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Some Results of Reverses Young's Inequalities

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Abstract. In this paper, we present some refinements of reverse Young's inequalities. Among other results, a refinement of reverse operator Young inequalities says

$$A\nabla_v B + 2\lambda(A\nabla B - A\sharp B) \le \frac{m\nabla_\lambda M}{m\sharp_\lambda M} A\sharp_v B,$$

where $0 < mI \le A, B \le MI$, $\lambda = \min\{v, 1-v\}$ and $v \in [0,1]$, extending a key result in [J. Math. Anal. Appl. 465 (2018) 267-280] and [Linear Multilinear Algebra 67 (2019) 1567-1578]. Furthermore, we give a reverse of Young's inequalities due to [Math. Slovaca 70 (2020), 453-466]. Moreover, we give a generalization of reverse Young-type inequality, and we also show a new Young-type inequality which is either better or not uniformly better than the main results in [Rocky Mountain J. Math. 46 (2016), 1089-1105]. As applications of these results, we obtain some inequalities for operators, Hilbert-Schmidt norms, unitarily invariant norms and determinants.

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and let $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. A self adjoint operator A is said to be positive if $\langle Ax, x\rangle \ge 0$ for all $x \in \mathcal{H}$, while it is said to be strictly positive if A is positive and invertible. As usual, we say that $A > B$ when $A - B > 0$ and $A \ge B$ when $A - B \ge 0$, respectively.

Moreover, $\mathbb{M}_n$ denotes the space of all $n \times n$ complex matrices. The unitarily invariance of the $\|\cdot\|_u$ on $\mathbb{M}_n$ means that $\|UAV\|_u = \|A\|_u$ for any $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. For $A = [a_{ij}] \in \mathbb{M}_n$, the Hilbert-Schmidt norm of A is defined by $\|A\|_2^2 = \sum\limits_{i,j=1}^{n} |a_{ij}|^2$, it is well know that $\|\cdot\|_2$ is unitarily invariant. The singular values of A, that is, the eigenvalues of the positive semi-definite matrix $|A| = (A^*A)^{\frac{1}{2}}$, is denoted by $s_j(A)$, $j = 1, 2, \cdots, n$, and arranged in a non-increasing order.

In addition, the Kantorovich constant and the Specht's ratio are defined by

$$K(h) = \frac{(h+1)^2}{4h} \text{ for } h > 0 \qquad \text{and} \qquad S(h) = \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\log\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0,1) \cup (1,\infty), \\ 1 & \text{if } h = 1. \end{cases}$$

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For the notations adopted in this paper, we denote v-weighted operator arithmetic mean (AM) and geometric mean (GM) as follows

$$A\nabla_v B = (1-v)A + vB \quad \text{and} \quad A\sharp_v B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}},$$

where $A, B \in \mathbb{B}(\mathcal{H})$ are strictly positive operators and $v \in \mathbb{R}$. Denoted by $A\nabla B$ and $A\sharp B$ for brevity respectively when $v = \frac{1}{2}$ for the sake of our convenience.

The operator Young's inequality or v-weighted AM-GM mean inequalities states that

$$A\sharp_v B \leq A\nabla_v B, \tag{1}$$

where $A, B \in \mathbb{B}(\mathcal{H})$ are strictly positive operators and $v \in [0,1]$. Refining this inequality has taken great attention of a considerable number of researchers in this field.

Furuichi [6] and Tominaga [21] obtained respectively a refinement and reverse with Specht's ratio

$$S((h')^r)A\sharp_v B \leq A\nabla_v B \leq S(h)A\sharp_v B, \tag{2}$$

where $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$, $h' = \frac{M'}{m'}$, $h = \frac{M}{m}$, $r = \min\{v, 1-v\}$ and $v \in [0,1]$.

Later, Zuo et. al. [25] and Liao et. al. [13] proved that if $A, B \in \mathbb{B}(\mathcal{H})$ are such that $0 < mI \leq A \leq m'I < M'I \leq B \leq MI$ or $0 < mI \leq B \leq m'I < M'I \leq A \leq MI$, then

$$K(h')^r A\sharp_v B \leq A\nabla_v B \leq K(h)^R A\sharp_v B, \tag{3}$$

where $h' = \frac{M'}{m'}$, $h = \frac{M}{m}$, $r = \min\{v, 1-v\}$, $R = \max\{v, 1-v\}$ and $v \in [0,1]$. The authors [25] also showed $S(h^r) \leq K(h)^r$ for $h > 0$, which implies the first inequality in (3) is better than the first one of (2).

