Filomat 36:9 (2022), 2991–3011
https://doi.org/10.2298/FIL2209991C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Twisted Partial Actions of Monoidal Hom-Hopf Algebras

Quan-Guo Chen[a], Wen-Jing Cheng[b]

[a]*School of Mathematical Sciences, Qufu Normal University, Shandong, Qufu 273165, China*
[b]*Zibo 11th Middle School, Shandong, Zibo 255035, China*

Abstract. In this work, the notion of a twisted partial Hom-Hopf action is introduced, and the conditions on partial cocycles are established in order to construct partial Hom-crossed products. Also, the equivalence of partial Hom-crossed products is discussed. Finally, we shall describe the coquasitriangular structures on partial Hom-crossed products.

1. Introduction

As mentioned in [33], the history of mathematics is composed of many moments in which an apparent dead end just opens new doors and triggers further progress. The origins of partial actions of groups can be considered as one such case. The notion of a partial group action was introduced by Exel [2] in order to endow important classes of C^*- algebras generated by partial isometries with a structure of a more general crossed product. In geometric terms, partial group actions describe no global symmetries of a space, but only local symmetries of certain subspaces. The historical developments and more recent advances of the theory of partial actions in operator algebras and dynamical systems which are systems in which a function describes the time dependence of a point in a geometrical space are explained in Exel's book [3]. The study of partial dynamical systems, that is, dynamical systems originating from the action of a partially defined homeomorphism on a topological space, was benefited strongly from the theory of partial group actions and partial crossed products. After the release of the seminal paper by Exel and Dokuchaev [3], partial group actions began to draw the attention of algebraists. A treatment from a purely algebraic point of view was given recently in ([5]-[8]). The Galois theory for partial group actions [7] motivated the introduction of partial actions for Hopf algebras [9], which opens a new door and triggers further progress. Recently, great progress has been made in the theory of partial actions of Hopf algebras in the literatures([10]-[16] and [19, 20]). For example, Batista and Vercruysse [34] introduced the notion of a partial action of an algebraic group on an affine space, and used such a partial action allows in a natural way to construct examples of partial coactions on the coordinate algebras of these spaces which interpreted partial coactions of Hopf algebras as partial symmetries in noncommutative geometry.

It is well known that quasitriangular bialgebras (or Hopf algebras) are fundamental in the theory of quantum groups and R-matrices, which are remarkable tool for studying the quantum Yang-Baxter

2020 *Mathematics Subject Classification*. Primary 16W30
Keywords. monoidal Hom-Hopf algebra, Hom-crossed product, coquasitriangular structure
Received: 31 October 2018; Accepted: 23 July 2022
Communicated by Dragan S. Djordjević
Research supported by the Natural Science Foundation of China (Nos.11871301 and 11801304) and the Natural Science Foundation of Shandong Province (No. ZR2022MA002).
Email addresses: cqg211@163.com (Quan-Guo Chen), 1757422505@qq.com (Wen-Jing Cheng)

equation. The notion of quasitriangular Hopf algebras, or quantum groups, in its turn, is due to Drinfeld [27] as an abstraction of structures implicit in the studies of Sklyanin [32] and Jimbo [28]. The dual concept, namely that of a coquasitriangular Hopf algebra (also called braided Hopf algebras in [29] was first introduced by Majid [30] and independently by Larson and Towber [29]. Subsequently, (Co)quasitriangular Hopf algebras generated an explosion of interest and were studied for their implications in quantum groups, the construction of invariants of knots and 3-manifolds, statistical mechanics and quantum mechanics, for example, using quasitriangular Hopf algebras, or quantum groups constructs a class of TQFT which provide models for different frameworks such as quantum gravity and topological quantum information. Based on this background, (co)quasitriangular bialgebras (Hopf algebras) have been a subject of research.

The aim of this paper is to develop the theory of partial actions of monoidal Hom-Hopf algebras. We can introduce the notion of a partial Hom-crossed product, and use it as a tool to construct the Hom-braided structure.

This paper is organized as follows.

In Section 2, we review the basic issues about monoidal Hom-Hopf algebras. The definitions of twisted partial Hom-Hopf actions on algebras are given in Section 3, including that of a partial Hom-crossed product. The cocycle and normalization conditions are needed in order to make the partial Hom-crossed product be both Hom-associative and Hom-unital. As a special case, partial Hom-Hopf actions on algebras and the associated partial Hom-smash products are obained. We shall give necessary and sufficient conditions for two partial Hom-crossed products to be isomorphic in Section 4, establishing an analogue of a corresponding result known in classical Hopf algebra theory. The last Section 5 is dedicated to the coquasitriangular structures on partial Hom-crossed products.

