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Existence and Nonexistence of Nontrivial Solutions for a Class of p-Kirchhoff Type Problems with Critical Sobolev Exponent

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Abstract. In this work, by using variational methods we study the existence of nontrivial positive solutions for a class of p-Kirchhoff type problems with critical Sobolev exponent.

1. Introduction

In this paper, we consider the existence and nonexistence of nontrivial positive solution to the following p-Kirchhoff type problem with critical exponent

$$\begin{cases} -M\left(\|u\|^p\right)\Delta_p u = u^{p^*-1} + \lambda f(x,u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad (\mathcal{P}_\lambda)$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 3$, $1 < p < N$, λ a real parameter, $M : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ and $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}^+$ is a continuous function with $f(x,t) = 0$ for all $t \leq 0$. The operator Δ_p is the p-Laplacian one that is,

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)$$

$p^* = pN/(N-p)$ is the critical exponent of Sobolev embedding and $\|.\|$ is the usual norm in $W_0^{1,p}(\Omega)$ defined by

$$\|u\|^p = \int_\Omega |\nabla u|^p dx.$$

Kirchhoff type problems are often referred to as being nonlocal because of the presence of the term $M\left(\|u\|^p\right)$ which implies that the equation in $(\mathcal{P}_\lambda)$ is no longer a pointwise identity. In the case $p = 2$, it is analogous to the stationary version of equations that arise in the study of string or membrane vibrations, namely,

$$u_{tt} - M\left(\|u\|^2\right)\Delta u = g(x,u),$$

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where u denotes the displacement and $g(x,u)$ is the external force. Equations of this type were first proposed by Kirchhoff in 1883 to describe the transversal oscillations of a stretched string.

These problems serve also to model other physical phenomena as biological systems where u describes a process which depends on the average of itself (for example, population density).

In recent years, Kirchhoff type problems received much attention, mainly after the famous article of Lions [10]; they have been studied in many papers by using variational methods. Some interesting studies can be found in [1, 3, 5, 6, 7, 8, 10, 11].

The problem $(\mathcal{P}_\lambda)$ with $p = 2$ and without the nonlocal term $M(\|u\|^p)$ has been treated by Brezis and Nirenberg [4].

Recently D. Naimen generalized the results of [4] to the nonlocal problem $(\mathcal{P}_\lambda)$ with $N = 3$, $p = 2$, $M\left(\|u\|^2\right) = a + b\|u\|^2$, $a, b \geq 0$ and $a + b > 0$.

On the other hand, G. M. Figueiredo in [11] considered the problem $(\mathcal{P}_\lambda)$ with $p = 2$, he proved the existence of a positive solution and studied the asymptotic behavior of this solution when λ converges to infinity.

The p-Kirchhoff problem $(\mathcal{P}_\lambda)$ has been studied in [7] and [8], where the authors imposed a relation between f and M. In [8] the authors showed the existence of $\lambda^* > 0$ such that $(\mathcal{P}_\lambda)$ has a nontrivial solution for $\lambda > \lambda^*$ under the following conditions:

(F_1) $f(x,t) = o(|t|^{p-1})$ as $t \to 0$, uniformly for $x \in \Omega$.

(F_2) There exists $q \in (p, p^*)$ such that $\lim\limits_{|t|\to+\infty} \dfrac{f(x,t)}{|t|^{q-2}t} = 0$, uniformly for $x \in \Omega$.

(F_3) There exists $\theta \in (p/\sigma, p^*)$ such that $0 < \theta F(x,t) \leq t f(x,t)$ for all $x \in \Omega$ and $t \neq 0$, where $F(x,t) = \int_0^t f(x,s)\,ds$ and σ is given by (G_2) below.

(G_1) There exists $\alpha_0 > 0$ such that $M(t) \geq \alpha_0$ for all $t \geq 0$.

(G_2) There exists $\sigma > p/p^*$ such that $\widehat{M}(t) \geq \sigma M(t)t$ for all $t \geq 0$, where $\widehat{M}(t) = \int_0^t M(s)\,ds$.

On the other hand, under the conditions $(G_1) - (G_2)$ and

$(\tilde{F}_1)$ $f(x,u) \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $f(x,-u) = -f(x,u)$ for all $u \in \mathbb{R}$,

$(\tilde{F}_2)$ $\lim\limits_{|t|\to+\infty} \dfrac{f(x,t)}{|t|^{p^*-2}t} = 0$ uniformly for $x \in \Omega$,

$(\tilde{F}_3)$ $\lim\limits_{|t|\to0} \dfrac{f(x,t)}{t^{p/(\sigma-1)}} = \infty$ uniformly for $x \in \Omega$,

the authors in [7] showed the existence of $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, problem $(\mathcal{P}_\lambda)$ has a sequence of nontrivial solutions $\{u_n\}$ and $u_n \to 0$ as $n \to \infty$.

