lambda^k\sum_{n=-\infty}^{\infty}\overline{\hat{\psi}(n)}z^{k-n}) \\
&= C_{\phi}^*P(\mu\lambda^k\sum_{n=-\infty}^{\infty}\overline{\hat{\psi}(-n)}z^{n+k}) \\
&= \mu\lambda^k C_{\phi}^*(\sum_{n=-k}^{\infty}\overline{\hat{\psi}(-n)}z^{n+k}). \qquad (4)
\end{aligned}$$

On using Theorem 2.1 for $a \neq 0$, $c = 0$ and $d = 1$, we obtain that $C_{\phi}^* = M_g C_{\sigma}$ where $g(z) = (1-\overline{b}z)^{-1}$ and $\sigma(z) = \frac{\overline{a}z}{1-\overline{b}z}$. Since $|a|+|b| \leq 1$ from Lemma 2.2, so $|b| < 1$ and hence, $\frac{1}{(1-\overline{b}z)^i} = \sum_{j=0}^{\infty}\binom{j+i-1}{j}(\overline{b}z)^j$ for $z \in \mathbb{D}$.

Therefore, from (4) we get that

$$\begin{aligned}(T_\psi C_\phi)^* C_{\xi,\theta} z^k &= \mu\lambda^k M_g C_\sigma\Big(\sum_{n=-k}^{\infty} \overline{\hat{\psi}(-n)} z^{n+k}\Big)\\ &= \mu\lambda^k M_g\Big(\sum_{n=-k}^{\infty} \overline{\hat{\psi}(-n)} \left(\frac{\bar{a}z}{1-\bar{b}z}\right)^{n+k}\Big)\\ &= \mu\lambda^k\Big(\sum_{n=-k}^{\infty} \overline{\hat{\psi}(-n)}\bar{a}^{n+k} \left(\frac{1}{1-\bar{b}z}\right)^{n+k+1} z^{n+k}\Big)\\ &= \mu\sum_{j=0}^{\infty}\Big(\sum_{n=-k}^{\infty}\binom{n+k+j}{j}\overline{\hat{\psi}(-n)}\bar{a}^{n+k}\bar{b}^{j}\lambda^k z^{n+k+j}\Big). \qquad (5)\end{aligned}$$

It follows from (2) that for each $k \in \mathbb{N}\cup\{0\}$, we have

$$\sum_{m=0}^{k}\Big(\sum_{n=-m}^{\infty}\binom{k}{m}\overline{\hat{\psi}(n)}\bar{a}^{m}\bar{b}^{k-m}\lambda^{m+n}z^{m+n}\Big) = \sum_{j=0}^{\infty}\Big(\sum_{n=-k}^{\infty}\binom{n+k+j}{j}\overline{\hat{\psi}(-n)}\bar{a}^{n+k}\bar{b}^{j}\lambda^k z^{n+k+j}\Big). \qquad (6)$$

Thus, the coefficient of z^p where $p \in \mathbb{N}\cup\{0\}$ must be equal on the both sides of (6). On comparing the coefficients of $1, z, z^2, z^3$ and so on, on the both sides of (6), we observe that

$$\sum_{n=-k+p}^{p}\binom{k}{p-n}\overline{\hat{\psi}(n)}\bar{a}^{p-n}\bar{b}^{n+k-p}\lambda^p = \sum_{n=-k}^{-k+p}\binom{p}{p-n-k}\overline{\hat{\psi}(-n)}\bar{a}^{n+k}\bar{b}^{p-n-k}\lambda^k \qquad (7)$$

for each $k,\ p \in \mathbb{N}\cup\{0\}$.

Conversely, let us suppose that (7) holds for each $k, p \in \mathbb{N}\cup\{0\}$. Then from (3) and (5), we have

$$\begin{aligned}(C_{\xi,\theta}T_\psi C_\phi - (T_\psi C_\phi)^* C_{\xi,\theta})z^k &= \mu\Big(\sum_{m=0}^{k}\Big(\sum_{n=-m}^{\infty}\binom{k}{m}\overline{\hat{\psi}(n)}\bar{a}^{m}\bar{b}^{k-m}\lambda^{m+n}z^{m+n}\Big)\Big)\\ &\quad-\mu\Big(\sum_{j=0}^{\infty}\Big(\sum_{n=-k}^{\infty}\binom{n+k+j}{j}\overline{\hat{\psi}(-n)}\bar{a}^{n+k}\bar{b}^{j}\lambda^k z^{n+k+j}\Big)\Big)\\ &= 0.\end{aligned}$$