Recently, Furuichi et. al. [8] and Gümüş et. al. [9] showed

$$A\nabla_v B \leq \frac{m\nabla_\lambda M}{m\sharp_\lambda M} A\sharp_v B, \tag{4}$$

where $0 < mI \leq A, B \leq MI$, $\lambda = \min\{v, 1-v\}$ and $v \in [0,1]$. Furthermore, they [9] also explained that (4) is better than the corresponding one in (3) and the constant $\frac{m\nabla_\lambda M}{m\sharp_\lambda M}$ is best possible.

Very recently, Beiranvand and Ghazanfari [4] gave a new refinements of (1), which reads as

$$\left(K(h^{2^{-n}})\right)^{\lambda_n} A\sharp_v B \leq \sum_{i=0}^{2^n-1}\left[(i+1-2^n v)A\sharp_{2^{-n}i}B + (2^n v - i)A\sharp_{2^{-n}(i+1)}B\right]\chi_{A_{n,i}} \leq A\nabla_v B, \tag{5}$$

where $n \in \mathbb{N}\cup 0$, $v \in [0,1]$, A, B are two invertible positive operators in $\mathbb{B}(\mathcal{H})$ and h is a positive real number such that either $A < hB \leq B$ or $A > hB \geq B$, and $\lambda_n = \sum_{i=0}^{2^n-1}\min\{i+1-2^n v, 2^n v - i\}\chi_{A_{n,i}}$.

Moreover, Bakherad, Krnić and Moslehian [3] presented a reverse Young-type inequality

$$(1-v)a + vb \leq a^{1-v}b^v, \tag{6}$$

where $a, b > 0$ and $v \in (-\infty, 0) \cup (1, \infty)$. In the same paper, the authors [3] also showed

$$(1-v)a + vb - v(\sqrt{a} - \sqrt{b})^2 \leq a^{1-v}b^v, \tag{7}$$

where $a, b > 0$ and $v \in (-\infty, 0) \cup (\frac{1}{2}, \infty)$. It is clear that the inequality (7) can be regarded as a refinement of (6) when $v \in (-\infty, 0)$.

For more information about Young's inequalities, we refer the readers to [1, 2, 7, 10–12, 14–20, 22–24] and references therein.

In this paper, we shall present a further refinement of reverse operator Young inequality (4). Moreover, we also give some reverses of the first inequalities in (5). In addition, we show a new Young-type inequality, which is either better or not uniformly better than (7) under the same conditions. In the end of this paper, we give a generalization of the inequality (6). As applications of these results, we obtain some inequalities for Hilbert-Schmidt norms, unitarily invariant norms and determinants.

2. Main results

First of all, we present a further refinement of reverse AM-GM operator inequality (4).

Theorem 2.1. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be such that $0 < mI \le A, B \le MI$ for some scalars $0 < m < M$. Then*

$$A\nabla_v B + 2\lambda(A\nabla B - A\sharp B) \le \frac{m\nabla_\lambda M}{m\sharp_\lambda M} A\sharp_v B,$$

where $\lambda = \min\{v, 1-v\}$ and $v \in [0,1]$.

Proof. For the sake of our convenience, we define

$$f(x) = \frac{(1-v) + vx + \lambda(\sqrt{x}-1)^2}{x^v},$$

where $v \in [0,1]$ and $x > 0$. Then we have

$$f'(x) = \begin{cases} v(\sqrt{x}-1)((2-2v)\sqrt{x}+1)x^{-v-1} & \text{for } 0 \le v \le \frac{1}{2}, \\ (1-v)(\sqrt{x}-1)(\sqrt{x}+2v)x^{-v-1} & \text{for } \frac{1}{2} \le v \le 1. \end{cases}$$

So, $f'(x) > 0$ when $x > 1$ and $f'(x) < 0$ when $0 < x < 1$. Therefore, $f(x) \le \max\{f(\frac{m}{M}), f(\frac{M}{m})\}$. We now compare between $f(\frac{m}{M})$ and $f(\frac{M}{m})$, letting

$$g(h) = \frac{(1-v) + vh + \lambda(\sqrt{h}-1)^2}{h^v} - \frac{(1-v) + vh^{-1} + \lambda(\sqrt{h^{-1}}-1)^2}{h^{-v}}.$$

Direct calculations show that

$$g'(h) = \begin{cases} v(\sqrt{h}-1)\Big((2-2v)h^{\frac{3}{2}} + h - (2-2v)h^{2v} - h^{2v+\frac{1}{2}}\Big)h^{-v-2} & \text{for } 0 \le v \le \frac{1}{2}, \\ (1-v)(\sqrt{h}-1)\Big(h^{\frac{3}{2}} + 2vh - 2vh^{2v+\frac{1}{2}} - h^{2v}\Big)h^{-v-2} & \text{for } \frac{1}{2} \le v \le 1. \end{cases}$$

Without lose of generality, we may assume $h = \frac{M}{m} > 1$.