The goal of this paper is to study the p-Laplace problem $(\mathcal{P}_\lambda)$ without relation between f and M (σ appears in (G_2) (F_3) and $(\tilde{F}_3)$). We consider the existence and nonexistence of nontrivial positive solution. Moreover, we study the asymptotic behavior of the solution of problem $(\mathcal{P}_\lambda)$ when λ converges to infinity.

Before stating our results, we introduce the following conditions on f and M.

(M_1) M is increasing and $M(0) > 0$.

(f_1) $\lim\limits_{t\to0} \dfrac{f(x,t)}{t} = 0$ and $\lim\limits_{t\to+\infty} \dfrac{f(x,t)}{t^{p^*}} = 0$ uniformly on $x \in \Omega$,

(f_2) There exists a reel θ such that $p < \theta < p^*$ and

$$0 < \theta F(x,t) = \theta \int_0^t f(x,s)\,ds \leq t f(x,t), \text{ for all } x \in \Omega.$$

Our main results are the following.

Theorem 1.1. *Assume that Ω is a star-shaped domain in $\mathbb{R}^N$, M satisfies (M_1) and $f(x,t) = u^{q-1}$ with $p < q < p^*$. Then $(\mathcal{P}_\lambda)$ has no nontrivial positive solution for all $\lambda \leq 0$.*

Theorem 1.2. *Assume that M satisfies (M_1) and f satisfies (f_1) and (f_2). Then there exists $\lambda_* > 0$ such that $(\mathcal{P}_\lambda)$ has a nontrivial solution for any $\lambda > \lambda_*$.*

This paper is organized as follows. In Section 2, based on a Pohozaev identity, we obtain a nonexistence result for problem $(\mathcal{P}_\lambda)$ when $\lambda \leq 0$. In Section 3, we construct a suitable truncation of M in order to use variational methods, first, we get the existence of a local Palais Smale sequence for the truncated problem by verifying the geometric conditions of the Mountain Pass Theorem [2], after that we use the concentration compactness principle, give some abstract conditions when the Palais Smale condition is satisfied and deduce by contradiction the existence of a nontrivial solution for the truncated problem. In Section 4, we prove Theorem 1.2.

Throughout this paper we use the following notation: S is the best Sobolev constant defined by $S = \inf_{u \in W^{1,p}(\mathbb{R}^N)} \|u\|^p \left(\int_{\mathbb{R}^N} |u|^{p^*} dx \right)^{-p/p^*}$, $B_\rho(x)$ is the ball centred at x and of radius ρ, $\to$ (resp.$\rightharpoonup$) denotes strong (resp. weak) convergence, $u^\pm = \max(\pm u,\ 0)$, C, C_1, C_2....are positive constants and $\circ_n(1)$ denotes $\circ_n(1) \to 0$ as $n \to \infty$.

2. Proof of Theorem 1.1

Let $u \in W_0^{1,p}(\Omega)$, $u > 0$ and

$$-M(\|u\|^p)\, \Delta_p u = u^{p^*-1} + \lambda u^{q-1}. \tag{1}$$

Multiplying the equation (1) by $\langle x, \nabla u \rangle$ on both sides and integrating by parts, we obtain

$$M(\|u\|^p) \left[\frac{p-1}{p} \int_{\partial\Omega} |\nabla u|^p \langle x, \nabla u \rangle\, dx + \frac{N-p}{p} \int_\Omega |\nabla u|^p dx \right] = \lambda \frac{N}{q} \int_\Omega |u|^q dx + \frac{N}{p^*} \int_\Omega |u|^{p^*} dx.$$

On the other hand, multiplying the equation (1) by u and integrating, we get

$$M(\|u\|^p) \int_\Omega |\nabla u|^p dx = \lambda \int_\Omega |u|^q dx + \int_\Omega |u|^{p^*} dx.$$

Putting the two identities together, we have

$$\begin{aligned} \frac{p-1}{p} M(\|u\|^p) \int_{\partial\Omega} |\nabla u|^p \langle x, \nabla u \rangle\, dx &= \lambda \left(\frac{N}{q} - \frac{N-p}{p} \right) \int_\Omega |u|^q dx + \left(\frac{N}{p^*} - \frac{N-p}{p} \right) \int_\Omega |u|^{p^*} dx \\ &= \lambda \left(\frac{N}{q} - \frac{N-p}{p} \right) \int_\Omega |u|^{q+1} dx. \end{aligned}$$

As $M(\|u\|^p) > 0, \langle x, \nabla u \rangle > 0, p < q < p^*$ and $\lambda \leq 0$, then the problem $(\mathcal{P}_\lambda)$ has no nontrivial positive solution.