2. Preliminaries

Throughout this paper, k will be field. More knowledge about monoidal Hom-(co)algebra, monoidal Hopf Hom-algebra, etc. can be found in ([17]-[26]). Let $\mathcal{M} = (\mathcal{M}, \otimes, k, a, l, r)$ be the monoidal category of vector spaces over k. We can construct a new monoidal category $\mathcal{H}(\mathcal{M})$ whose objects are ordered pairs (M, μ) with $M \in \mathcal{M}$ and $\mu \in Aut(M)$ and morphisms $f : (M, \mu) \to (N, \nu)$ are morphisms $f : M \to N$ in $\mathcal{M}$ satisfying $\nu \circ f = f \circ \mu$. The monoidal structure is given by $(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$ and (k, id_k). All monoidal Hom-structures are objects in the tensor category $\widetilde{\mathcal{H}}(\mathcal{M}) = (\mathcal{H}(\mathcal{M}), \otimes, (k, id_k), \widetilde{a}, \widetilde{l}, \widetilde{r})$ introduced in ([17]) with the associativity and unit constraints given by

$$\widetilde{a}_{M,N,C}((m \otimes n) \otimes p) = \mu(m) \otimes (n \otimes \gamma^{-1}(c)),$$

$$\widetilde{l}(x \otimes m) = \widetilde{r}(m \otimes x) = x\mu(m),$$

for $(M, \mu), (N, \nu)$ and (C, γ). The category $\widetilde{\mathcal{H}}(\mathcal{M})$ is termed Hom-category associated to $\mathcal{M}$.

2.1. Monoidal Hom-algebra

Recall from [17] that a monoidal Hom-algebra is an object $(A, \alpha) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $m_A : A \otimes A \to A$, $m_A(a \otimes b) = ab$ and an element $1_A \in A$ such that

$$\alpha(ab) = \alpha(a)\alpha(b), \alpha(a)(bc) = (ab)\alpha(c), \tag{1}$$

$$\alpha(1_A) = 1_A,\ a1_A = \alpha(a) = 1_A a, \tag{2}$$

for all $a, b, c \in A$.

A left (A, α)-Hom-module consists of an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $\psi : A \otimes M \to M, \psi(a \otimes m) = ma$ satisfying the following conditions:

$$(ab)\mu(m) = \alpha(a)(bm), 1_A m = \mu(m), \tag{3}$$

for all $m \in M$ and $a, b \in A$. For ψ to be a morphism in $\widetilde{\mathcal{H}}(\mathcal{M})$ means

$$\mu(am) = \alpha(a)\mu(m). \tag{4}$$

We call that ψ is a left Hom-action of (A, α) on (M, μ).

Let (M, μ) and (M', μ') be two left (A, α)-Hom-modules. We call a morphism $f : M \to M'$ right (A, α)-linear, if $f \circ \mu = \mu' \circ f$ and $f(am) = af(m)$. ${}_A\mathcal{M}$ denotes the category of all left (A, α)-Hom-modules.

2.2. Monoidal Hom-coalgebras

Recall from [17] that a monoidal Hom-coalgebra is an object $(C, \gamma) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with two linear maps $\Delta_C : C \to C \otimes C$, $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ (summation implicitly understood) and $\varepsilon_C : C \to k$ such that

$$\gamma^{-1}(c_{(1)}) \otimes \Delta_C(c_{(2)}) = c_{(1)(1)} \otimes (c_{(1)(2)} \otimes \gamma^{-1}(c_{(2)})), \Delta_C(\gamma(c)) = \gamma(c_{(1)}) \otimes \gamma(c_{(2)}), \tag{5}$$

$$\varepsilon_C(\gamma(c)) = \varepsilon_C(c),\ c_{(1)}\varepsilon_C(c_{(2)}) = \gamma^{-1}(c) = \varepsilon_C(c_{(1)})c_{(2)}, \tag{6}$$

for all $c \in C$.

A right (C, γ)-Hom-comodule consists of an object $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M})$ together with a linear map $\rho_M : M \to M \otimes C$, $\rho_M(m) = m_{[0]} \otimes m_{[1]}$ (summation implicitly understood) satisfying the following conditions:

$$\mu^{-1}(m_{[0]}) \otimes \Delta(m_{[1]}) = m_{[0][0]} \otimes (m_{[0][1]} \otimes \gamma^{-1}(m_{[1]})), \tag{7}$$

$$m_{[0]}\varepsilon_C(m_{[1]}) = \gamma^{-1}(m), \tag{8}$$

$$\mu(m)_{[0]} \otimes \mu(m)_{[1]} = \mu(m_{[0]}) \otimes \gamma(m_{[1]}), \tag{9}$$

for all $m \in M$. We call that ρ_M is a right Hom-coaction of (A, α) on (M, μ).

