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Extrapolated Diagonal and Off-Diagonal Splitting Iteration Method

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Abstract. Recently, Dehghan et al. presented the diagonal and off-diagonal splitting (DOS) iteration method for solving the linear systems $\mathcal{A}x = b$ [3]. In this paper, we improve its convergence rate with extrapolation. Also convergence analysis of extrapolated DOS (EDOS) iterative method is studied by giving an upper bound of the extrapolation parameter, then consistency of EDOS and its optimal extrapolation parameter are discussed. Finally, several numerical examples are given to show the efficiency of the presented method.

1. Introduction

Solving the linear system

$$\mathcal{A}x = b, \tag{1}$$

is one of the most important problems in numerical analysis, where $\mathcal{A} \in \mathbb{C}^{n\times n}$ is a nonsingular matrix with non-vanishing diagonal entries, and $x, b \in \mathbb{C}^n$. In iteration method

$$x^{(p+1)} = \mathcal{T}x^{(p)} + \mathcal{C}, \qquad p = 0, 1, 2, ..., \tag{2}$$

for solving (1), we usually split the coefficient matrix $\mathcal{A}$, as

$$\mathcal{A} = \mathcal{E} + \mathcal{D} + \mathcal{F},$$

where $\mathcal{D}$ is a diagonal matrix, $\mathcal{E}$ is a strictly lower triangular matrix and $\mathcal{F}$ is a general matrix. In [3], authors introduced a new splitting iteration method for solving (1) based on the diagonal and off-diagonal splitting (DOS) iterative method, as follows:
The DOS iterative method: Given an initial guess $x^{(0)} \in \mathbb{C}^{n\times n}$ for $p = 0, 1, 2, ...$ until $\{x^{(p)}\}$ converges, compute the next iterate $x^{(p+1)}$ according to the following procedure:

$$\begin{cases} \mathcal{D}x^{(p+\frac{1}{2})} = [\theta_1\mathcal{D} + (\theta_1 - 1)\mathcal{E} + (\theta_1 - 1)\mathcal{F}]x^{(p)} + (1-\theta_1)b, \\ (\mathcal{D} + \theta_2\mathcal{E})x^{(p+1)} = [(1-\theta_2)\mathcal{D} - \theta_2\mathcal{F}]x^{(p+\frac{1}{2})} + \theta_2 b, \end{cases} \tag{3}$$

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where θ_1 and θ_2 are given constants. We can rewrite the DOS method as

$$x^{(p+1)} = \mathcal{T}(\theta_1, \theta_2)x^{(p)} + C(\theta_1, \theta_2)b,$$

where

$$\mathcal{T}(\theta_1, \theta_2) = (\mathcal{D} + \theta_2\mathcal{E})^{-1}[(1-\theta_2)\mathcal{D} - \theta_2\mathcal{F}]\mathcal{D}^{-1}[\theta_1\mathcal{D} + (\theta_1 - 1)\mathcal{E} + (\theta_1 - 1)\mathcal{F}],$$
$$C(\theta_1, \theta_2) = (\mathcal{D} + \theta_2\mathcal{E})^{-1}[(1-\theta_1)[(1-\theta_2)\mathcal{D} - \omega_2\mathcal{F}]\mathcal{D}^{-1} + \theta_2 I].$$

Formal manipulation reduces this to

$$\mathcal{T}(\theta_1, \theta_2) = I + Q(\theta_1, \theta_2) \tag{4}$$

where

$$Q(\theta_1, \theta_2) = \mathcal{D}^{-1}(\theta_1 - 1)\mathcal{A} - (\mathcal{D} + \theta_2\mathcal{E})^{-1}\theta_2\mathcal{A} - (\mathcal{D} + \theta_2\mathcal{E})^{-1}\mathcal{D}^{-1}\theta_2(\theta_1 - 1)\mathcal{A}^2. \tag{5}$$

Several methods have been devised to accelerate the convergence of the iterative process (2). One of the most powerful method is the extrapolation. That is defined by

$$x^{(p+1)} = \beta(\mathcal{T}x^{(p)} + C) + (1-\beta)x^{(p)}, \qquad p = 0, 1, 2, ..., \tag{6}$$

where $\beta \in \mathbb{R} - \{0\}$ is the extrapolation parameter. The extrapolated method converges independently of whether the original iteration method is convergent or not. The iteration matrix of method (6) is

$$\mathcal{T}_\beta = \beta\mathcal{T} + (1-\beta)I, \tag{7}$$

so, Eq.(6) is rewritten by

$$x^{(p+1)} = \mathcal{T}_\beta x^{(p)} + C_\beta, \qquad p = 0, 1, 2, ..., \tag{8}$$

where $C_\beta = \beta C$. We obtain a range for the parameter β so that (8) be faster than (2). Also, we determine the optimum β, say β^*, which minimizes the spectral radius of $\mathcal{T}_\beta$.

In [11], Missirlis and Evans defined extrapolated GS method ($E\mathcal{GS}$) and extrapolated $\mathcal{SOR}$ ($E\mathcal{SOR}$) method for the numerical solution of linear systems. Also, Evans [4] introduced extrapolated ($\mathcal{AOR}$) ($E\mathcal{AOR}$). Hadjidimos [6] studied determining an optimum value of extrapolation parameter for complex systems. Yeyios [15] derived ranges for the extrapolation parameter and Cao [2] obtained convergence conditions for extrapolation methods. Song and Wang [12–14] presented the sufficient and necessary conditions for semi-convergence of the extrapolated iterative methods for singular problems, also they obtained the upper bounds and optimum extrapolation parameter. Recently, in order to improve the efficiency of the MHSS iteration method, Zeng and Zhang [19] presented the CMHSS method for solving both singular and nonsingular complex systems.