Putting $s_1(v) = (2-2v)h^{\frac{3}{2}} + h - (2-2v)h^{2v} - h^{2v+\frac{1}{2}}$ and $s_2(v) = h^{\frac{3}{2}} + 2vh - 2vh^{2v+\frac{1}{2}} - h^{2v}$.

Then we have $s_1'(v) = -2h^{\frac{3}{2}} + 2h^{2v} - 2(2-2v)h^{2v}\log h - 2h^{2v+\frac{1}{2}}\log h$, and $s_1''(v) = 4h^{2v}\Big(2(1-\log h) + (2v - \sqrt{h})\log h\Big)\log h < 0$ when $0 \le v \le \frac{1}{2}$ and $h > 1$. So $s_1'(v) < s_1'(0) = -2h^{\frac{3}{2}} + 2 - 4\log h - 2\sqrt{h}\log h < 0$. Then we have $s_1(v) > s_1(\frac{1}{2}) = 0$; Similarly, we get $s_2'(v) = 2h - 2h^{2v+\frac{1}{2}} - 4vh^{2v+\frac{1}{2}}\log h - 2h^{2v}\log h < 0$ when $\frac{1}{2} \le v \le 1$ and $h > 1$, we have $s_2(v) < s_2(\frac{1}{2}) = 0$.

That is $g'(h) > 0$ when $0 \le v \le \frac{1}{2}$ and $g'(h) < 0$ when $\frac{1}{2} \le v \le 1$. Therefore, we have $g(h) > g(1) = 0$ when $0 \le v \le \frac{1}{2}$, and $g(h) < g(1) = 0$ when $\frac{1}{2} \le v \le 1$. It follows that

$$\max_{x \in [\frac{m}{M}, \frac{M}{m}]} f(x) = \begin{cases} \frac{m\nabla_v M}{m\sharp_v M} & \text{for } 0 \le v \le \frac{1}{2}, \\ \\ \frac{M\nabla_v m}{M\sharp_v m} & \text{for } \frac{1}{2} \le v \le 1. \end{cases}$$

Which is equivalent to saying

$$(1-v) + vx + \lambda(x + 1 - 2\sqrt{x}) \le \frac{m\nabla_\lambda M}{m\sharp_\lambda M} x^v, \tag{8}$$

putting $0 < \frac{m}{M}I \le x = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \le \frac{M}{m}I$, we can get Theorem 2.1 by (8) with a standard functional calculus. $\square$

Putting $x = \left(\frac{b}{a}\right)^2$ in (8), we can get the following corollary.

Corollary 2.1. *Let* $0 < m \leq a, b \leq M$, $\lambda = \min\{v, 1-v\}$ *and* $v \in [0,1]$. *Then we have*

$$\Big((1-v)a + vb\Big)^2 + (\lambda + v - v^2)(a-b)^2 \leq \Big(\frac{(m^2)\nabla_\lambda(M^2)}{(m\sharp_\lambda M)^2}\Big)(a^{1-v}b^v)^2. \tag{9}$$

In the next result, we give inequality (9) for Hilbert-Schmidt norms.

Theorem 2.2. *Let* $A, B, X \in \mathbb{M}_n$ *be such that* $0 < mI \leq A, B \leq MI$. *We have*

$$\|(1-v)AX + vXB\|_2^2 + (\lambda + v - v^2)\|AX - XB\|_2^2 \leq \Big(\frac{(m^2)\nabla_\lambda(M^2)}{(m\sharp_\lambda M)^2}\Big)\|A^{1-v}XB^v\|_2^2,$$

where $\lambda = \min\{v, 1-v\}$ *and* $v \in [0,1]$.