Let (M, μ) and (M', μ') be two right (C, γ)-Hom-comodules. We call a morphism $f : M \to M'$ right (C, γ)-colinear, if $f \circ \mu = \mu' \circ f$ and $f(m)_{[0]} \otimes f(m)_{[1]} = f(m_{[0]}) \otimes m_{[1]}$. $\mathcal{M}^C$ denotes the category of all right (C, γ)-Hom-comodules.

2.3. Monoidal Hom-Hopf algebra

A monoidal Hom-bialgebra $H = (H, \beta, m_H, 1_H, \Delta_H, \varepsilon_H)$ is a bialgebra in the category $\widetilde{\mathcal{H}}(\mathcal{M})$. This means that $(H, \beta, m_H, 1_H)$ is a monoidal Hom-algebra and $(H, \beta, \Delta_H, \varepsilon_H)$ is a monoidal Hom-coalgebra such that Δ_H and ε_H are Hom-algebra maps, that is, for any $h, g \in H$,

$$\Delta_H(hg) = \Delta_H(h)\Delta_H(g), \Delta_H(1_H) = 1_H \otimes 1_H, \tag{10}$$

$$\varepsilon_H(hg) = \varepsilon_H(h)\varepsilon_H(g), \varepsilon_H(1_H) = 1. \tag{11}$$

A monoidal Hom-bialgebra (H, β) is called a monoidal Hom-Hopf algebra, if there exists a morphism (called the Hom-antipode) $S_H : H \to H$ in $\widetilde{\mathcal{H}}(\mathcal{M})$ such that

$$S_H(h_{(1)})h_{(2)} = \varepsilon_H(h)1_A = h_{(1)}S(h_{(2)}), \tag{12}$$

for all $h \in H$.

3. Twisted partial Hom-actions and partial Hom-crossed products

Definition 3.1. *Let (H,β) be a monoidal Hom-Hopf algebra, (A,α) a monoidal Hom-algebra with an element 1_A. Let $\psi : H\otimes A \to A$ and $\omega : H\otimes H \to A$ be two linear maps such that $\psi\circ(\beta\otimes\alpha) = \alpha\circ\psi$ and $\omega\circ(\beta\otimes\beta) = \alpha\circ\omega$. We will write $\psi(h\otimes a) = h\cdot a$, and $\omega(h\otimes l) = \omega(h,l)$, where $a\in A$ and $h,l\in H$. The pair $(\cdot,\omega)$ is called a twisted partial Hom-action of H on A, If the following conditions hold:*

$$1_H\cdot a = \alpha(a), \tag{13}$$

$$h\cdot(ab) = (h_{(1)}\cdot a)(h_{(2)}\cdot b), \tag{14}$$

$$(\beta(h_{(1)})\cdot(l_{(1)}\cdot\alpha^{-1}(a)))\alpha(\omega(h_{(2)},l_{(2)})) = \alpha(\omega(h_{(1)},l_{(1)}))(h_{(2)}l_{(2)}\cdot a), \tag{15}$$

$$\omega(h,l) = \omega(h_{(1)},l_{(1)})(\beta^{-1}(h_{(2)}l_{(2)})\cdot 1_A), \tag{16}$$

for all $a,b\in A$ and $h,l\in H$. We shall say that $(A,\alpha,\cdot,\omega)$ is a twisted partial (H,β)-Hom-module algebra.

Proposition 3.2. *Let $(A,\alpha,\cdot,\omega)$ be a twisted partial (H,β)-Hom-module algebra. Then the following identities hold:*

$$\omega(h,l) = (h_{(1)}\cdot(\beta^{-1}(l_{(1)})\cdot 1_A))\omega(h_{(2)},l_{(2)}) = (h_{(1)}\cdot 1_A)\omega(h_{(2)},\beta^{-1}(l)). \tag{17}$$

Proof. Taking $a = 1_A$ in (15), we have the first identity. For the second identity, we compute as follows:

$$\begin{aligned}
&\underline{(h_{(1)}\cdot(\beta^{-1}(l_{(1)})\cdot 1_A))}\omega(h_{(2)},l_{(2)})\\
(14) &= ((\underline{h_{(1)(1)}}\cdot 1_A)(\underline{h_{(1)(2)}}\cdot(\beta^{-2}(l_{(1)})\cdot 1_A)))\omega(\underline{h_{(2)}},l_{(2)})\\
(5) &= ((\beta^{-1}(h_{(1)})\cdot 1_A)(h_{(2)(1)}\cdot(\beta^{-2}(l_{(1)})\cdot 1_A)))\omega(\beta(h_{(2)(2)}),l_{(2)})\\
(1) &= (h_{(1)}\cdot 1_A)((h_{(2)(1)}\cdot(\beta^{-2}(l_{(1)})\cdot 1_A))\omega(h_{(2)(2)},\beta^{-1}(l_{(2)})))\\
&= (h_{(1)}\cdot 1_A)((h_{(2)(1)}\cdot(\beta^{-1}(\beta^{-1}(l)_{(1)})\cdot 1_A))\omega(h_{(2)(2)},\beta^{-1}(l)_{(2)}))\\
&= (h_{(1)}\cdot 1_A)\omega(h_{(2)},\beta^{-1}(l)),
\end{aligned}$$

as desired. □

We say that the map ω is trivial, if the following condition holds:

$$h\cdot(\beta^{-1}(l)\cdot 1_A) = \omega(h,l) = (h_{(1)}\cdot 1_A)(\beta^{-1}(h_{(2)})\beta^{-2}(l)\cdot 1_A). \tag{18}$$