In this work, we extrapolate the DOS iterative method, called EDOS iterative method. Our new method accelerates the convergence of the DOS iterative method. We study some theories about convergence of the extrapolated iteration method and its optimum parameter. We find the upper bound for extrapolation parameter β, such that the spectral radius of the DOS iteration matrix is greater than the our method. Finally, the EDOS is compared with the DOS method. By numerical experiments and theoretic analysis, we conclude that the proposed method is superior to some existence methods.

We write $\Lambda(\mathcal{A})$ *and* I_n to denote the spectrum of the matrix $\mathcal{A}$ and identity matrix of size $n \times n$, respectively. We denote the Kronecker product of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A} \otimes \mathcal{B} = [a_{ij}\mathcal{B}]$.

The rest of the paper is organized as follows. Section 2, is the preliminaries. In Section 3, we describe the EDOS method. Section 4, is devoted to discuss about the convergence and consistency of the EDOS method. Numerical results are discussed in section 5. Finally, in Section 6 some conclusions are given.

2. Preliminaries

In this section we give some definitions and results which are utilized during the paper.

Definition 2.1. *An $n \times n$ matrix $\mathcal{A} = (a_{ij})$ is said to be strictly diagonally dominant (SDD) if*

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad for\ i = 1, ..., n.$$

Definition 2.2. *The matrix $\mathcal{A}$ is called a $\mathcal{Z}$-matrix if $a_{ij} \leq 0$ for $i, j = 1, 2, 3, ..., n$ $(i \neq j)$. A $\mathcal{Z}$-matrix with positive diagonal elements is named an $\mathcal{L}$-matrix.*

Definition 2.3. *Let $\mathcal{A}$ be an $\mathcal{L}$-matrix. Then the matrix $\mathcal{A}$ is an $\mathcal{M}$-matrix if $\mathcal{A}$ is nonsingular and $\mathcal{A}^{-1} \geq 0$.*

Definition 2.4. *A complex matrix $\mathcal{A} = (a_{ij})$ is an $\mathcal{H}$-matrix, if its comparison matrix $< \mathcal{A} >= (m_{ij})$ is an $\mathcal{M}$-matrix, where $m_{ii} = |a_{ii}|$ and $m_{ij} = -|a_{ij}|$, $i \neq j$.*

Definition 2.5. *Let $\mathcal{A}$ be a real matrix. Then $\mathcal{A} = \mathcal{M} - \mathcal{N}$ is called a splitting of $\mathcal{A}$ if $\mathcal{M}$ is a nonsingular matrix, the splitting is called $\mathcal{M}$-splitting if $\mathcal{M}$ is a nonsingular $\mathcal{M}$-matrix and $\mathcal{N} > 0$.*

Lemma 2.6. *Let $\mathcal{A} = \mathcal{M} - \mathcal{N}$ be an $\mathcal{M}$-splitting of $\mathcal{A}$. Then $\rho(\mathcal{M}^{-1}N) < 1$ if and only if $\mathcal{A}$ is a nonsingular $\mathcal{M}$-matrix.*

Definition 2.7. *[5] The splitting $\mathcal{A} = \mathcal{M} - \mathcal{N}$ is called an $\mathcal{H}$-splitting if $< \mathcal{M} > -|\mathcal{N}|$ is an $\mathcal{M}$-matrix.*

Lemma 2.8. *[5] If $\mathcal{A} = \mathcal{M} - \mathcal{N}$ be an $\mathcal{H}$-splitting, then $\mathcal{A}$ and $\mathcal{M}$ are $\mathcal{H}$-matrices and $\rho(\mathcal{M}^{-1}\mathcal{N}) \leq \rho(< \mathcal{M} >^{-1} |\mathcal{N}|) < 1$.*

Lemma 2.9. *[5] Let $\mathcal{A} \in \mathbb{R}^{n\times n}$. If $\mathcal{A}$ is an $\mathcal{H}$-matrix, then $|\mathcal{A}^{-1}| \leq < \mathcal{A} >^{-1}$.*

Theorem 2.10. *If $\mathcal{A}$ be an $\mathcal{H}$-matrix, then Gauss-Seidel method converges for any initial guess $x^{(0)}$.*

Proof. *Let $\mathcal{A}$ be an $\mathcal{H}$-matrix. Consider the Gauss-Seidel splitting of $\mathcal{A} = D - \mathcal{E} - \mathcal{F}$. Let $\mathcal{M} = \mathcal{D} - \mathcal{E}$ and $\mathcal{N} = \mathcal{F}$. Suppose $< \mathcal{A} >$ be the comparison matrix of $\mathcal{A}$, so that $< \mathcal{A} >$ is an $\mathcal{M}$-matrix. Note that $< \mathcal{A} >= |\mathcal{D}| - |\mathcal{E}| - |\mathcal{F}|$, so $\rho((|\mathcal{D}| - |\mathcal{E}|)^{-1}|\mathcal{F}|) < 1$. Hence by Lemma 2.9, we have*

$$|(\mathcal{D} - \mathcal{E})^{-1}\mathcal{F}|) \leq |(\mathcal{D} - \mathcal{E})^{-1}||\mathcal{F}| \leq < \mathcal{D} - \mathcal{E} >^{-1} |\mathcal{F}| = (\mathcal{D} - |\mathcal{E}|)^{-1}|\mathcal{F}|.$$

Then

$$\rho((\mathcal{D} - \mathcal{E})^{-1}\mathcal{F}) \leq \rho(|(\mathcal{D} - \mathcal{E})^{-1}\mathcal{F}|) \leq \rho((\mathcal{D} - |\mathcal{E}|)^{-1}|\mathcal{F}|) < 1.$$

This shows that, $\rho((\mathcal{D} - \mathcal{E})^{-1}\mathcal{F}) < 1$, and GS method converges. □

The proof of the following theorem is similar to Theorem 2.10.