Case (ii) : If $b = 0$, then

$$\begin{aligned}C_{\xi,\theta}T_\psi C_\phi z^k &= C_{\xi,\theta}T_\psi(\phi(z))^k\\ &= C_{\xi,\theta}T_\psi(az)^k\\ &= C_{\xi,\theta}P\Big(\sum_{n=-\infty}^{\infty}\hat{\psi}(n)a^k z^{n+k}\Big)\\ &= C_{\xi,\theta}\Big(\sum_{n=-k}^{\infty}\hat{\psi}(n)a^k z^{n+k}\Big)\\ &= e^{i\xi}\sum_{n=-k}^{\infty}\overline{\hat{\psi}(n)}\bar{a}^k e^{-i(n+k)\theta}z^{n+k}\\ &= \mu\sum_{n=-k}^{\infty}\overline{\hat{\psi}(n)}\bar{a}^k\lambda^{n+k}z^{n+k}. \qquad (8)\end{aligned}$$

For $a \neq 0$, $b = c = 0$ and $d = 1$, we get from Theorem 2.1 that $g(z) = h(z) = 1$ and $\sigma(z) = \bar{a}z$. Thus, $C_\phi^* = C_\sigma$. We compute

$$\begin{aligned}(T_\psi C_\phi)^* C_{\xi,\theta} z^k &= C_\phi^* T_\psi^* C_{\xi,\theta} z^k \\ &= C_\sigma T_{\overline{\psi}}(\mu \lambda^k z^k) \\ &= C_\sigma P(\mu \sum_{n=-\infty}^{\infty} \overline{\hat{\psi}(n)} \lambda^k z^{k-n}) \\ &= C_\sigma P(\mu \sum_{n=-\infty}^{\infty} \overline{\hat{\psi}(-n)} \lambda^k z^{n+k}) \\ &= \mu C_\sigma (\sum_{n=-k}^{\infty} \overline{\hat{\psi}(-n)} \lambda^k z^{n+k}) \\ &= \mu \sum_{n=-k}^{\infty} \overline{\hat{\psi}(-n)} \lambda^k \bar{a}^{n+k} z^{n+k}. \end{aligned} \tag{9}$$

Since the equation (2) holds, on equating the expressions (8) and (9), we obtain that $\overline{\hat{\psi}(n)}\lambda^n = \overline{\hat{\psi}(-n)}\bar{a}^n$ for every $n \in \mathbb{Z}$. Conversely, let us assume that $\overline{\hat{\psi}(n)}\lambda^n = \overline{\hat{\psi}(-n)}\bar{a}^n$ for every $n \in \mathbb{Z}$. Then (8) and (9) implies that $(C_{\xi,\theta} T_\psi C_\phi - (T_\psi C_\phi)^* C_{\xi,\theta}) z^k = 0$. Thus, $T_\psi C_\phi$ is complex symmetric with conjugation $C_{\xi,\theta}$. □

Example 2.7. *Let $\psi(z) = z + \bar{z} \in L^\infty$. Then, $\hat{\psi}(n) = \hat{\psi}(-n)$ for all $n \in \mathbb{Z}$. Let $\phi(z) = iz$. Then $\phi(z)$ is a self-analytic map on $\mathbb{D}$. Consider the conjugation $C_{\xi,\theta}$ where we choose $\theta = \pi/2$. Then $\lambda = e^{-i\theta} = -i$. On taking $a = i$, $b = 0$ and $\lambda = -i$ in Theorem 2.6 , we get that $\overline{\hat{\psi}(n)}\lambda^n = \overline{\hat{\psi}(-n)}\bar{a}^n$ for every $n \in \mathbb{Z}$ and hence, $C_{\xi,\theta} T_\psi C_\phi = (T_\psi C_\phi)^* C_{\xi,\theta}$. Therefore, the operator $T_\psi C_\phi$ is complex symmetric with respect to the conjugation $C_{\xi,\pi/2}$.*

In the light of the above example, an interesting observation has been made in the following Corollary:

Corollary 2.8. *Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n) z^n \in L^\infty$ and $\phi(z) = az$ be a self-analytic map on $\mathbb{D}$ where $a = e^{i\theta}$ for $\theta \in \mathbb{R}$. Then $T_\psi C_\phi$ is complex symmetric with respect to the conjugation $C_{\xi,\theta}$ if and only if $\hat{\psi}(n) = \hat{\psi}(-n)$ for all $n \in \mathbb{Z}$.*

Proof. It follows from Theorem 2.6 that $T_\psi C_\phi$ is complex symmetric with respect to the conjugation $C_{\xi,\theta}$ if and only if $\overline{\hat{\psi}(n)}\lambda^n = \overline{\hat{\psi}(-n)}\bar{a}^n$ if and only if $\hat{\psi}(n) = \hat{\psi}(-n)$ for all $n \in \mathbb{Z}$ where $a = e^{i\theta}$ and $\lambda = e^{-i\theta}$. □

An operator $T : \mathcal{H} \to \mathcal{H}$ where $\mathcal{H}$ denotes a Hilbert space is said to be *hyponormal* if $T^*T \geq TT^*$ or equivalently, $\|Tx\| \geq \|T^*x\|$ for every $x \in \mathcal{H}$. Our next goal is to find out the conditions under which a Toeplitz composition operator $T_\psi C_\phi$ becomes a normal operator. The proof involves the technique followed in [Proposition 2.2, [2]].

Theorem 2.9. *Let $\psi \in L^\infty$ and let ϕ be any self-analytic mapping from $\mathbb{D}$ into itself. If the operator $T_\psi C_\phi : \mathcal{H}^2 \to \mathcal{H}^2$ is hyponormal and complex symmetric with conjugation $C_{\xi,\theta}$, then $T_\psi C_\phi$ is a normal operator on $\mathcal{H}^2$.*

Proof. Since $T_\psi C_\phi$ is complex symmetric with respect to the conjugation $C_{\xi,\theta}$, this gives that $(T_\psi C_\phi)^* = C_{\xi,\theta} T_\psi C_\phi C_{\xi,\theta}$. On using the isometry of $C_{\xi,\theta}$, we obtain that

$$\|(T_\psi C_\phi)^* f\| = \|C_{\xi,\theta} T_\psi C_\phi C_{\xi,\theta} f\| = \|T_\psi C_\phi C_{\xi,\theta} f\| \text{ for every } f \in \mathcal{H}^2.$$

By hypothesis, $T_\psi C_\phi$ is a hyponormal operator on $\mathcal{H}^2$ and thus, $\|T_\psi C_\phi f\| \geq \|(T_\psi C_\phi)^* f\|$ for every $f \in \mathcal{H}^2$. Therefore, $\|(T_\psi C_\phi)^* f\| = \|T_\psi C_\phi C_{\xi,\theta} f\| \geq \|(T_\psi C_\phi)^* C_{\xi,\theta} f\| = \|C_{\xi,\theta} T_\psi C_\phi f\| = \|T_\psi C_\phi f\|$ for every $f \in \mathcal{H}^2$. Hence, $\|(T_\psi C_\phi)^* f\| \geq \|T_\psi C_\phi f\|$ and this together with the hyponormality of $T_\psi C_\phi$ implies that $\|(T_\psi C_\phi)^* f\| = \|T_\psi C_\phi f\|$ for every $f \in \mathcal{H}^2$ which proves that $T_\psi C_\phi$ is a normal operator. □

In the following theorem, the conditions under which the Toeplitz composition operator $T_\psi C_\phi$ commutes with the conjugation $C_{\xi,\theta}$ has been investigated which further provides us with a criteria which together with the complex symmetry of $T_\psi C_\phi$ makes the operator $T_\psi C_\phi$ a normal operator.