If (18) holds, for all $h,l\in H$ and $a\in A$, we have

$$\begin{aligned}
h\cdot(\beta^{-1}(l)\cdot a) &= h\cdot((\beta^{-1}(l_{(1)})\cdot\alpha^{-1}(a))(\beta^{-1}(l_{(2)})\cdot 1_A))\\
(14) &= (h_{(1)}\cdot(\beta^{-1}(l_{(1)})\cdot\alpha^{-1}(a)))\underline{(h_{(2)}\cdot(\beta^{-1}(l_{(2)})\cdot 1_A))}\\
(18) &= (h_{(1)}\cdot(\beta^{-1}(l_{(1)})\cdot\alpha^{-1}(a)))\omega(h_{(2)},l_{(2)})\\
(15) &= \omega(h_{(1)},l_{(1)})(\beta^{-1}(h_{(2)})\beta^{-1}(l_{(2)})\cdot a)\\
(18) &= ((h_{(1)(1)}\cdot 1_A)(\beta^{-1}(h_{(1)(2)})\beta^{-2}(l_{(1)})\cdot 1_A))(\beta^{-1}(h_{(2)})\beta^{-1}(l_{(2)})\cdot a)\\
(1) &= (\underline{\beta(h_{(1)(1)})}\cdot 1_A)((\underline{\beta^{-1}(h_{(1)(2)})}\beta^{-2}(l_{(1)})\cdot 1_A)(\underline{\beta^{-2}(h_{(2)})}\beta^{-2}(l_{(2)})\cdot\alpha^{-1}(a)))\\
(5) &= (h_{(1)}\cdot 1_A)((\beta^{-1}(h_{(2)(1)})\beta^{-2}(l_{(1)})\cdot 1_A)(\beta^{-1}(h_{(2)(2)})\beta^{-2}(l_{(2)})\cdot\alpha^{-1}(a)))\\
&= (h_{(1)}\cdot 1_A)(\beta^{-1}(h_{(2)})\beta^{-2}(l)\cdot a).
\end{aligned}$$

If ω is trivial, then the condition (16) is superfluous, and we can restate Definition 3.1 and get the definition of a partial (H,β)-Hom-module algebra.

Definition 3.3. *Let (H,β) be a monoidal Hom-Hopf algebra, (A,α) a monoidal Hom-algebra with an element 1_A. Let $\psi : H\otimes A \to A$, $\psi(h\otimes a) = h\cdot a$ be a linear map such that $\psi\circ(\beta\otimes\alpha) = \alpha\circ\psi$. (A,α) is called a (left) partial (H,β)-Hom-module algebra, if the following conditions hold:*

$$1_H\cdot a = \alpha(a), \tag{19}$$

$$h\cdot(ab) = (h_{(1)}\cdot a)(h_{(2)}\cdot b), \tag{20}$$

$$h\cdot(l\cdot a) = (h_{(1)}\cdot 1_A)(\beta^{-1}(h_{(2)}l)\cdot a), \tag{21}$$

for all $a,b\in A$ and $h,l\in H$.

Example 3.4. *Let (B,α) be the monoidal Hom-algebra as the vector space over the complex number field $\mathbb{C}$ generated by the generators $1_B, g, x$ which are subject to the following relationships*

$$1_B1_B = 1_B, 1_Bx = -x, 1_Bg = g, 1_B(xg) = -xg, xg = gx, g^2 = 1_B, x^2 = 0,$$

where α is given via

$$\alpha : 1_B\mapsto 1_B, x\mapsto -x, g\mapsto g, xg\mapsto -xg.$$

Let $H = \mathbb{C}\langle 1_H, h\rangle$ be the group Hopf algebra with $h^2 = 1_H$, $\Delta_H(h) = h\otimes h$, $S_H(h) = h = h^{-1}$, $\varepsilon_H(h) = 1$, so we have a monoidal Hom-Hopf algebra (H, id). Define the action of H on A as follows:

$$\begin{aligned} &1_H \blacktriangleright 1_B = 1_B, 1_H\blacktriangleright x = -x, 1_H\blacktriangleright g = g, 1_H\blacktriangleright xg = -xg,\\ &\quad h\blacktriangleright 1_B = 1_B, h\blacktriangleright g = g, h\blacktriangleright x = -x, h\blacktriangleright xg = -xg.\end{aligned}$$