Theorem 2.11. *If $\mathcal{A}$ be an $\mathcal{H}$-matrix, then Jacobi method is converge for any initial guess $x^{(0)}$.*

Corollary 2.12. *[4] If the matrix $\mathcal{A}$ is a nonsingular $\mathcal{H}$-matrix, the $E\mathcal{AOR}$ method converges if*

$$0 < r < \frac{2}{1 + \rho(|\mathcal{E}| + |\mathcal{F}|)} \quad and \quad r^2 < \omega^2 < \frac{2r}{1 + \rho(|\mathcal{E}| + |\mathcal{F}|)}.$$

Theorem 2.13. *[16] If $\mathcal{A}$ is a strictly diagonally dominant matrix by rows, the Jacobi and Gauss-Seidel methods are convergent.*

Theorem 2.14. *[8] If $\mathcal{A}$ is a strictly diagonally dominant matrix, then a sufficient condition for the convergence of the $\mathcal{AOR}$ method is*

$$0 < r < \frac{2}{1 + \max_i(e_i + f_i)} \quad and \quad 0 < \omega < \frac{2r}{1 + \rho(L_{r,r})},$$

where e_i, f_i are respectively, the sum of the absolute values of the ith row of $\mathcal{E}$ and $\mathcal{F}$, respectively. Also, $L_{r,r}$ is the iteration matrix $\mathcal{AOR}$ method.

Theorem 2.15. *[8] If $\mathcal{A}$ is a strictly diagonally dominant matrix, the $\mathcal{SOR}$ method is converges if*

$$0 < \omega < \frac{2}{1 + \max_i(e_i + f_i)},$$

where e_i, f_i are respectively the i-row sums of the absolute of the entries of $\mathcal{E}$ and $\mathcal{F}$, respectively.

Theorem 2.16. *[7] Let $\mathcal{A}$ be an $\mathcal{H}$-matrix. Then $\mathcal{AOR}$ method is convergent.*

Theorem 2.17. *[17] Let $\mathcal{A} \in \mathbb{R}^{n\times n}$ be nonsingular. Then there exist nonsingular matrices P and Q such that $P\mathcal{A}Q$ is diagonally dominant.*

Theorem 2.18. *[17] Let $\mathcal{A} \in \mathbb{R}^{n\times n}$ be nonsingular. Then there exists nonsingular matrices P such that $P\mathcal{A}$ is diagonally dominant.*

Corollary 2.19. *[17] The Jacobi, the Gauss-Seidel, the $\mathcal{SOR}$ and the $\mathcal{AOR}$ methods are convergent for all nonsingular linear systems in the sense of preconditioned version.*

3. The EDOS iterative method

In this section, we extrapolate DOS iterative method and obtain the following EDOS iterative method as follows.

The EDOS iterative method: Given an initial guess $x^{(0)}$ for $p = 0, 1, 2, ...$ until $\{x^{(p)}\}$ converges, compute the next iterate $x^{(p+1)}$ according to the following procedure:

$$\begin{cases} \mathcal{D}x^{(p+\frac{1}{2})} = [\theta_1\mathcal{D} + (\theta_1 - 1)\mathcal{E} + (\theta_1 - 1)\mathcal{F}]x^{(p)} + (1 - \theta_1)b, \\ (\mathcal{D} + \theta_2\mathcal{E})\acute{x}^{(p+1)} = [(1 - \theta_2)\mathcal{D} - \theta_2\mathcal{F}]x^{(p+\frac{1}{2})} + \theta_2 b, \\ x^{(p+1)} = (1 - \beta)x^{(p)} + \beta\acute{x}^{(p+1)}, \end{cases} \tag{9}$$

where $\beta \in \mathbb{R} - \{0\}$ is called the extrapolation parameter.

At each step of the EDOS iteration method, we require solutions of two systems whose coefficient matrices are $\mathcal{D}$ and $\mathcal{D} + \theta_2\mathcal{E}$. The first linear subsystem is easy to implement since $\mathcal{D}$ is a diagonal matrix, and in the second system, it is a lower triangular matrix we can use the forward substitution methods. In matrix-vector form, the above EDOS iterative method can be rewritten as

$$x^{(p+1)} = \mathcal{T}(\beta, \theta_1, \theta_2)x^{(p)} + \mathcal{G}(\beta, \theta_1, \theta_2)b,$$

where

$$\mathcal{T}(\beta, \theta_1, \theta_2) = (1 - \beta)I + \beta\mathcal{T}(\theta_1, \theta_2) \quad and \quad \mathcal{G}(\beta, \theta_1, \theta_2) = \beta C(\theta_1, \theta_2). \tag{10}$$

When $\mathcal{F}$ is strictly upper triangular matrix, we observe that for specific values of the parameters θ_1, θ_2 the EDOS iterative reduces to extrapolated well-known methods, for instance:

$\mathcal{T}(\beta, 0, 0)$ is the iteration matrix of the extrapolated Jacobi (EJ) method,

$\mathcal{T}(\beta, 1, 1)$ is the iteration matrix of the extrapolated Gauss-Seidel ($E\mathcal{GS}$) method. We will use the numerical example 5.4 to show that the $E\mathcal{GS}$ and EDOS ($\theta_1 = 1, \theta_2 = 1$) methods are the same.