3. Truncated problem

To use variational methods we make a truncation on M.

Let $k > 0$ be a real number, there exists $t_0 \in \mathbb{R}^+$ such that $k = M(t_0)$. We consider the function

$$M_k(t) = \begin{cases} M(t) & \text{if } 0 \leq t \leq t_0 \\ k & t \geq t_0, \end{cases}$$

and we study the truncated problem associated to M_k

$$\begin{cases} -M_k(\|u\|^p)\, \Delta_p u = u^{p^*-1} + \lambda f(x,u), \quad u > 0 & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{T}_\lambda$}$$

The main result of this section, is the following theorem whose proof will be given later.

Theorem 3.1. *Suppose that* (f_1), (f_2) *and* (M_1) *hold. If*

$$M(0) < k < \frac{\theta}{p} M(0).$$

then there exists $\lambda_0 > 0$ *such that problem* $(\mathcal{T}_\lambda)$ *has a nontrivial positive solution for any* $\lambda > \lambda_0$.

Since our approach is variational, we define the energy functional I_λ by

$$I_\lambda(u) = \frac{1}{p}\widehat{M}_k\left(\|u\|^p\right) - \frac{1}{p^*}\int_\Omega \left(u^+\right)^{p^*} dx - \lambda \int_\Omega F(x,u)\, dx, \quad \forall u \in W_0^{1,p}(\Omega),$$

where $\widehat{M}_k(t) = \int_0^t M_k(s)\, ds$. It is clear that I_λ is well defined in $W_0^{1,p}(\Omega)$ and belongs to $C^1\left(W_0^{1,p}(\Omega),\ \mathbb{R}\right)$. $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is said to be a weak solution of problem $(\mathcal{T}_\lambda)$ if it satisfies $u \geq 0$ and

$$M_k\left(\|u\|^p\right) \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi\, dx - \int_\Omega \left(u^+\right)^{p^*-2} u^+ \varphi\, dx - \lambda \int_\Omega f(x,u)\, \varphi\, dx = 0,$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

We first verify that I_λ satisfies the geometric conditions of the Mountain Pass Theorem.

Lemma 3.2. *Suppose that* (f_1) *and* (M_1) *hold. Then there exist* $u_1 \in W_0^{1,p}(\Omega)$, $\rho_1 \in \mathbb{R}$ *and* $\delta_1 \in \mathbb{R}$ *such that*

- *(i)* $I_\lambda(u) \geq \delta_1 > 0$, *for all* $u \in B_{\rho_1}(0)$,
- *(ii)* $I_\lambda(u_1) < 0$ *with* $\|u_1\|^p > \rho_1 > 0$.

Proof. *(i)* Let $\varepsilon > 0$ and $u \in W_0^{1,p}(\Omega) \setminus \{0\}$, by (f_1) there exists $C_\varepsilon > 0$ such that

$$F(x,u) \leq \frac{\varepsilon}{p}|u|^p + \frac{C_\varepsilon}{p^*}|u|^{p^*}.$$

So, by (M_1) and Sobolev's inequality, we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{M(0)}{p}\|u\|^p - \frac{S^{-p^*/p}}{p^*}\|u\|^{p^*} - \lambda C_1 \frac{\varepsilon}{p}\|u\|^p - C_2\frac{C_\varepsilon}{p^*}\|u\|^{p^*} \\ &\geq \left(\frac{M(0)}{p} - \lambda C_1 \frac{\varepsilon}{p}\right)\|u\|^p - \left(\frac{S^{-p^*/p}}{p^*} + C_2 \frac{C_\varepsilon}{p^*}\right)\|u\|^{p^*} \\ &\geq C_3\|u\|^p - C_4\|u\|^{p^*} \end{aligned}$$

for ε small enough. Thus the result follows.