Define $\mathbb{C}$-bilinear maps $\sigma : H\otimes H\mapsto A$ as follows:

$$\sigma(1_H,1_H) = \sigma(1_H,h) = \sigma(h,1_H) = 1_A, \sigma(h,h) = g.$$

It is easily to check that the quadruple $(H,B,\blacktriangleright,\sigma)$ satisfies the conditions (13), (14) and (15). Observe that $e = \frac{1}{2}(1_B+g)$ is a central idempotent of B which satisfies $\alpha(e) = e$. Then, we consider the monoidal Hom-subalgebra (A,α), where A is the sub-vector space of B generated by e, it follows easily that $A = \mathbb{C}\langle 1_B+g, x-xg\rangle$ with $1_A = e$. Using $\blacktriangleright$, we can define a map $\cdot : H\otimes A\mapsto A$ by $h\cdot a = e(h\blacktriangleright\alpha(a))$, for all $a\in A$, concretely,

$$\begin{aligned} &1_H\cdot(1_B+g) = 1_B+g, 1_H\cdot(x-xg) = -(x-xg),\\ &\quad h\cdot(1_B+g) = 1_B+g, h\cdot(x-xg) = -(x-xg).\end{aligned}$$

In view of conditions (15) and (16), we define $\omega : H\otimes H\to A$ as follows:

$$\omega(h,1_H) = \omega(1_H,h) = 1_B+g = \omega(h,h).$$

So we have a twisted partial (H,id)-Hom-module algebra $(A,\alpha,\cdot,\omega)$.

Given any two linear maps $\psi : H\otimes A\to A$ and $\omega : H\otimes H\to A$ such that $\psi\circ(\beta\otimes\alpha) = \alpha\circ\psi$ and $\omega\circ(\beta\otimes\beta) = \alpha\circ\omega$, we can define a product on the vector space $A\otimes H$, given by the multiplication

$$(a\otimes h)(b\otimes l) = (\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b)))\omega(\beta(h_{(2)(1)}), l_{(1)})\otimes\beta^2(h_{(2)(2)})\beta(l_{(2)}), \tag{22}$$

for all $a,b\in A$ and $h,l\in H$. let $A\sharp_\omega H$ be the subspace of $A\otimes H$ generated by the elements of the form $a\sharp h = \alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot 1_A)\otimes\beta(h_{(2)})$.

Proposition 3.5. *Let (H,β) be a monoidal Hom-Hopf algebra, (A,α) a monoidal Hom-algebra with an element $1_A\in A$, $\omega : H\otimes H\to A$ and $\psi : H\otimes A\to A, h\otimes a\mapsto h\cdot a$, two linear maps satisfying $\psi\circ(\beta\otimes\alpha) = \alpha\circ\psi$, $\omega\circ(\beta\otimes\beta) = \alpha\circ\omega$ and the conditions (13), (14) and (16).*

1. $1_A\sharp 1_H$ *satisfies* $(1_A\sharp 1_H)(a\sharp h) = (a\sharp h)(1_A\sharp 1_H) = (\alpha\otimes\beta)((a\sharp h))$ *if and only if,*

$$\omega(1_H, h) = h\cdot 1_A = \omega(h, 1_H). \tag{23}$$

2. *Suppose that* $\omega(h, 1_H) = h\cdot 1_A$*, for all* $h\in H$*. Then* $A\otimes H$ *satisfies (1) if and only if the condition (15) holds and, for all* $h, g, l\in H$*,*

$$(\beta(h_{(1)})\cdot\omega(g_{(1)}, l_{(1)}))\omega(\beta(h_{(2)}), g_{(2)}l_{(2)}) = \omega(\beta(h_{(1)}), \beta(g_{(1)}))\omega(h_{(2)}g_{(2)}, l). \tag{24}$$

Proof. (1) Assume that $\omega(h, 1_H) = \omega(1_H, h) = h\cdot 1_A$. Then we have

$$\begin{aligned}(1_A\sharp 1_H)(a\sharp h) &=(1_A\otimes 1_H)(\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot 1_H)\otimes\beta(h_{(2)}))\\ &=(\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot 1_H))(\beta(h_{(2)(1)})\cdot 1_A)\otimes\beta^3(h_{(2)(2)})\\ (1)\ &=a((\underline{\beta^{-1}(h_{(1)})}\cdot 1_H)(\underline{h_{(2)(1)}}\cdot 1_A))\otimes\underline{\beta^3(h_{(2)(2)})}\\ (5)\ &=a(\underline{(h_{(1)(1)}\cdot 1_H)(h_{(1)(2)}\cdot 1_A)})\otimes\beta^2(h_{(2)})\\ (14)\ &=a(h_{(1)}\cdot 1_H)\otimes\beta^2(h_{(2)})\\ &=(\alpha\otimes\beta)(\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot 1_H)\otimes\beta(h_{(2)}))\\ &=(\alpha\otimes\beta)(a\sharp h).\end{aligned}$$