$\mathcal{T}(\beta, 1 - \theta_1, 0)$ is the iteration matrix of the extrapolated Simultaneous Over-relaxation method,

$\mathcal{T}(\beta, 1, free)$ is the iteration matrix of the extrapolated Successive Over-relaxation ($E\mathcal{SOR}$) method.

4. Convergence analysis and consistency of the EDOS method

In this section, we discuss the consistency of EDOS iterative method and indicate that the EDOS iterative method converges to the unique solution of the system (1).

Theorem 4.1. *[16] If $\mathcal{A}$ is a nonsingular matrix, then the iteration method (8) is consistent with (1) if and only if $C = (I - \mathcal{T})\mathcal{A}^{-1}b$.*

Theorem 4.2. *[16] If $\mathcal{A}$ is a nonsingular matrix, then the iteration method (8) is completely consistent with (1) if and only if it is consistent and $(I - \mathcal{T})$ is nonsingular.*

Since

$$\begin{aligned}(I - \mathcal{T}(\beta, \theta_1, \theta_2))\mathcal{A}^{-1}b &= (I - (1-\beta)I - \beta\mathcal{T}(\theta_1, \theta_2))\mathcal{A}^{-1}b = \beta(I - \mathcal{T}(\theta_1, \theta_2))\mathcal{A}^{-1}b \\ &= \beta C(\theta_1, \theta_2) = \mathcal{G}(\beta, \theta_1, \theta_2),\end{aligned}$$

and

$$\det(I - \mathcal{T}(\beta, \theta_1, \theta_2)) = \det(I - (1-\beta)I - \beta\mathcal{T}(\theta_1, \theta_2)) = \beta^n \det(I - \mathcal{T}(\theta_1, \theta_2)) \neq 0.$$

It follows that the EDOS iterative method is completely consistent with the system (1).

Lemma 4.3. *[1]. Let $\mathcal{A} \in \mathbb{C}^n$, $\mathcal{A} = \mathcal{M}_i - \mathcal{N}_i$ $(i = 1, 2)$ are two splitting of the matrix $\mathcal{A}$, and let $x^{(0)} \in \mathbb{C}^n$ be a given initial vector. If $\{x^{(p)}\}$ is a two-step iteration sequence defined by*

$$\begin{cases} \mathcal{M}_1 x^{(p+\frac{1}{2})} = \mathcal{N}_1 x^{(p)} + b, \\ \mathcal{M}_2 x^{(p+1)} = \mathcal{N}_2 x^{(p+\frac{1}{2})} + b, \quad p = 0, 1, 2, ..., \end{cases}$$

then

$$x^{(p+1)} = \mathcal{M}_2^{-1}\mathcal{N}_2\mathcal{M}_1^{-1}\mathcal{N}_1 x^{(p)} + \mathcal{M}_2^{-1}(I + \mathcal{N}_2\mathcal{M}_1^{-1})b, \quad p = 0, 1, 2, \dots .$$

Moreover, if the spectral radius $\rho(\mathcal{M}_2^{-1}\mathcal{N}_2\mathcal{M}_1^{-1}\mathcal{N}_1) < 1$, then the iterative sequence $\{x^{(p)}\}$ converges to the unique solution $x^{(*)} \in \mathbb{C}^n$ of the system of linear equations (1) for all initial vectors $x^{(0)} \in \mathbb{C}^n$.

Theorem 4.4. *Let $\gamma = \Re(\gamma) + i\Im(\gamma)$, ν are eigenvalues of $Q(\theta_1, \theta_2)$ and $\mathcal{T}(\theta_1, \theta_2)$, respectively. Iteration of scheme (3) is convergent for $0 \le \theta_1 \le 1$, $0 < \theta_2 \le 1$, if and only if $-1 < \Im(\gamma) < 1$ and $-1 - \sqrt{1 - \Im(\gamma)^2} < \Re(\gamma) < -1 + \sqrt{1 - \Im(\gamma)^2}$.*

Proof. From (4), we have $\nu = 1 + \gamma = 1 + \Re(\gamma) + i\Im(\gamma)$, so $|\nu|^2 = (1 + \Re(\gamma))^2 + (\Im(\gamma))^2 = \Re(\gamma)^2 + 2\Re(\gamma) + \Im(\gamma)^2 + 1$. Since $|\nu| \le \rho(\mathcal{T}(\theta_1, \theta_2)) < 1$ thus $|\nu| < 1$, therefore we have $-1 - \sqrt{1 - \Im(\gamma)^2} < \Re(\gamma) < -1 + \sqrt{1 - \Im(\gamma)^2}$ and $-1 \le \Im(\gamma) \le 1$. □

Corollary 4.5. *Let $0 \le \theta_1 \le 1$ and $0 < \theta_2 \le 1$. If the real parts of the eigenvalues of matrix $Q(\theta_1, \theta_2)$ defined in (5) are all greater than zero, then iteration scheme (3) is divergent.*