Theorem 2.10. *Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n) z^n \in L^\infty$ and $\phi(z) = az + b$ $(a \neq 0)$ be a linear fractional transformation mapping $\mathbb{D}$ into itself. Then the Toeplitz composition operator $T_\psi C_\phi$ commutes with the conjugation $C_{\xi,\theta}$ on $\mathcal{H}^2$ if and only if for each $m, k \in \mathbb{N} \cup \{0\}(0 \leq m \leq k)$ and $n \in \mathbb{Z}$, we have:*
(i) $\hat{\psi}(n) a^m b^{k-m} \lambda^k = \overline{\hat{\psi}(n)}\,\overline{a}^m \overline{b}^{k-m} \lambda^{m+n}$ *if $b \neq 0$ and,*
(ii) $\hat{\psi}(n) a^k = \overline{\hat{\psi}(n)}\,\overline{a}^k \lambda^n$ *if $b = 0$.*

Proof. If the operator $T_\psi C_\phi$ commutes with $C_{\xi,\theta}$, then for each $k \in \mathbb{N} \cup \{0\}$, we have $T_\psi C_\phi C_{\xi,\theta} z^k = C_{\xi,\theta} T_\psi C_\phi z^k$. We consider the following two cases:
Case (i) : Let us suppose that $b \neq 0$. Since for each $k \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} T_\psi C_\phi C_{\xi,\theta} z^k &= T_\psi C_\phi (e^{i\xi} e^{-ik\theta} z^k) \\ &= P(\psi(z) \cdot \mu \lambda^k (az+b)^k) \\ &= \mu \lambda^k P(\sum_{m=0}^{k} (\sum_{n=-\infty}^{\infty} \binom{k}{m} \hat{\psi}(n) a^m b^{k-m} z^{m+n})) \\ &= \mu \lambda^k (\sum_{m=0}^{k} P(\sum_{n=-\infty}^{\infty} \binom{k}{m} \hat{\psi}(n) a^m b^{k-m} z^{m+n})) \\ &= \mu \lambda^k \sum_{m=0}^{k} (\sum_{n=-m}^{\infty} \binom{k}{m} \hat{\psi}(n) a^m b^{k-m} z^{m+n}) \end{aligned}$$

and

$$\begin{aligned} C_{\xi,\theta} T_\psi C_\phi z^k &= C_{\xi,\theta} T_\psi ((az+b)^k) \\ &= C_{\xi,\theta} P(\sum_{m=0}^{k} (\sum_{n=-\infty}^{\infty} \binom{k}{m} \hat{\psi}(n) a^m b^{k-m} z^{m+n})) \\ &= C_{\xi,\theta} (\sum_{m=0}^{k} P(\sum_{n=-\infty}^{\infty} \binom{k}{m} \hat{\psi}(n) a^m b^{k-m} z^{m+n})) \\ &= C_{\xi,\theta} (\sum_{m=0}^{k} (\sum_{n=-m}^{\infty} \binom{k}{m} \hat{\psi}(n) a^m b^{k-m} z^{m+n})) \\ &= \mu \sum_{m=0}^{k} (\sum_{n=-m}^{\infty} \binom{k}{m} \overline{\hat{\psi}(n)}\,\overline{a}^m \overline{b}^{k-m} \lambda^{m+n} z^{m+n}) \, ; \end{aligned}$$

we obtain that $\hat{\psi}(n) a^m b^{k-m} \lambda^k = \overline{\hat{\psi}(n)}\,\overline{a}^m \overline{b}^{k-m} \lambda^{m+n}$ for each $n \in \mathbb{Z}$ and $m \in \mathbb{N} \cup \{0\}(0 \leq m \leq k)$.

Conversely, if for each $n \in \mathbb{Z}$ and $m, k \in \mathbb{N} \cup \{0\}$, we have $\hat{\psi}(n) a^m b^{k-m} \lambda^k = \overline{\hat{\psi}(n)}\,\overline{a}^m \overline{b}^{k-m} \lambda^{m+n}$, then $(T_\psi C_\phi C_{\xi,\theta} - C_{\xi,\theta} T_\psi C_\phi) z^k = 0$ which proves that $T_\psi C_\phi$ commutes with $C_{\xi,\theta}$.
Case (ii) : Let $b = 0$. Then $T_\psi C_\phi C_{\xi,\theta} z^k = C_{\xi,\theta} T_\psi C_\phi z^k$ if and only if $P(\psi(z) \cdot \mu \lambda^k (az)^k) = C_{\xi,\theta} P(\psi(z) \cdot (az)^k)$ if and only if $P(\sum_{n=-\infty}^{\infty} \hat{\psi}(n) \mu \lambda^k a^k z^{n+k}) = e^{i\xi} \sum_{n=-k}^{\infty} \overline{\hat{\psi}(n)}\,\overline{a}^k e^{-i(n+k)\theta} z^{n+k}$ if and only if $\sum_{n=-k}^{\infty} \hat{\psi}(n) \lambda^k a^k z^{n+k} = \sum_{n=-k}^{\infty} \overline{\hat{\psi}(n)}\,\overline{a}^k \lambda^{n+k} z^{n+k}$ if and only if $\hat{\psi}(n) a^k = \overline{\hat{\psi}(n)}\,\overline{a}^k \lambda^n$ for every $n \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$. □