Similarly, we can prove that $(a\sharp h)(1_A\sharp 1_H) = (\alpha\otimes\beta)(a\sharp h)$.
Conversely, if $1_A\sharp 1_H$ satisfies $(1_A\sharp 1_H)(a\sharp h) = (a\sharp h)(1_A\sharp 1_H) = (\alpha\otimes\beta)((a\sharp h))$. Then, we have

$$(\beta(h_{(1)})\cdot 1_A)\otimes\beta^2(h_{(2)}) = (1_A\sharp 1_H)(1_A\sharp h) = \omega(1_H, \beta(h_{(1)}))\otimes\beta^2(h_{(2)}). \tag{25}$$

Applying $id\otimes\varepsilon_H$ to both sides of (25) yields $\omega(1_H, h) = h\cdot 1_A$. $\omega(h, 1_H) = h\cdot 1_A$ follows similarly.
(2) We have by assumption that

$$(1_A\otimes\beta(h))[(1_A\otimes g)(a\otimes 1_H)] = [(1_A\otimes h)(1_A\otimes g)](\alpha(a)\otimes 1_H) \tag{26}$$

and

$$(1_A\otimes\beta(h))[(1_A\otimes g)(1_A\otimes l)] = [(1_A\otimes h)(1_A\otimes g)](1_A\otimes\beta(l)). \tag{27}$$

Since

$$\begin{aligned}\text{LHS of (26)} &= (1_A\otimes\beta(h))((\underline{g_{(1)}}\cdot\alpha^{-1}(a))(\underline{\beta(g_{(2)(1)})}\cdot 1_A)\otimes\underline{\beta^3(g_{(2)(2)})})\\ (5)\ &= (1_A\otimes\beta(h))(\underline{(\beta(g_{(1)})_{(1)}\cdot\alpha^{-1}(a))(\beta(g_{(1)})_{(2)}\cdot 1_A)}\otimes\beta^2(g_{(2)}))\\ (14)\ &= (1_A\otimes\beta(h))(\beta(g_{(1)})\cdot a\otimes\beta^2(g_{(2)}))\\ &= (\beta(h_{(1)})\cdot(g_{(1)}\cdot\alpha^{-1}(a)))\omega(\beta^2(h_{(2)(1)}), \beta^2(g_{(2)(1)}))\otimes\beta^3(h_{(2)(2)}g_{(2)(2)})\end{aligned}$$

and

$$\begin{aligned}\text{RHS of (26)} &= (\underline{(\beta(h_{(1)})_{(1)}\cdot 1_A)\omega(\beta(h_{(1)})_{(2)}, g_{(1)})}\otimes\beta(h_{(2)})\beta(g_{(2)}))(\alpha(a)\otimes 1_H)\\ (17)\ &= (\omega(\beta(h_{(1)}), \beta(g_{(1)}))\otimes\beta(h_{(2)}g_{(2)}))(\alpha(a)\otimes 1_H)\\ &= \omega(\beta(h_{(1)}), \beta(g_{(1)}))(\beta(h_{(2)(1)}g_{(2)(1)})\cdot a)\otimes\beta^3(h_{(2)(2)}g_{(2)(2)}),\end{aligned}$$

it follows that

$$\begin{aligned}&(\beta(h_{(1)})\cdot(g_{(1)}\cdot\alpha^{-1}(a)))\omega(\beta^2(h_{(2)(1)}), \beta^2(g_{(2)(1)}))\otimes\beta^3(h_{(2)(2)}g_{(2)(2)})\\ &\qquad= \omega(\beta(h_{(1)}), \beta(g_{(1)}))(\beta(h_{(2)(1)}g_{(2)(1)})\cdot a)\otimes\beta^3(h_{(2)(2)}g_{(2)(2)}).\end{aligned}$$