Theorem 4.6. *[3] Let $\mathcal{A}$ be a strictly diagonally dominant, $0 \le \theta_1 \le 1$ and $0 < \theta_2 \le 1$, $\mathcal{E} = (e_{ij})$, $\mathcal{F} = (f_{ij})$ and $e_{ij}f_{ij} \ge 0$, then the DOS iterative method is convergent to the exact solution of the linear system (1).*

Theorem 4.7. *[3] Suppose $\mathcal{A}$ be an $\mathcal{H}$-matrix, $\mathcal{A} = \mathcal{E} + \mathcal{D} + \mathcal{F}$, where $\mathcal{D}$ is a diagonal matrix, $\mathcal{E}$ is a strictly lower triangular matrix, $\mathcal{F}$ is a general matrix, and $0 \le \theta_1 \le 1$, $0 < \theta_2 \le 1$, then the DOS iterative method is convergent to the exact solution of the linear system (1).*

Theorem 4.8. *If the DOS method converges, then the EDOS iterative method is convergent to the exact solution of the linear system (1) or equivalently, $\rho(\mathcal{T}(\beta,\theta_1,\theta_2)) < 1$ if and only if*

$$0 < \beta < \min_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{\frac{-2\Re(\gamma)}{\Re(\gamma)^2+\Im(\gamma)^2}\}, \quad where\ \gamma = \Re(\gamma) + i\Re(\gamma).$$

Proof. *Let λ is an eigenvalue of $\mathcal{T}(\beta,\theta_1,\theta_2)$, it follows from Eq.(10) that*

$$\lambda = 1-\beta+\beta\nu = 1-\beta+\beta(1+\Re(\gamma)+i\Im(\gamma)) = 1+\beta\Re(\gamma)+i\beta\Im(\gamma),$$

which implies that

$$|\lambda|^2 = (1+\beta\Re(\gamma))^2 + (\beta\Im(\gamma))^2 = \beta^2\Re(\gamma)^2 + 2\beta\Re(\gamma) + \beta^2\Im(\gamma)^2 + 1.$$

To get $\rho(\mathcal{T}(\beta,\theta_1,\theta_2)) < 1$, it is enough $|\lambda| < 1$, therefore,

$$\beta^2\Re(\gamma)^2 + 2\beta\Re(\gamma) + \beta^2\Im(\gamma)^2 < 0. \tag{11}$$

We distinguish two cases according to whether β is less or greater than zero.
Case I: Assuming that $\beta > 0$ using (11), we have

$$\beta\Re(\gamma)^2 + 2\Re(\gamma) + \beta\Im(\gamma)^2 < 0,$$

we obtain

$$\beta < \frac{-2\Re(\gamma)}{\Re(\gamma)^2+\Im(\gamma)^2},$$

since $\Re(\gamma) < 0$, we have $0 < \beta < \frac{-2\Re(\gamma)}{\Re(\gamma)^2+\Im(\gamma)^2}$. Hence, $\rho(\mathcal{T}(\beta,\theta_1,\theta_2)) < 1$ if and only if

$$0 < \beta < \min_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{\frac{-2\Re(\gamma)}{\Re(\gamma)^2+\Im(\gamma)^2}\}.$$

Case II: Let $\beta < 0$, then (11) gives

$$\beta\Re(\gamma)^2 + 2\Re(\gamma) + \beta\Im(\gamma)^2 > 0.$$

We have $\beta > \frac{-2\Re(\gamma)}{\Re(\gamma)^2+\Im(\gamma)^2}$, which contradicts to the Theorem 4.4. So there is no convergence for $\beta < 0$. □

Theorem 4.9. *Let conditions of Theorem 4.8 hold. If β^* is optimum extrapolated parameter, then $\beta^* = \frac{-2}{\Re_{\max}(\gamma)+\Re_{\min}(\gamma)}$, where $\Re_{\max}(\gamma) = \max_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{\Re(\gamma)\}$, and $\Re_{\min}(\gamma) = \min_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{\Re(\gamma)\}$.*

Proof. *Suppose λ is an eigenvalue of $\mathcal{T}(\beta,\theta_1,\theta_2)$, it follows from Eq. (10) that*

$$\lambda = 1-\beta+\beta\nu = 1-\beta+\beta(1+\Re(\gamma)+i\Im(\gamma)) = 1+\beta\Re(\gamma)+i\beta\Im(\gamma),$$

then we have

$$\begin{aligned}\rho(\mathcal{T}(\beta,\theta_1,\theta_2))^2 &= \max_{\lambda\in\Lambda(\mathcal{T}(\beta,\theta_1,\theta_2))}\{|\lambda|^2\} = \max_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{|1+\beta\Re(\gamma)|^2 + |\beta\Im(\gamma)|^2\}\\ &\le \max_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}|1+\beta\Re(\gamma)|^2 + \max_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\beta^2|\Im(\gamma)|^2,\end{aligned} \tag{12}$$

where $\Lambda(\mathcal{T}(\beta,\theta_1,\theta_2))$ is the spectrum of matrix $\mathcal{T}(\beta,\theta_1,\theta_2)$. There exists $\beta^* > 0$, such that

$$\max_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{|1+\beta\Re(\gamma)|\} = \begin{cases} 1+\beta\Re_{max}(\gamma) & if \quad 0 < \beta \le \beta^*,\\ -1-\beta\Re_{min}(\gamma) & if \quad \beta \ge \beta^*,\end{cases}$$

where $1+\beta^*\mathfrak{R}_{\max}(\gamma) = -1-\beta^*\mathfrak{R}_{\min}(\gamma)$*, therefore* $\beta^* = \frac{-2}{\mathfrak{R}_{\max}(\gamma)+\mathfrak{R}_{\min}(\gamma)}$*. That* β^* *is optimum extrapolated parameter. Thus, from (12) we have*

$$\rho(\mathcal{T}(\beta,\theta_1,\theta_2))^2 \le \begin{cases} (1+\beta\mathfrak{R}_{\max}(\gamma))^2+\beta^2\mathfrak{I}_{max}(\gamma)^2 & if \quad 0<\beta\le\beta^*, \\ (-1-\beta\mathfrak{R}_{\min}(\gamma))^2+\beta^2\mathfrak{I}_{max}(\gamma)^2 & if \quad \beta\ge\beta^*. \end{cases}$$