Corollary 2.11. *Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n) z^n \in L^\infty$ and $\phi(z) = az + b$ $(a \neq 0)$ be a linear fractional transformation mapping $\mathbb{D}$ into itself. Then the Toeplitz composition operator $T_\psi C_\phi$ commutes with the conjugation $C_{0,0}$ on $\mathcal{H}^2$ if and only if for each $n \in \mathbb{Z}$ and $m, k \in \mathbb{N} \cup \{0\}(0 \leq m \leq k)$, we have:*

(i) *$\hat{\psi}(n)a^m b^{k-m} \in \mathbb{R}$ if $b \neq 0$, and*
(ii) *$\hat{\psi}(n)a^k \in \mathbb{R}$ if $b = 0$.*

The following theorem is in general valid for any linear operator T on a Hilbert space $\mathcal{H}$ which is complex symmetric with respect to any conjugation C defined on $\mathcal{H}$ such that T commutes with C.

Theorem 2.12. *Let $\psi \in L^\infty$ and let ϕ be any self-analytic mapping from $\mathbb{D}$ into itself. Suppose that $T_\psi C_\phi$ is a complex symmetric operator with conjugation $C_{\xi,\theta}$ on $\mathcal{H}^2$ and further, suppose that $T_\psi C_\phi$ commutes with $C_{\xi,\theta}$. Then $T_\psi C_\phi$ is a normal operator on $\mathcal{H}^2$.*

Proof. By hypothesis, $T_\psi C_\phi$ is a complex symmetric operator with conjugation $C_{\xi,\theta}$ such that it commutes with $C_{\xi,\theta}$ which implies that $T_\psi C_\phi$ is a self-adjoint operator. That is,

$$(T_\psi C_\phi)^* = C_{\xi,\theta} T_\psi C_\phi C_{\xi,\theta} = C_{\xi,\theta} C_{\xi,\theta} T_\psi C_\phi = T_\psi C_\phi. \tag{10}$$

Hence, $T_\psi C_\phi$ is a normal operator on $\mathcal{H}^2$. □

Corollary 2.13. *Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n) z^n \in L^\infty$ and $\phi(z) = az + b$ $(a \neq 0)$ be a linear fractional transformation mapping $\mathbb{D}$ into itself. Suppose that $T_\psi C_\phi : \mathcal{H}^2 \to \mathcal{H}^2$ is a complex symmetric operator with conjugation $C_{0,0}$ and $\hat{\psi}(n)a^m b^{k-m} \in \mathbb{R}$ for each $n \in \mathbb{Z}$ and $m, k \in \mathbb{N} \cup \{0\}(0 \leq m \leq k)$. Then $T_\psi C_\phi$ is a normal operator on $\mathcal{H}^2$.*

Proof. From Corollary 2.11, we obtain that $T_\psi C_\phi$ commutes with the conjugation $C_{0,0}$ as $\hat{\psi}(n)a^m b^{k-m} \in \mathbb{R}$ for each $n \in \mathbb{Z}$ and $m, k \in \mathbb{N} \cup \{0\}(0 \leq m \leq k)$. Thus, we get that $T_\psi C_\phi$ is a normal operator on $\mathcal{H}^2$ by Theorem 2.12. □

3. Normality Of Toeplitz Composition Operators

In this section we discuss the normality of the Toeplitz composition operators on $\mathcal{H}^2$. We explore the conditions under which the operator $T_\psi C_\phi$ becomes normal and further we discover the necessary and sufficient conditions for the operator $T_\psi C_\phi$ to be Hermitian.