Proof. Since A, B are positive definite matrices, it follows by spectral theorem that there exist unitary matrices $U, V \in M_n$, such that $A = U\Lambda_1 U^*$ and $B = V\Lambda_2 V^*$, where $\Lambda_1 = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$ and $\Lambda_2 = diag(\nu_1, \nu_2, \cdots, \nu_n)$ for λ_i, ν_i are eigenvalues of A, B respectively, so $\lambda_i, \nu_i > 0$, $i = 1, 2, \cdots, n$. Let $Y = U^*XV = [y_{il}]$. Then $(1-v)AX + vXB = U[(1-v)\Lambda_1 Y + vY\Lambda_2]V^* = U[((1-v)\lambda_i + v\nu_l)y_{il}]V^*$ and $A^{1-\nu}XB^\nu = U[(\lambda_i^{1-v}\nu_l^v)y_{il}]V^*$. By (9) and the unitarily invariance of the Hilbert-Schmidt norm, we have

$$\begin{aligned}
\Big(\frac{(m^2)\nabla_\lambda(M^2)}{(m\sharp_\lambda M)^2}\Big)\|A^{1-v}XB^v\|_2^2 &= \Big(\frac{(m^2)\nabla_\lambda(M^2)}{(m\sharp_\lambda M)^2}\Big)\sum_{i,l=1}^n \Big(\lambda_i^{1-v}\nu_l^v\Big)^2 |y_{il}|^2 \\
&= \sum_{i,l=1}^n \Big[\Big(\frac{(m^2)\nabla_\lambda(M^2)}{(m\sharp_\lambda M)^2}\Big)(\lambda_i^{1-v}\nu_l^v)^2\Big]|y_{il}|^2 \\
&\geq \sum_{i,l=1}^n \Big[\Big((1-v)\lambda_i + v\nu_l\Big)^2 + (\lambda + v - v^2)(\lambda_i - \nu_l)^2\Big]|y_{il}|^2 \\
&= \sum_{i,l=1}^n \Big((1-v)\lambda_i + v\nu_l\Big)^2 |y_{il}|^2 + (\lambda + v - v^2)\sum_{i,l=1}^n (\lambda_i - \nu_l)^2 |y_{il}|^2 \\
&= \|(1-v)AX + vXB\|_2^2 + (\lambda + v - v^2)\|AX - XB\|_2^2.
\end{aligned}$$

Here we complete the proof of Theorem 2.2. □

Next, we show a unitarily invariant norm inequality involving operator monotone functions.

Theorem 2.3. *Let* $0 < mI \leq A, B \leq MI$ *and* $f : [0, \infty) \to [0, \infty)$ *be an operator monotone function. Then for every unitarily invariant norm* $\|\cdot\|_u$, *we have*

$$\frac{\|f(A)\sharp_v f(B)\|_u}{\|A\sharp_v B\|_u} \leq \frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big\|\frac{f(A\sharp_v B)}{A\sharp_v B}\Big\|_u,$$

where $\lambda = \min\{v, 1-v\}$ *and* $v \in [0,1]$.

Proof. Compute

$$\begin{aligned}
\frac{\|f(A)\sharp_v f(B)\|_u}{\|A\sharp_v B\|_u} &\leq \frac{\|\frac{m\nabla_\lambda M}{m\sharp_\lambda M} f(A\sharp_v B)\|_u}{\|A\sharp_v B\|_u} \\
&\leq \Big\|\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\frac{f(A\sharp_v B)}{A\sharp_v B}\Big\|_u \\
&= \frac{m\nabla_\lambda M}{m\sharp_\lambda M}\Big\|\frac{f(A\sharp_v B)}{A\sharp_v B}\Big\|_u,
\end{aligned}$$

where the first inequality is by Theorem 2 in [9], and the second one is due to the submultiplicativity property of unitarily invariant norm. □

Next, we present inequalities (8) for determinants.

Theorem 2.4. *Let $A, B \in \mathbb{M}_n$ be such that $0 < mI \le A, B \le MI$. Then we have*

$$\det(A\nabla_v B) + \lambda^n \det(2A\nabla B - 2A\sharp B) \le \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^n \det(A\sharp_v B),$$

where $\lambda = \min\{v, 1-v\}$ and $v \in [0,1]$.