With defining the functions $f_1(\beta)$ *and* $f_2(\beta)$ *with*

$$f_1(\beta) = (1+\beta\mathfrak{R}_{max}(\gamma))^2+\beta^2\mathfrak{I}_{\max}(\gamma)^2 \quad and \quad f_2(\beta) = (-1-\beta\mathfrak{R}_{min}(\gamma))^2+\beta^2\mathfrak{I}_{\max}(\gamma)^2,$$

we have

$$\min_\beta \rho(\mathcal{T}(\beta,\theta_1,\theta_2))^2 \le \begin{cases} \min_\beta f_1(\beta) & if \quad 0<\beta\le\beta^*, \\ \min_\beta f_2(\beta) & if \quad \beta\ge\beta^*. \end{cases}$$

In order to find $\min_{0<\beta\le\beta^*} f_1(\beta)$ *and* $\min_{\beta\ge\beta^*} f_2(\beta)$*, after differentiating the functions* $f_1(\beta), f_2(\beta)$*, we obtain that* $f_1'(\beta)\ge 0$ *iff* $\beta\ge\beta_1$ *and* $f_2'(\beta)\ge 0$ *iff* $\beta\ge\beta_2$ *where*

$$\beta_1 = \frac{-\mathfrak{R}_{\max}(\gamma)}{\mathfrak{R}_{\max}(\gamma)^2+\mathfrak{I}_{\max}(\gamma)^2} \quad and \quad \beta_2 = \frac{-\mathfrak{R}_{\min}(\gamma)}{\mathfrak{R}_{\min}(\gamma)^2+\mathfrak{I}_{\max}(\gamma)^2}.$$

Therefore

$$\min_\beta \rho(T(\beta,\theta_1,\theta_2)) \le \begin{cases} \frac{\mathfrak{I}_{\max}(\gamma)}{\sqrt{\mathfrak{R}_{\max}(\gamma)^2+\mathfrak{I}_{\max}(\gamma)^2}} & if \quad 0<\beta\le\beta^*, \\ \\ \frac{\mathfrak{I}_{\max}(\gamma)}{\sqrt{\mathfrak{R}_{\min}(\gamma)^2+\mathfrak{I}_{\max}(\gamma)^2}} & if \quad \beta\ge\beta^*, \end{cases}$$

where $\mathfrak{I}_{\max}(\gamma) = \max_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{\mathfrak{I}(\gamma)\}$. □

Theorem 4.10. *If* $\rho(\mathcal{T}(\theta_1,\theta_2))>1$*, then we have the following statements.*
Case I: If $\mathfrak{R}(\gamma)>0$ *for all* $\gamma\in\Lambda(Q(\theta_1,\theta_2))$*, then* $\max_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{\frac{-2\mathfrak{R}(\gamma)}{\mathfrak{R}(\gamma)^2+\mathfrak{I}(\gamma)^2}\}<\beta<0$.
Case II: If $\mathfrak{R}(\gamma)<0$ *for all* $\gamma\in\Lambda(Q(\theta_1,\theta_2))$*, then* $0<\beta<\min_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{\frac{-2\mathfrak{R}(\gamma)}{\mathfrak{R}(\gamma)^2+\mathfrak{I}(\gamma)^2}\}$.

Proof. *Similar to the proof of Theorem 4.8, we have*

$$\beta^2\mathfrak{R}(\gamma)^2+2\beta\mathfrak{R}(\gamma)+\beta^2\mathfrak{I}(\gamma)^2<0. \tag{13}$$

Let $\beta<0$*, then (13) gives* $\beta\mathfrak{R}(\gamma)^2+2\mathfrak{R}(\gamma)+\beta\mathfrak{I}(\gamma)^2>0$*, so when* $\mathfrak{R}(\gamma)>0$ *for all* $\gamma\in\Lambda(Q(\theta_1,\theta_2))$*, we can write* $\frac{-2\mathfrak{R}(\gamma)}{\mathfrak{R}(\gamma)^2+\mathfrak{I}(\gamma)^2}<\beta<0$*, therefore we have* $\max_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{\frac{-2\mathfrak{R}(\gamma)}{\mathfrak{R}(\gamma)^2+\mathfrak{I}(\gamma)^2}\}<\beta<0$.
Now, Let $\beta>0$*, then (13) gives* $\beta\mathfrak{R}(\gamma)^2+2\mathfrak{R}(\gamma)+\beta\mathfrak{I}(\gamma)^2<0$*, so when* $\mathfrak{R}(\gamma)<0$ *for all* $\gamma\in\Lambda(Q(\theta_1,\theta_2))$*, we can write* $0<\beta<\frac{-2\mathfrak{R}(\gamma)}{\mathfrak{R}(\gamma)^2+\mathfrak{I}(\gamma)^2}$*, therefore we will have* $0<\beta<\min_{\gamma\in\Lambda(Q(\theta_1,\theta_2))}\{\frac{-2\mathfrak{R}(\gamma)}{\mathfrak{R}(\gamma)^2+\mathfrak{I}(\gamma)^2}\}$. □