(ii) Let $v \in C_0^\infty(\Omega)$ with $v \geq 0$ and $\|v\| = 1$. Then, for $t > 0$ we have

$$I_\lambda(tv) \leq \frac{k}{p}t^p - \frac{t^{p^*}}{p^*}\int_\Omega v^{p^*} dx.$$

Choosing $u_1 = t_1 v$ with t_1 large enough we get our conclusion. □

By Lemma 3.2 we get a Palais Smale sequence $(u_n) \subset W_0^{1,p}(\Omega)$ with

$$I_\lambda(u_n) \longrightarrow c_\lambda \text{ and } I'_\lambda(u_n) \longrightarrow 0,$$

where

$$c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t))$$

and

$$\Gamma = \left\{ \gamma \in C\left([0,1], W_0^{1,p}(\Omega)\right); \gamma(0) = 0 \text{ and } \gamma(1) = u_1 \right\}.$$

Lemma 3.3. *We have*

$$\lim_{\lambda \to +\infty} c_\lambda = 0.$$

Proof. Let $v \in C_0^\infty(\Omega)$ with $v \geq 0$ and $\|v\| = 1$. Then there exists $t_\lambda > 0$ such that

$$\max_{t \geq 0} I_\lambda(tv) = I_\lambda(t_\lambda v),$$

that is

$$t_\lambda^p M_k\left(t_\lambda^p\right) = t_\lambda^{p^*} \int_\Omega v^{p^*} dx + \lambda \int_\Omega f(x, t_\lambda v)\, t_\lambda v \, dx. \tag{2}$$

Since

$$k t_\lambda^p \geq t_\lambda^p M_k\left(t_\lambda^p\right) \geq t_\lambda^{p^*} \int_\Omega v^{p^*} dx,$$

it follows that

$$t_\lambda \leq \left[\frac{k}{\int_\Omega v^{p^*} dx}\right]^{1/\left(p^*-p\right)} < +\infty.$$

Then there exist $\lambda_n, T \geq 0$ such that

$$\lim_{n \to +\infty} \lambda_n = +\infty \text{ and } \lim_{n \to +\infty} t_{\lambda_n} = T,$$

which implies that

$$t_{\lambda_n}^p M_k\left(t_{\lambda_n}^p\right) \leq C, \forall n \in \mathbb{N},$$

for some $C > 0$. Hence, from (2) we have

$$t_{\lambda_n}^{p^*} \int_\Omega v^{p^*} dx + \lambda_n \int_\Omega f(x, t_{\lambda_n} v)\, t_{\lambda_n} v \, dx \leq C,$$

so, as $\lim_{n \to +\infty} \lambda_n = +\infty$ we conclude that $T = 0$.
Therefore, we have $\lim_{\lambda \to +\infty} \widehat{M}_k\left(t_\lambda^p\right) = 0$ and

$$0 \leq c_\lambda \leq \max_{t \geq 0} I_\lambda(tv) = I_\lambda(t_\lambda v) \leq \frac{1}{p} \widehat{M}_k\left(t_\lambda^p\right).$$

Then $\lim_{\lambda \to +\infty} c_\lambda = 0$. □

Next, we prove an important lemma which ensures the local compactness of the Palais Smale sequence for I_λ.

Lemma 3.4. *Let $(u_n) \subset W_0^{1,p}(\Omega)$ be a Palais Smale sequence for I_λ, namely $I_\lambda(u_n) \to c_\lambda < +\infty$ and $I'_\lambda(u_n) \to 0$. If*

$$c_\lambda < \left(\frac{1}{\theta} - \frac{1}{p^*}\right)(M(0)\, S)^{p^*/\left(p^*-p\right)},$$

then $u_n \longrightarrow u$ in $W_0^{1,p}(\Omega)$ for some $u \in W_0^{1,p}(\Omega)$.

Proof. We have

$$c_\lambda + \circ_n(1) = I_\lambda(u_n) \text{ and } \circ_n(1) = \left\langle I'_\lambda(u_n), u_n \right\rangle, \tag{3}$$

that is

$$\begin{aligned} c_\lambda + \circ_n(1) &= I_\lambda(u_n) - \frac{1}{\theta}\left\langle I'_\lambda(u_n), u_n \right\rangle \\ &\geq \frac{1}{p}\widehat{M}_k(\|u_n\|^p) - \frac{1}{\theta}M_k(\|u_n\|^p) \\ &\geq \left(\frac{M(0)}{p} - \frac{k}{\theta}\right)\|u_n\|^p. \end{aligned}$$

Then (u_n) is bounded in $W_0^{1,p}(\Omega)$. Up to a subsequence if necessary, we obtain