Proof. It is a fact that the determinant of a positive definite matrix is product of its singular values, by (8), we obtain

$$\begin{aligned}\left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}\right)^n \det(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v &= \det\left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v\right)\\ &= \prod_{j=1}^n s_j\Big(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v\Big)\\ &= \prod_{j=1}^n \left(\frac{m\nabla_\lambda M}{m\sharp_\lambda M}s_j^v(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)\\ &\ge \prod_{j=1}^n \left((1-v) + vs_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) + \lambda\Big(s_j^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) - 1\Big)^2\right)\\ &\ge \prod_{j=1}^n \left((1-v) + vs_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right) + \prod_{j=1}^n \left(\lambda\Big(s_j^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) - 1\Big)^2\right)\\ &= \det\left((1-v)I_n + vA^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) + \lambda^n \det\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} - I_n\right)^2.\end{aligned}$$

Multiplying $\det(A^{\frac{1}{2}})$ on both sides of inequalities above, we can get Theorem 2.4 directly, as desired. □

Next, we give some reverses operator Young's inequalities (5) as promised. Before that, we list a lemma due to Dragomir [5].

Lemma 2.1. *For $i = 1, 2, \cdots, n$, we consider $p_i > 0$ with $\sum_{i=1}^n p_i = 1$. If f is a convex function on a fixed closed interval I, then*

$$\sum_{i=1}^n p_i f(x_i) - f\Big(\sum_{i=1}^n p_i x_i\Big) \le n\lambda\Big[\sum_{i=1}^n \frac{1}{n} f(x_i) - f\Big(\frac{1}{n}\sum_{i=1}^n x_i\Big)\Big],$$

where $\lambda = \max\{p_1, p_2, \cdots, p_n\}$.

If we take $f(x) = -\log x$ in Lemma 2.1, then we obtain the following corollary.

Corollary 2.2. *Let $x_i \in I \subset [a,b]$ and $\sum_{i=1}^n p_i = 1$ with $0 < a < b$, $p_i > 0$. Then we have*

$$\frac{\sum_{i=1}^n p_i x_i}{\prod_{i=1}^n x_i^{p_i}} \le \left(\frac{\frac{1}{n}\sum_{i=1}^n x_i}{\prod_{i=1}^n x_i^{\frac{1}{n}}}\right)^{n\lambda},$$

where $\lambda = \max\{p_1, p_2, \cdots, p_n\}$ and $i = 1, 2, \cdots, n$.

Following an idea of Beiranvand and Ghazanfari [4], we suppose that f is a real convex function on $[0,1]$ and $n \in \mathbb{N} \cup 0$. Let $A_{0,0} = [0,1]$ and for $n = 1, 2, \cdots,\ i = 0, 1, 2, \cdots, 2^n - 1$,

$A_{n,i} = [2^{-n}i, 2^{-n}(i+1))$,

$f_n(v) = \sum_{i=0}^{2^n-1} \Big[(i+1-2^n v) f(2^{-n}i) + (2^n v - i) f(2^{-n}(i+1))\Big] \chi_{A_{n,i}}(v)$.

It can be easily shown that f_n is continuous on [0,1] for every $n \in \mathbb{N}$, and $\{f_n\}$ is a decreasing sequence that converges pointwise to f.

Theorem 2.5. *Let $A, B \in \mathbb{M}_n$ be such that either $0 < A < B < hA$ or $0 < hA < B < A$. Then*

$$\Big(K(h^{2^{-n}})\Big)^{\lambda_n} A \sharp_v B \geq \sum_{i=0}^{2^n-1} \Big[(i+1-2^n v) A \sharp_{2^{-n}i} B + (2^n v - i) A \sharp_{2^{-n}(i+1)} B\Big] \chi_{A_{n,i}}(v) \geq A \sharp_v B,$$

where $n \in \mathbb{N} \cup 0$, $v \in [0,1]$ and $\lambda_n = \sum_{i=0}^{2^n-1} \max\{i+1-2^n v, 2^n v - i\} \chi_{A_{n,i}}$.

Proof. Taking $f(v) = a^v b^{1-v}$, $x_1 = f(2^{-n}i)$, $x_2 = f(2^{-n}(i+1))$, $p_1 = i+1-2^n v$ and $p_2 = 2^n v - i$ when $v \in [2^{-n}i, 2^{-n}(i+1))$ in Corollary 2.2, we get

$$f(2^{-n}i)^{i+1-2^n v} f(2^{-n}(i+1))^{2^n v - i} \left(\frac{\frac{1}{2}\Big(f(2^{-n}i) + f(2^{-n}(i+1))\Big)}{f(2^{-n}i)^{\frac{1}{2}} f(2^{-n}(i+1))^{\frac{1}{2}}} \right)^{2\lambda_n} \geq (i+1-2^n v) f(2^{-n}i) + (2^n v - i) f(2^{-n}(i+1)).$$