Theorem 3.1. *Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n) z^n \in L^\infty$ and $\phi(z) = az + b$ $(a \neq 0)$ be a linear fractional transformation mapping $\mathbb{D}$ into itself. Let the operator $T_\psi C_\phi$ on $\mathcal{H}^2$ be hyponormal. Then we have the following:*
(i) *If $b \neq 0$, then $\sum_{n=0}^{\infty}\{|\hat{\psi}(n)|^2 - \sum_{m=0}^{\infty}((\binom{m+n}{m})|\hat{\psi}(-n)||a|^n|b|^m)^2\} \geq 0$.*
(ii) *If $b = 0$, then $\sum_{n=0}^{\infty}\{|\hat{\psi}(n)|^2 - |\hat{\psi}(-n)|^2|a|^{2n}\} \geq 0$.*

Proof. By the hyponormality of $T_\psi C_\phi$ on $\mathcal{H}^2$, we have $\|T_\psi C_\phi f\|^2 \geq \|(T_\psi C_\phi)^* f\|^2$ for every $f \in \mathcal{H}^2$. In particular, on taking $f \equiv 1$, we obtain that

$$\|T_\psi C_\phi(1)\|^2 \geq \|(T_\psi C_\phi)^*(1)\|^2. \tag{11}$$

Then $\|T_\psi C_\phi(1)\|^2 = \|P(\sum_{n=-\infty}^{\infty} \hat{\psi}(n) z^n)\|^2 = \|\sum_{n=0}^{\infty} \hat{\psi}(n) z^n\|^2 = \sum_{n=0}^{\infty} |\hat{\psi}(n)|^2$. It can be noted that the function $\psi(z)$ can be expressed as

$$\psi(z) = \psi_+(z) + \psi_0(z) + \overline{\psi_-(z)}$$

where $\psi_+(z) = \sum_{n=1}^{\infty} \hat{\psi}(n) z^n$, $\psi_-(z) = \sum_{n=1}^{\infty} \overline{\hat{\psi}(-n)} z^n$ and $\psi_0(z) = \hat{\psi}(0)$. It follows that $P(\overline{\psi(z)}) = P(\overline{\psi_+(z)} + \overline{\psi_0(z)} + \psi_-(z)) = \sum_{n=0}^{\infty} \overline{\hat{\psi}(-n)} z^n$.

Let us first assume that $b \neq 0$. Since $C_{\phi}^{*} = M_g C_{\sigma}$ where $g(z) = (1 - \overline{b}z)^{-1}$ and $\sigma(z) = \frac{\overline{a}z}{1-\overline{b}z}$, it is obtained that

$$\begin{aligned}
\|(T_{\psi}C_{\phi})^{*}(1)\|^2 = \|C_{\phi}^{*}T_{\overline{\psi}}(1)\|^2 &= \|M_g C_{\sigma} P(\overline{\psi(z)})\|^2 \\
&= \|M_g C_{\sigma}(\sum_{n=0}^{\infty} \overline{\hat{\psi}(-n)} z^n)\|^2 \\
&= \|\sum_{n=0}^{\infty} \overline{\hat{\psi}(-n)} \frac{\overline{a}^n z^n}{(1-\overline{b}z)^{n+1}}\|^2 \\
&= \|\sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} \binom{m+n}{m} \overline{\hat{\psi}(-n)}\,\overline{a}^n \overline{b}^m z^{m+n}\Big)\|^2 \\
&= \sum_{n=0}^{\infty} \Big(\sum_{m=0}^{\infty} \Big(\binom{m+n}{m} |\hat{\psi}(-n)||a|^n |b|^m\Big)^2\Big).
\end{aligned}$$

Hence, it follows from (11) that $\sum_{n=0}^{\infty}\{|\hat{\psi}(n)|^2 - \sum_{m=0}^{\infty}(\binom{m+n}{m}|\hat{\psi}(-n)||a|^n|b|^m)^2\} \geq 0$.

If $b = 0$, then $C_{\phi}^{*} = C_{\sigma}$ where $\sigma(z) = \overline{a}z$. This implies that $\|(T_{\psi}C_{\phi})^{*}(1)\|^2 = \|C_{\sigma}T_{\overline{\psi}}(1)\|^2 = \sum_{n=0}^{\infty} |\hat{\psi}(-n)|^2|a|^{2n}$. Thus, from (11), we get that $\sum_{n=0}^{\infty}\{|\hat{\psi}(n)|^2 - |\hat{\psi}(-n)|^2|a|^{2n}\} \geq 0$. □