Applying $id \otimes \varepsilon_H$ to both sides of the above equality, it follows that the equality (15) holds. Since

$$\begin{aligned}
\text{LHS of (27)} &= (1_A \otimes \beta(h))((\underline{g_{(1)}} \cdot 1_A)\omega(\underline{\beta(g_{(2)(1)})}, l_{(1)}) \otimes \underline{\beta^2(g_{(2)(2)})}\beta(l_{(2)})) \\
(5) &= (1_A \otimes \beta(h))(\underline{(\beta(g_{(1)})_{(1)} \cdot 1_A)\omega(\beta(g_{(1)})_{(2)}, l_{(1)})} \otimes \beta(g_{(2)})\beta(l_{(2)})) \\
(17) &= (1_A \otimes \beta(h))(\omega(\beta(g_{(1)}), \beta(l_{(1)})) \otimes \beta(g_{(2)})\beta(l_{(2)})) \\
&= (\beta(h_{(1)}) \cdot \omega(g_{(1)}, l_{(1)})) \\
&\quad \times \omega(\beta^2(h_{(2)(1)}), \beta(g_{(2)(1)}l_{(2)(1)})) \otimes \beta^2(h_{(2)(2)}g_{(2)(2)})\beta^3(l_{(2)(2)})
\end{aligned}$$

and

$$\begin{aligned}
\text{RHS of (27)} &= (\omega(\beta(h_{(1)}), \beta(g_{(1)})) \otimes \beta(h_{(2)}g_{(2)}))(1_A \otimes \beta(l)) \\
&= \omega(\beta(h_{(1)}), \beta(g_{(1)}))\omega(\beta(h_{(2)(1)}g_{(2)(1)}), \beta(l_{(1)})) \otimes \beta^2(h_{(2)(2)}g_{(2)(2)})\beta^2(l_{(2)}),
\end{aligned}$$

it follows that

$$\begin{aligned}
&(\beta(h_{(1)}) \cdot \omega(g_{(1)}, l_{(1)}))\omega(\beta^2(h_{(2)(1)}), \beta(g_{(2)(1)}l_{(2)(1)})) \otimes \beta^2(h_{(2)(2)}g_{(2)(2)})\beta^3(l_{(2)(2)}) \\
&\quad = \omega(\beta(h_{(1)}), \beta(g_{(1)}))\omega(\beta(h_{(2)(1)}g_{(2)(1)}), \beta(l_{(1)})) \otimes \beta^2(h_{(2)(2)}g_{(2)(2)})\beta^2(l_{(2)}).
\end{aligned}$$

Applying $id \otimes \varepsilon_H$ to both sides of the above equality, we obtain the equality (24).

Assume that (15) and (24) hold. Then, for all $a, b, c \in A$ and $h, g, l \in H$, we have