By some complex and direct computations, we have

$$K(h)^{\lambda_n} a^v b^{1-v} \geq (i+1-2^n v) f(2^{-n}i) + (2^n v - i) f(2^{-n}(i+1)).$$

Moreover, using the well known Young's inequality, we also have

$$(i+1-2^n v) f(2^{-n}i) + (2^n v - i) f(2^{-n}(i+1)) \geq f(2^{-n}i)^{i+1-2^n v} f(2^{-n}(i+1))^{2^n v - i} = a^v b^{1-v}.$$

That is

$$K(h^{2^{-n}})^{\lambda_n} a^v b^{1-v} \geq (i+1-2^n v) a^{2^{-n}i} b^{1-2^{-n}i} + (2^n v - i) a^{2^{-n}(i+1)} b^{1-2^{-n}(i+1)} \geq a^v b^{1-v}. \tag{10}$$

Putting $b = 1$ in (10) and letting $X = A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$, then we have $hI > X > I$ or $0 < hI < X < I$. Since $K(h)$ is continuous and monotone increasing on $h \in (1, \infty)$ and decreasing when $h \in (0,1)$, so we can complete the proof easily with a standard functional calculus. □

Theorem 2.6. *Let $A, B, X \in \mathbb{M}_n$ be such that A, B are positive definite. Let $Sp(A) = \{\tau_1, \cdots, \tau_n\}$ be the spectrum of A, and $Sp(B) = \{\mu_1, \cdots, \mu_n\}$ be the spectrum of B. If $v \in [0,1]$, then*

$$\begin{aligned}
&K_n^{2\lambda_n} \|A^v X B^{1-v}\|_2^2 \\
&\geq \sum_{i=0}^{2^n-1} \|(i+1-2^n v) A^{2^{-n}i} X B^{1-2^{-n}i} + (2^n v - i) A^{2^{-n}(i+1)} X B^{1-2^{-n}(i+1)}\|_2^2 \chi_{A_{n,i}} \\
&\geq \|A^v X B^{1-v}\|_2^2,
\end{aligned}$$

where $K_n = \max\left\{\Big(K(\frac{\tau_i}{\mu_j})^{2^{-n}}\Big) : k, j = 1, \cdots, m\right\}$, and $\lambda_n = \sum_{i=0}^{2^n-1} \max\{i+1-2^n v, 2^n v - i\} \chi_{A_{n,i}}$.

Proof. Using the same technique as in Theorem 2.2, we can similarly get the proof of Theorem 2.6 with (10). So we omit the details for the sake of simplicity and unnecessary repetition of the article. □

In the next step, we shall give a new reverse Young-type inequality, which is either better or not uniformly better than (7) under the same conditions. Firstly, we show some scalars inequalities.

Theorem 2.7. *Let $a, b > 0$. If $v \in (-\infty, 0) \cup (\frac{1}{2}, \infty)$, then*

$$(1-v)^2 a + v^2 b - v^2(\sqrt{a} - \sqrt{b})^2 \leq v^{2v} a^{1-v} b^v. \tag{11}$$

Proof. Compute

$$\begin{aligned}(1-v)^2 a + v^2 b - v^2(\sqrt{a} - \sqrt{b})^2 &= (1-2v)a + 2v(v\sqrt{ab}) \\ &\leq a^{(1-2v)}(v\sqrt{ab})^{2v} \\ &= v^{2v} a^{1-v} b^v,\end{aligned}$$

where the inequality is by (6). □

Our next intention is to compare Theorem 2.7 with inequality (7). Taking
$f(v) = \left((1-v)^2 a + v^2 b - v^2(\sqrt{a} - \sqrt{b})^2\right) - \left((1-v)a + vb - v(\sqrt{a} - \sqrt{b})^2\right) = 2v(v-1)\sqrt{ab};$
$g(v) = v^{2v} a^{1-v} b^v - a^{1-v} b^v = \left(v^{2v} - 1\right) a^{1-v} b^v.$
Then we have
– $f(v) \geq 0$ and $g(v) \geq 0$ when $v \geq 1$;
– $f(v) < 0$ and $g(v) < 0$ when $v \in (\frac{1}{2}, 1)$;
– $f(v) > 0$ and $g(v) > 0$ when $v \in (-1, 0)$;
– $f(v) > 0$ and $g(v) = 0$ when $v = -1$;
– $f(v) > 0$ and $g(v) < 0$ when $v \in (-\infty, -1)$;

Obviously, inequality (11) is not uniformly better than (7) when $v \in (-1, 0) \cup (\frac{1}{2}, \infty)$, and inequality (11) is better than (7) when $v \in (-\infty, -1]$. However, the inequality (7) is less precise than (6) when $v \in (\frac{1}{2}, \infty)$. So we take $v \in (-\infty, 0)$ in the following discussion.