Corollary 3.2. *Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n)z^n \in L^{\infty}$ and $\phi(z) = az + b$ $(a \neq 0)$ be a linear fractional transformation mapping $\mathbb{D}$ into itself. Let the operator $T_{\psi}C_{\phi}$ on $\mathcal{H}^2$ be normal. Then we have the following:*

(i) *If $b \neq 0$, then $\sum_{n=0}^{\infty}\{|\hat{\psi}(n)|^2 - \sum_{m=0}^{\infty}(\binom{m+n}{m}|\hat{\psi}(-n)||a|^n|b|^m)^2\} = 0$.*

(ii) *If $b = 0$, then $\sum_{n=0}^{\infty}\{|\hat{\psi}(n)|^2 - |\hat{\psi}(-n)|^2|a|^{2n}\} = 0$.*

The condition obtained above in Corollary 3.2 is necessary but not sufficient which can be observed through the following example:

Example 3.3. *Let $\psi(z) = z + \overline{z}$ and $\phi(z) = iz$. Then, for $a = i$, $b = 0$, $\hat{\psi}(-1) = \hat{\psi}(1) = 1$ and $\hat{\psi}(-n) = \hat{\psi}(n) = 0$ where $n \in \mathbb{Z} - \{0\}$, the condition $\sum_{n=0}^{\infty}\{|\hat{\psi}(n)|^2 - |\hat{\psi}(-n)|^2|a|^{2n}\} = 0$ is satisfied. But the Toeplitz composition operator $T_{\psi}C_{\phi}$ is not normal as $(T_{\psi}C_{\phi})(T_{\psi}C_{\phi})^{*}(z) = z^3 + 2z$ whereas $(T_{\psi}C_{\phi})^{*}(T_{\psi}C_{\phi})(z) = -z^3 + 2z$.*

Next we investigate the necessary and sufficient conditions under which the operator $T_{\psi}C_{\phi}$ becomes Hermitian.

Theorem 3.4. *Let $\psi(z) = \sum_{n=-\infty}^{\infty} \hat{\psi}(n)z^n \in L^{\infty}$ and $\phi(z) = az + b$ $(a \neq 0)$ be a linear fractional transformation mapping $\mathbb{D}$ into itself. Then the Toeplitz composition operator $T_{\psi}C_{\phi}$ on $\mathcal{H}^2$ is Hermitian if and only if for each $k, p \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{Z}$, we have :*

(i) $\sum_{n=-k+p}^{p} \binom{k}{p-n}\hat{\psi}(n)a^{p-n}b^{n+k-p} = \sum_{n=-k}^{-k+p} \binom{p}{p-n-k}\overline{\hat{\psi}(-n)}\,\overline{a}^{n+k}\overline{b}^{p-n-k}$ *when $b \neq 0$*

and, ***(ii)*** $a^k\hat{\psi}(n) = \overline{a}^{n+k}\overline{\hat{\psi}(-n)}$ *when $b = 0$.*

Proof. Let us suppose that the operator $T_{\psi}C_{\phi}$ is Hermitian on $\mathcal{H}^2$. This implies that $T_{\psi}C_{\phi}z^k = (T_{\psi}C_{\phi})^{*}z^k$ for every $k \in \mathbb{N} \cup \{0\}$. Let us suppose $b \neq 0$. Since

$$\begin{aligned} T_\psi C_\phi z^k &= T_\psi(\phi(z))^k \\ &= P(\psi(z) \cdot \sum_{m=0}^{k} \binom{k}{m} a^m b^{k-m} z^m) \\ &= P(\sum_{m=0}^{k} (\sum_{n=-\infty}^{\infty} \binom{k}{m} \hat{\psi}(n) a^m b^{k-m} z^{m+n})) \\ &= \sum_{m=0}^{k} P(\sum_{n=-\infty}^{\infty} \binom{k}{m} \hat{\psi}(n) a^m b^{k-m} z^{m+n}) \\ &= \sum_{m=0}^{k} (\sum_{n=-m}^{\infty} \binom{k}{m} \hat{\psi}(n) a^m b^{k-m} z^{m+n}) \end{aligned}$$