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega),\ u_n \rightharpoonup u \text{ in } L^{p^*}(\Omega),\ u_n \to u \text{ a.e. in } \Omega, \|u_n\|^p \rightharpoonup \alpha\ (\alpha \geq 0).$$

Therefore, by using the concentration compactness principle of Lions [9], there exists a subsequence (still denoted by $\{u_n\}$) which satisfies

$$|\nabla u_n|^p \longrightarrow |\nabla u|^p + \mu \text{ and } |u_n|^{p^*} \longrightarrow |u|^{p^*} + \nu,$$

with

$$\mu \geq \sum_{i\in I}\mu_i\delta_{x_i},\ \nu = \sum_{i\in I}\nu_i\delta_{x_i} \text{ and } \mu_i \geq S\nu_i^{p/p^*}.$$

First, we prove by contradiction that $I = \emptyset$. Let $i \in I$, $\psi \in C_0^\infty(\Omega, [0,1])$, $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\Omega \setminus B_2(0)$, $\left|\nabla\psi\right|_\infty < 1$ and $\psi_\rho(x) = \psi((x - x_i)/\rho)$ where $\rho > 0$.
We have $\psi_\rho u_n$ is bounded. Thus

$$\begin{aligned} \circ_n(1) &= \left\langle I'_\lambda(u_n), \psi_\rho u_n \right\rangle \\ &= M_k(\|u_n\|^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\left(\psi_\rho u_n\right) dx - \int_\Omega (u_n^+)^{p^*}\psi_\rho\, dx \\ &\quad -\lambda\int_\Omega f(x,u_n)\psi_\rho u_n\, dx \\ &= M_k(\|u_n\|^p)\int_\Omega \psi_\rho |\nabla u_n|^p\, dx - \int_\Omega (u_n^+)^{p^*}\psi_\rho\, dx \\ &\quad +M_k(\|u_n\|^p)\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\rho dx \\ &\quad -\lambda\int_\Omega f(x,u_n)\psi_\rho u_n\, dx. \end{aligned}$$

We have by Hölder inequality,

$$M_k(\|u_n\|^p)\int_\Omega u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\psi_\rho dx \leq M_k(\|u_n\|^p)\left(\int_{B_{2\rho}(x_0)}|u_n|^p\, dx\right)^{1/p}\|u_n\|^{p-1}.$$

By the dominated convergence Theorem, we obtain

$$\lim_{\rho\to 0}\lim_{n\to+\infty}\int_{B_{2\rho}(x_0)}|u_n|^p\, dx = 0.$$

Thus

$$\lim_{\rho\to 0}\lim_{n\to+\infty} M_k\left(\|u_n\|^p\right)\int_\Omega u_n\left|\nabla u_n\right|^{p-2}\nabla u_n\nabla\psi_\rho dx=0. \tag{4}$$

On the other hand, we have by (f_1)

$$\int_\Omega f\left(x,u_n\right)\psi_\rho u_n\,dx\le\varepsilon\int_{B_{2\rho}(x_0)}|u_n|^{p^*}\psi_\rho\,dx+C_\varepsilon\int_{B_{2\rho}(x_0)}u_n^2\psi_\rho\,dx.$$

So

$$\lim_{\rho\to 0}\lim_{n\to+\infty}\int_{B_{2\rho}(x_0)}u_n^2\psi_\rho\,dx=0$$

and as ε is arbitrary, we get

$$\lim_{\rho\to 0}\lim_{n\to+\infty}\varepsilon\int_{B_{2\rho}(x_0)}|u_n|^{p^*}\psi_\rho\,dx=0.$$

Therefore

$$\lim_{\rho\to 0}\lim_{n\to+\infty}\int_\Omega f\left(x,u_n\right)\psi_\rho u_n\,dx=0. \tag{5}$$

From (4) and (5) we obtain

$$\begin{aligned}0&=\lim_{\rho\to 0}\lim_{n\to+\infty}\left\langle I_\lambda'\left(u_n\right),\psi_\rho u_n\right\rangle\\&=\lim_{\rho\to 0}\lim_{n\to+\infty}\left(M_k\left(\|u_n\|^p\right)\int_\Omega|\nabla u_n|^p\psi_\rho\,dx-\int_\Omega\left(u_n^+\right)^{p^*}\psi_\rho\,dx\right)\\&\ge M_k\left(\alpha\right)\mu_i-\nu_i,\end{aligned}$$