By a standard functional calculus, we can easily get the following operators mean inequality with Theorem 2.7.

Theorem 2.8. *Let A, B are two invertible positive operators in $\mathbb{B}(\mathcal{H})$ and $v \in (-\infty, 0)$. Then*

$$(1-v)^2 A + v^2 B - 2v^2(A \nabla B - A \sharp B) \leq v^{2v} A \sharp_v B.$$

Corollary 2.3. *Let $a, b > 0$ and $v \in (-\infty, 0)$. Then we have*

$$\left((1-v)a + vb\right)^2 - v^2(a-b)^2 \leq v^{2v}(a^{1-v} b^v)^2 + 2v(1-v)ab.$$

Proof. The proof come from Theorem 2.7 directly. □

Next, we give an inequality for Hilbert-Schmidt norms using Corollary 2.3.

Theorem 2.9. *Let $v \in (-\infty, 0)$ and let $A, B, X \in \mathbb{M}_n$ be such that $0 < A, B$. We have*

$$\|(1-v)AX + vXB\|_2^2 - v^2\|AX - XB\|_2^2 \leq v^{2v}\|A^{1-v} X B^v\|_2^2 + 2v(1-v)\|A^{\frac{1}{2}} X B^{\frac{1}{2}}\|_2^2.$$

Proof. Using the same technique as in Theorem 2.2, we can easily get the proof of Theorem 2.9. So we omit the details. □

Next, we give an inequality for determinant by Theorem 2.7.

Theorem 2.10. *Let $v \in (-\infty, 0)$ and let $A, B \in \mathbb{M}_n$ be such that $0 < B \leq 4A$. Then we have*

$$(1-v)^{2n} \det(B) + v^{2n} \det(2A \sharp B - B) \leq v^{2vn} \det(B \sharp_v A).$$

Proof. Taking $b = 1$ and $a = s_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$ in Theorem 2.7. Since $0 < B \leq 4A$, which means that $0 < s_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \leq 4$, so we have

$$\begin{aligned}
v^{2vn}\det(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v} &= \det\left(v^{2v}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-v}\right) \\
&= \prod_{j=1}^{n}\left(v^{2v}s_j^{1-v}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right) \\
&\geq \prod_{j=1}^{n}\left((1-v)^2 s_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) + v^2 - v^2\left(s_j^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) - 1\right)^2\right) \\
&= \prod_{j=1}^{n}\left((1-v)^2 s_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) + v^2\left(2s_j^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) - s_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)\right) \\
&\geq \prod_{j=1}^{n}\left((1-v)^2 s_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right) + \prod_{j=1}^{n}\left(v^2\left(2s_j^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) - s_j(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\right)\right) \\
&= (1-v)^{2n}\det\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) + v^{2n}\det\left(2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} - A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right).
\end{aligned}$$

Multiplying $\det(A^{\frac{1}{2}})$ on both sides of inequalities above, we get

$$v^{2vn}\det(B\sharp_v A) = v^{2vn}\det(A\sharp_{1-v}B) \geq (1-v)^{2n}\det(B) + v^{2n}\det(2A\sharp B - B),$$

as desired. □

In the end of this paper, we give a natural generalization of the inequality (6), which can be regarded as a reverse of the classical weighted arithmetic-geometric mean inequality

$$\sum_{i=1}^{n} p_i a_i \geq \prod_{i=1}^{n} a_i^{p_i},$$

where $a_i, p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$. Our generalization can be stated as follows.

Theorem 2.11. *Let $a_i > 0$ and let $p_i \notin [0,1]$ be such that $\sum_{i=1}^{n} p_i = 1$. Then we have*

$$\sum_{i=1}^{n} p_i a_i \leq \prod_{i=1}^{n} a_i^{p_i}, \tag{12}$$

where $p_i \geq p_j$ when $1 \leq i < j \leq n$, and with equality if $a_1 = a_2 = \cdots = a_n$.