Theorem 4.11. *Let* $\mathcal{A}$ *be an* $\mathcal{H}$*-matrix. Then sufficient condition for* $\rho(\mathcal{T}(\beta,0,0))<1$ *is*

$$0<\beta<\min_{\gamma\in\Lambda(Q(0,0))}\{\frac{-2\mathfrak{R}(\gamma)}{\mathfrak{R}(\gamma)^2+\mathfrak{I}(\gamma)^2}\}.$$

Proof. *Since* $\mathcal{A}$ *is an* $\mathcal{H}$*-matrix, therefore Theorem 2.11 implies that Jacobi method converges. So from Theorem 4.8 the proof is complete.* □

Theorem 4.12. *Let* $\mathcal{A}$ *be an* $\mathcal{H}$*-matrix. Then sufficient condition for* $\rho(\mathcal{T}(\beta,1,1))<1$ *is*

$$0<\beta<\min_{\gamma\in\Lambda(Q(1,1))}\{\frac{-2\mathfrak{R}(\gamma)}{\mathfrak{R}(\gamma)^2+\mathfrak{I}(\gamma)^2}\}.$$

Proof. *Since $\mathcal{A}$ is an $\mathcal{H}$-matrix, therefore Theorem 2.11 implies that Gauss-Seidel method converges. This result, together with Theorem 4.8 proves the validity of the theorem.* □

Theorem 4.13. *Let $\mathcal{A}$ be a strictly diagonally dominant matrix. Assume that $0 \le \theta_1 \le 1$ and $0 < \theta_2 \le 1$, then extrapolated successive over-relaxation $E\mathcal{SOR}$ method converges or equivalently, $\rho(\mathcal{T}(\beta, 1, free)) < 1$ if*

$$0 < \beta < \min_{\gamma \in \Lambda(Q(1,free))} \{ \frac{-2\Re(\gamma)}{\Re(\gamma)^2 + \Im(\gamma)^2} \}.$$

Theorem 4.14. *Let $\mathcal{A}$ be an $\mathcal{H}$-matrix. Then sufficient condition for $\rho(\mathcal{T}(\beta, 1-\theta_1, 0)) < 1$ is*

$$0 < \beta < \min_{\gamma \in \Lambda(Q(1-\theta_1,0))} \{ \frac{-2\Re(\gamma)}{\Re(\gamma)^2 + \Im(\gamma)^2} \}.$$

Using the singular value decomposition we can convert a nonsingular matrix $\mathcal{A}$ to a strictly diagonally dominant matrix. We can find nonsingular matrices P and Q using the $\mathcal{SVD}$ decomposition such that $P\mathcal{A}Q$ is strictly diagonally dominant [17, 18]. Also, Yuan in Theorem 2.18 showed that there exists a nonsingular matrix P such that $P\mathcal{A}$ is strictly diagonally dominant.

As said in [3], the DOS iteration method converges unconditionally when $\mathcal{A}$ is strictly diagonally dominant, for $0 \le \theta_1 \le 1$ and $0 < \theta_2 \le 1$. It is obvious that after finding P and Q such that $P\mathcal{A}Q$ is strictly diagonally dominant, instead of solving (1) we can solve $P\mathcal{A}Qy = Pb,\ x = Qy$.

5. Numerical experiments

In this section, we will use four examples to exhibit the effectiveness of our method. We also compare the performance of the EDOS method with the DOS method from the point of view of the iteration counts (denoted as "IT"), CPU times (denoted as "CPU") and the spectral radius (denoted as "ρ"). We denote "β^*", the optimal value β. The numerical experiments were computed in double precision in Matlab R2016b on a PC computer with Intel(R) Core (TM) i7-7700k CPU 4.20GHz, 8.00 GB memory with machine precision and Windows 10 operating system. In our implementations, the initial guess $x^{(0)}$ is chosen zero vector. In all examples, the stopping criterion is $\frac{\|b-\mathcal{A}x^{(p)}\|_2}{\|b\|_2} < 10^{-5}$.
In our tests, we take $h = \frac{1}{m+1}$, $n = m^2$, $\theta_1 = 0.25$, $\theta_2 = 1$ and for the tests reported in this section, $\mathcal{F}$ is strictly upper triangular matrix.

Example 5.1. *[3] Consider the linear system*

$$(\pi\mathcal{K}_V + \mathcal{K}_H)x = b,$$

where $\mathcal{K}_V$ and $\mathcal{K}_H$ are the viscous and hysteretic damping matrices, respectively. Here, $\mathcal{K}_V = 10I_n$, $\mathcal{K}_H = 0.02\mathcal{W}$, $\mathcal{W} = I_m \otimes \mathcal{V}_m + \mathcal{V}_m \otimes I_m$, $\mathcal{V}_m = h^{-2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$. We take $b = (-\pi^2 I_n + \mathcal{W} + \pi\mathcal{K}_V + \mathcal{K}_H)B$, where $B = (1, 1, ..., 1)^T$.