then

$$\nu_i\ge\left(M\left(0\right)S\right)^{p^*/\left(p^*-p\right)}.$$

Therefore

$$\begin{aligned}c_\lambda+\circ_n\left(1\right)&=I_\lambda\left(u_n\right)-\frac{1}{\theta}I_\lambda'\left(u_n\right)\\&\ge\left(\frac{1}{p}-\frac{k}{\theta}\right)\|u_n\|^p+\left(\frac{1}{\theta}-\frac{1}{p^*}\right)\int_\Omega\left(u_n^+\right)^{p^*}\,dx\\&\ge\left(\frac{1}{\theta}-\frac{1}{p^*}\right)\int_{B_\rho(x_0)}\left(u_n^+\right)^{p^*}\psi_\rho\,dx.\end{aligned}$$

As a conclusion we obtain

$$c_\lambda\ge\left(\frac{1}{\theta}-\frac{1}{p^*}\right)\left(M\left(0\right)S\right)^{p^*/\left(p^*-p\right)},$$

which is a contradiction with the hypothesis. Then $u_n\to u$ in $L^{p^*}\left(\Omega\right)$.
Now, we prove that $u_n\to u$ in $W_0^{1,p}\left(\Omega\right)$, we have for $p\ge 2$

$$\begin{aligned}M\left(0\right)C_p\|u_n-u\|^p&\le M_k\left(\|u_n\|^p\right)\left\langle|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u,\nabla u_n-\nabla u\right\rangle\\&=M_k\left(\|u_n\|^p\right)\left[\|u_n\|^p-\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\nabla u dx\right.\\&\qquad\left.-\int_\Omega|\nabla u|^{p-2}\nabla u\left(\nabla u_n-\nabla u\right)dx\right].\end{aligned}$$

Or

$$\begin{aligned}\lim_{n\to+\infty}\int_{\Omega}|\nabla u|^{p-2}\nabla u\,(\nabla u_n-\nabla u)\,dx &= 0,\ \lim_{n\to+\infty}\left\langle I'_{\lambda}(u_n),u_n\right\rangle=0 \text{ and}\\ \lim_{n\to+\infty}\left\langle I'_{\lambda}(u_n),u\right\rangle &= 0.\end{aligned}$$

That is

$$\begin{aligned}\lim_{n\to+\infty}M_k(\|u_n\|^p)\|u_n\|^p &= \int_{\Omega}f(x,u)\,u\,dx+\int_{\Omega}u^{p^*}\,dx\\ &= \lim_{n\to+\infty}M_k(\|u_n\|^p)\int_{\Omega}|\nabla u_n|^{p-2}\nabla u_n\nabla u dx\end{aligned}$$

Then we conclude that $\lim_{n\to+\infty}\|u_n-u\|^p=0.$ □

Proof of Theorem 3.1. By Lemma 3.2 there exists a Palais Smale sequence $\{u_n\}$

$$I_{\lambda}(u_n)\longrightarrow c_{\lambda}\text{ and }I'_{\lambda}(u_n)\longrightarrow 0,$$

from Lemma 3.4 $u_n\to u$ in $W_0^{1,p}(\Omega)$, by Lemma 2 there exists $\lambda_0>0$ such that

$$c_{\lambda}<\left(\frac{1}{\theta}-\frac{1}{p^*}\right)(M(0)\,S)^{p^*/(p^*-p)}$$

for all $\lambda\geq\lambda_0$. Then we deduce that u is a solution of $(\mathcal{T}_{\lambda})$. □

4. Existence result

Proof of Theorem 1.2. Let $\lambda_*\geq\lambda_0$ such that

$$c_{\lambda}<\min\left\{\left(\frac{1}{\theta}-\frac{1}{p^*}\right)(M(0)\,S)^{p^*/(p^*-p)},\left(\frac{k}{p}-\frac{k}{\theta}\right)t_0\right\},$$

for all $\lambda\geq\lambda_*$. Assume that $\|u\|^p\geq t_0$ for all $\lambda\geq\lambda_*$, then

$$\left(\frac{k}{p}-\frac{k}{\theta}\right)t_0>c_{\lambda}=I_{\lambda}(u)-\frac{1}{\theta}\left\langle I'_{\lambda}(u),u\right\rangle\geq\left(\frac{1}{p}-\frac{k}{\theta}\right)\|u\|^p\geq\left(\frac{k}{p}-\frac{k}{\theta}\right)t_0$$

which leads to a contradiction. Thus u is a solution of $(\mathcal{P}_{\lambda})$. □

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