and

$$\begin{aligned} (T_\psi C_\phi)^* z^k &= C_\phi^* T_{\overline{\psi}} z^k \\ &= C_\phi^* P(\sum_{n=-\infty}^{\infty} \overline{\hat{\psi}(-n)} z^{n+k}) \\ &= M_g C_\sigma(\sum_{n=-k}^{\infty} \overline{\hat{\psi}(-n)} z^{n+k}) \\ &= \sum_{n=-k}^{\infty} \overline{\hat{\psi}(-n)} \overline{a}^{n+k} \left(\frac{1}{1-\overline{b}z}\right)^{n+k+1} z^{n+k}) \\ &= \sum_{j=0}^{\infty} (\sum_{n=-k}^{\infty} \binom{n+k+j}{j} \overline{\hat{\psi}(-n)} \overline{a}^{n+k} \overline{b}^j z^{n+k+j}) \end{aligned}$$

where $g(z) = (1-\overline{b}z)^{-1}$ and $\sigma(z) = \frac{\overline{a}z}{1-\overline{b}z}$; it follows that the coefficient of z^p for $p \in \mathbb{N} \cup \{0\}$ in the expressions for $T_\psi C_\phi z^k$ and $(T_\psi C_\phi)^* z^k$ are equal for each $k \in \mathbb{N} \cup \{0\}$. Therefore, on comparing the coefficients of $1, z, z^2$, z^3 and so on in the expressions of $T_\psi C_\phi z^k$ and $(T_\psi C_\phi)^* z^k$, we obtain that for each $k, p \in \mathbb{N} \cup \{0\}$,

$$\sum_{n=-k+p}^{p} \binom{k}{p-n} \hat{\psi}(n) a^{p-n} b^{n+k-p} = \sum_{n=-k}^{-k+p} \binom{p}{p-n-k} \overline{\hat{\psi}(-n)} \overline{a}^{n+k} \overline{b}^{p-n-k}. \tag{12}$$

Conversely, let us assume that for each $k, p \in \mathbb{N} \cup \{0\}$, equation (12) holds. Then evaluating the expression $(T_\psi C_\phi - (T_\psi C_\phi)^*) z^k$ for each $k \in \mathbb{N} \cup \{0\}$ gives the value as zero. Hence, we obtain that the operator $T_\psi C_\phi$ is Hermitian on $\mathcal{H}^2$.

Now we take $b = 0$. Then it can be easily evaluated that $(T_\psi C_\phi - (T_\psi C_\phi)^*) z^k = 0$ if and only if $\sum_{n=-k}^{\infty} \hat{\psi}(n) a^k z^{n+k} - \sum_{n=-k}^{\infty} \overline{\hat{\psi}(-n)} \overline{a}^{n+k} z^{n+k} = 0$ if and only if $a^k \hat{\psi}(n) = \overline{a}^{n+k} \overline{\hat{\psi}(-n)}$ for every $n \in \mathbb{Z}$ and $k \in \mathbb{N} \cup \{0\}$. $\square$

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References

[1] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[2] S. R. Garcia and C. Hammond, Which weighted composition operators are complex symmetric?, in *Concrete operators, spectral theory, operators in harmonic analysis and approximation*, 171–179, Operator Theory: Advances and Applications, 236, Birkhäuser/Springer, Basel.

[3] S. R. Garcia and M. Putinar, Complex symmetric operators and applications, Transactions of the American Mathematical Society **358** (2006), no. 3, 1285–1315.

[4] S. R. Garcia and M. Putinar, Complex symmetric operators and applications. II, Transactions of the American Mathematical Society **359** (2007), no. 8, 3913–3931.

[5] A. Gupta and S. K. Singh, Toeplitz composition operators on the Fock space, Complex Variables and Elliptic Equations **64** (2019), no. 7, 1077–1092.

[6] P. R. Hurst, Relating composition operators on different weighted Hardy spaces, Archiv der Mathematik (Basel) **68** (1997), no. 6, 503–513.

[7] S. Jung, Y. Kim, E. Ko, and J. E. Lee, Complex symmetric weighted composition operators on $H^2(\mathbb{D})$, Journal of Functional Analysis **267** (2014), no. 2, 323–351.

[8] E. Ko and J. E. Lee, On complex symmetric Toeplitz operators, Journal of Mathematical Analysis and Applications **434** (2016), no. 1, 20–34.

[9] M. J. Martín and D. Vukotić, Adjoints of composition operators on Hilbert spaces of analytic functions, Journal of Functional Analysis **238** (2006), no. 1, 298–312.

[10] J. H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.