Table 1: Numerical results of Example 5.1

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	DOS			EDOS			
m	IT	CPU	$\rho(T(\theta_1,\theta_2))$	β^*	IT	CPU	$\rho(T(\beta,\theta_1,\theta_2))$
10	4	0.00039	0.03080	1.008	4	0.00034	0.0231
20	7	0.0009	0.1935	1.08	6	0.0008	0.1290
30	12	0.0028	0.4010	1.18	9	0.0017	0.2932
40	17	0.0058	0.5661	1.28	12	0.0034	0.4445
50	24	0.130	0.6808	1.48	15	0.0088	0.5276

Example 5.2. *[3] Consider the linear system*

$$(I_m \otimes V_m + V_m \otimes I_m)x = b,$$

where $V_m = tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$*. We take the right-hand side vector to be*

$$b = [10(I_m \otimes V_c + V_c \otimes I_m) + 9(e_1 e_m^T + e_m e_1^T) \otimes I_m - (I_m \otimes V_m + V_m \otimes I_m)]B,$$

where $V_c = V_m - e_1 e_m^T - e_m e_1^T \in \mathbb{R}^{m\times m}$*,* e_1 *and* e_m *are the first and the last unit vectors in* $\mathbb{R}^m$*, respectively, and* $B = (1, 1, ..., 1)^T$*.*

Table 2: Numerical results of Example 5.2.

	DOS			EDOS			
m	IT	CPU	$\rho(T(\theta_1, \theta_2))$	β^*	IT	CPU	$\rho(T(\beta, \theta_1, \theta_2))$
10	86	0.0057	0.8938	1.75	47	0.0032	0.8141
20	280	0.0302	0.9697	1.91	146	0.0167	0.9421
30	568	0.09798	0.9860	1.95	290	0.0515	0.9727
40	940	0.2410	0.9920	1.96	478	0.1333	0.9842
50	1391	0.5853	0.9948	1.97	705	0.2842	0.9898

Example 5.3. *[3] Letting*

$$(\mathcal{W} + \frac{3-\sqrt{3}}{h} I)x = b,$$

where $\mathcal{W} \in \mathbb{R}^{n\times n}$*,* $\mathcal{W} = I_m \otimes V_m + V_m \otimes I_m$*, with* $V_m = h^{-2}\, tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}$*.* $b = [b_s]$ *where*

$$b_s = \frac{s}{\tau(s+1)^2}, \; s = 1, 2, ..., n.$$

Table 3: Numerical results of Example 5.3

	DOS			EDOS			
m	IT	CPU	$\rho(T(\theta_1, \theta_2))$	β^*	IT	CPU	$\rho(T(\beta, \theta_1, \theta_2))$
10	57	0.0052	0.8280	1.71	31	0.0022	0.7059
20	140	0.0157	0.9308	1.85	73	0.0081	0.8720
30	229	0.0427	0.9589	1.86	121	0.0223	0.9235
40	319	0.0843	0.9712	1.87	168	0.0473	0.9462
50	408	0.1672	0.9780	1.88	207	0.0882	0.9571

In three examples, we compare the EDOS iteration method with the DOS iteration method. Numerical results for examples 1-3 are listed in Tables 1-3, respectively. We observe that the performance of EDOS iterative method is better than the DOS iteration method from the point of view of spectral radius, iteration numbers and CPU time.

Example 5.4. *[10] Consider linear equations (1) with Hermitian positive definite coefficient matrix, where* $\mathcal{A} = 0.1\pi I_n + 0.02\mathcal{K}_n$ *with* $\mathcal{K}_n = I_m \otimes F_m + F_m \otimes I_m$ *where* $F_m = (m+1)^2\, tridiag(-1, 2, 1) \in \mathbb{R}^{m\times m}$*, and* $b = (1, 1, ..., 1)^T$*.*

Numerical result for example (5.4) is listed in Table 4, it shows that the iteration numbers and CPU times with $EG\mathcal{S}$ and EDOS ($\theta_1 = 1, \theta_1 = 1$) are the same.

Table 4: Numerical results of Example 5.4

m		GS	EGS	DOS ($\theta_1 = 1$, $\theta_2 = 1$)	EDOS ($\theta_1 = 1$, $\theta_2 = 1$)	DOS ($\theta_1 = 0.25$, $\theta_2 = 1$)	EDOS ($\theta_1 = 0.25$, $\theta_2 = 1$)
40	IT	1294	20	1294	20	10	7
	CPU	1.2534	0.0019	0.1003	0.0019	0.0010	0.0007
	ρ	0.9895	0.4921	0.9895	0.4921	0.3790	0.2659
	β^*	——	0.75	——	0.75	——	0.8740
60	IT	2861	20	2861	20	10	7
	CPU	17.864	0.007	0.4660	0.007	0.0020	0.0014
	ρ	0.9952	0.4964	0.9952	0.4964	0.3809	0.2680
	β^*	——	0.75	——	0.75	——	0.8740
80	IT	5043	20	5043	20	10	7
	CPU	100.6645	0.0061	0.8313	0.0061	0.0043	0.0026
	ρ	0.9973	0.4980	0.9952	0.4980	0.3816	0.2688
	β^*	——	0.75	——	0.75	——	0.8740

6. Conclusions

We construct a new method, called the EDOS iterative method which obtained from the combination of DOS splitting iteration method and the extrapolation method for solving the nonsingular linear system. We demonstrate that EDOS method converges to the unique solution of (1). An upper bound for the extrapolation parameter is derived. We have compared the numerical results of the EDOS iterative method with the DOS iteration method. Numerical results show that the EDOS method is superior to the DOS method in terms of the iteration counts, the CPU times and spectral radius.

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