Proof. We use mathematical induction to prove the validity of the theorem. First, suppose that the inequality (12) is true for all positive integer $n = k \geq 2$. That is $\sum_{i=1}^{k} p_i a_i \leq \prod_{i=1}^{k} a_i^{p_i}$, and so we only need to prove it holds for

$n = k + 1$. Thus,

$$\begin{aligned}
\prod_{i=1}^{k+1} a_i^{p_i} &= a_1^{p_1} \times a_2^{p_2} \times \cdots \times a_k^{p_k} \times a_{k+1}^{p_{k+1}} \\
&= \left(a_1^{\frac{p_1}{p_1+p_2+\cdots+p_k}} \times a_2^{\frac{p_2}{p_1+p_2+\cdots+p_k}} \times \cdots \times a_k^{\frac{p_k}{p_1+p_2+\cdots+p_k}}\right)^{p_1+p_2+\cdots+p_k} \times a_{k+1}^{p_{k+1}} \\
&\geq (p_1 + p_2 + \cdots + p_k)\left(a_1^{\frac{p_1}{p_1+p_2+\cdots+p_k}} \times a_2^{\frac{p_2}{p_1+p_2+\cdots+p_k}} \times \cdots \times a_k^{\frac{p_k}{p_1+p_2+\cdots+p_k}}\right) + p_{k+1}a_{k+1} \\
&\geq (p_1 + p_2 + \cdots + p_k)\left(\frac{p_1}{p_1 + p_2 + \cdots + p_k}a_1 + \cdots + \frac{p_k}{p_1 + p_2 + \cdots + p_k}a_k\right) + p_{k+1}a_{k+1} \\
&= p_1a_1 + p_2a_2 + \cdots + p_{k+1}a_{k+1} \\
&= \sum_{i=1}^{k+1} p_ia_i,
\end{aligned}$$

where the first inequality is by (6) and the second one is due to our assumption. We completed the proof. □

Remark 1. *We require $p_i \geq p_j$ when $1 \leq i < j \leq n$ in Theorem 2.11. This is because that we have to make sure $p_1 + p_2 + \cdots + p_k > 0$ in the process of our proof. It may be wrong of Theorem 2.11 without this restriction. For example, letting $n = 3, a_1 = 3, a_2 = 2, a_3 = 4, p_1 = -2, p_2 = 1.1, p_3 = 1.9$, then we have $a_1^{p_1}a_2^{p_2}a_3^{p_3} \approx 3.317 \leq 3.8 = p_1a_1 + p_2a_2 + p_3a_3$.*

Motivated by the inequality (7), we now give a further refinement of Theorem 2.11.

Theorem 2.12. *Let $a_i > 0$ and let $p_i \notin [0, 1]$ be such that $\sum_{i=1}^{n} p_i = 1$. If $p_i \geq p_j$ when $1 \leq i < j \leq n$ such that $p_i - p_j \notin [0, 1]$, then we have*

$$\sum_{i=1}^{n} p_ia_i - p_n\left(\sum_{i=1}^{n} a_i - n\prod_{i=1}^{n} a_i^{\frac{1}{n}}\right) \leq \prod_{i=1}^{n} a_i^{p_i},$$

with equality if $a_1 = a_2 = \cdots = a_n$.

Proof. Compute

$$\begin{aligned}
\sum_{i=1}^{n} p_ia_i - p_n\left(\sum_{i=1}^{n} a_i - n\prod_{i=1}^{n} a_i^{\frac{1}{n}}\right) &= \sum_{i=1}^{n}\left(p_i - p_n\right)a_i + np_n\prod_{i=1}^{n} a_i^{\frac{1}{n}} \\
&= (p_1 - p_n)a_1 + (p_2 - p_n)a_2 + \cdots + (p_{n-1} - p_n)a_{n-1} + np_n\left(\prod_{i=1}^{n} a_i^{\frac{1}{n}}\right) \\
&\leq a_1^{p_1-p_n} \times a_2^{p_2-p_n} \times \cdots \times a_{n-1}^{p_{n-1}-p_n} \times \left(\prod_{i=1}^{n} a_i^{\frac{1}{n}}\right)^{np_n} \\
&= \prod_{i=1}^{n} a_i^{p_i},
\end{aligned}$$

where the inequality is by (12). So we completed the proof. □

Remark 2. *Letting $a_1 = a$, $a_2 = b$, $p_1 = 1 - v$, $p_2 = v$, we can get the inequality (7) by Theorem 2.12 when $n = 2$ and $v < 0$.*

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