$$
\begin{aligned}
& (\alpha(a) \otimes \beta(h))[(b \otimes g)(c \otimes l)] \\
& =(\alpha(a) \otimes \beta(h))[b((\alpha^{-1}(g_{(1)}) \cdot \alpha^{-2}(c))\omega(g_{(2)(1)}, \beta^{-1}(l_{(1)}))) \otimes \beta^2(g_{(2)(2)})\beta(l_{(2)})] \\
& =\alpha(a)[((h_{(1)(1)} \cdot \alpha^{-2}(b)) \\
& \quad \times ((h_{(1)(2)(1)} \cdot (\beta^{-3}(g_{(1)}) \cdot \alpha^{-4}(c)))(h_{(1)(2)(2)} \cdot \omega(\beta^{-2}(g_{(2)(1)}), \beta^{-3}(l_{(1)}))))) \\
& \quad \times \omega(\beta(h_{(2)(1)}), \beta(g_{(2)(2)(1)})l_{(2)(1)})] \otimes \beta^3(h_{(2)(2)})(\beta^3(g_{(2)(2)(2)})\beta^2(l_{(2)(2)})) \\
(1)\ & =\alpha(a)[((h_{(1)(1)} \cdot \alpha^{-2}(b)) \\
& \quad \times (\beta(h_{(1)(2)(1)}) \cdot (\beta^{-2}(g_{(1)}) \cdot \alpha^{-3}(c))))((\alpha(h_{(1)(2)(2)}) \cdot \omega(\beta^{-1}(g_{(2)(1)}), \beta^{-2}(l_{(1)}))) \\
& \quad \times \omega(h_{(2)(1)}, g_{(2)(2)(1)}\beta^{-1}(l_{(2)(1)})))] \otimes \beta^3(h_{(2)(2)})(\beta^3(g_{(2)(2)(2)})\beta^2(l_{(2)(2)})) \\
(5)\ & =\alpha(a)[((\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)) \\
& \quad \times (h_{(2)(1)} \cdot (\beta^{-2}(g_{(1)}) \cdot \alpha^{-3}(c))))(\underline{(\alpha^2(h_{(2)(2)(1)(1)}) \cdot \omega(g_{(2)(1)(1)}, \beta^{-1}(l_{(1)(1)})))} \\
& \quad \times \underline{\omega(\beta^2(h_{(2)(2)(1)(2)}), g_{(2)(1)(2)}\beta^{-1}(l_{(1)(2)})))}] \otimes \beta^4(h_{(2)(2)(2)})(\beta^2(g_{(2)(2)})\beta(l_{(2)})) \\
(24)\ & =\alpha(a)[((\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)) \\
& \quad \times (h_{(2)(1)} \cdot (\beta^{-2}(g_{(1)}) \cdot \alpha^{-3}(c))))(\omega(\beta^2(h_{(2)(2)(1)(1)}), \beta(g_{(2)(1)(1)})) \\
& \quad \times \omega(\beta(h_{(2)(2)(1)(2)})g_{(2)(1)(2)}, \beta^{-1}(l_{(1)})))] \otimes \beta^4(h_{(2)(2)(2)})(\beta^2(g_{(2)(2)})\beta(l_{(2)})) \\
(1)\ & =\alpha(a)[((\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)) \\
& \quad \times ((\beta^{-1}(h_{(2)(1)}) \cdot (\beta^{-3}(g_{(1)}) \cdot \alpha^{-4}(c)))\omega(\beta(h_{(2)(2)(1)(1)}), g_{(2)(1)(1)}))) \\
& \quad \times \omega(\beta^2(h_{(2)(2)(1)(2)})\beta(g_{(2)(1)(2)}), l_{(1)})] \otimes \beta^4(h_{(2)(2)(2)})(\beta^2(g_{(2)(2)})\beta(l_{(2)})) \\
(5)\ & =\alpha(a)[((\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)) \\
& \quad \times \underline{((\beta(h_{(2)(1)(1)(1)}) \cdot (\beta^{-1}(g_{(1)(1)(1)}) \cdot \alpha^{-4}(c)))\omega(\beta(h_{(2)(1)(1)(2)}), g_{(1)(1)(2)})})) \\
& \quad \times \omega(\beta(h_{(2)(1)(2)})g_{(1)(2)}, l_{(1)})] \otimes \beta^3(h_{(2)(2)})(\beta(g_{(2)})\beta(l_{(2)})) \\
(15)\ & =\alpha(a)[((\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)) \\
& \quad \times (\omega(\beta(h_{(2)(1)(1)(1)}), g_{(1)(1)(1)})(h_{(2)(1)(1)(2)}\beta^{-1}(g_{(1)(1)(2)}) \cdot \alpha^{-3}(c)))) \\
& \quad \times \omega(\beta(h_{(2)(1)(2)})g_{(1)(2)}, l_{(1)})] \otimes \beta^3(h_{(2)(2)})(\beta(g_{(2)})\beta(l_{(2)})) \\
(1)\ & =(a((\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)) \\
& \quad \times \omega(\beta^2(h_{(2)(1)(1)(1)}), \beta(g_{(1)(1)(1)}))))((\beta^2(h_{(2)(1)(1)(2)})\beta(g_{(1)(1)(2)}) \cdot \alpha^{-1}(c)) \\
& \quad \times \omega(\beta(h_{(2)(1)(2)})g_{(1)(2)}, l_{(1)})) \otimes \beta^3(h_{(2)(2)})(\beta(g_{(2)})\beta(l_{(2)})) \\
(5)\ & =(a((\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)) \\
& \quad \times \omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))))((\beta(h_{(2)(2)(1)})g_{(2)(1)} \cdot \alpha^{-1}(c)) \\
& \quad \times \omega(\beta^2(h_{(2)(2)(2)(1)})\beta(g_{(2)(2)(1)}), l_{(1)})) \otimes (\beta^4(h_{(2)(2)(2)(2)})\beta^3(g_{(2)(2)(2)}))\beta^2(l_{(2)}) \\
& =(a((\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))) \otimes \beta^2(h_{(2)(2)})\beta(g_{(2)}))(\alpha(c) \otimes \beta(l)) \\
& =((a \otimes h)(b \otimes g))(\alpha(c) \otimes \beta(l)).
\end{aligned}
$$

Thus the proof is completed. □

Given a twisted partial (H, β)-Hom-module algebra $(A, \alpha, \cdot, \omega)$, the induced monoidal Hom-algebra $(A\sharp_\omega H, \alpha \otimes \beta)$ is called a partial Hom-crossed product. In order to establish some notation, we give the following lemma.

Lemma 3.6. *In $A\sharp_\omega H$, we have the following identities:*

1. $a\sharp h = \alpha^{-1}(a)(\beta^{-1}(h_{(1)}) \cdot 1_A)\sharp\beta(h_{(2)})$,
2. $(a\sharp h)(b\sharp l) = (\alpha^{-1}(a)(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)))\omega(\beta(h_{(2)(1)}), l_{(1)})\sharp\beta^2(h_{(2)(2)})\beta(l_{(2)})$,

for all $a, b \in A$ and $h, l \in H$.

Proof. Straightforward. □

If ω is trivial, the multiplication in $A\sharp_\omega H$ becomes

$$\begin{aligned}(a\sharp h)(b\sharp l) =&(\alpha^{-1}(a)(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)))\underline{\omega(\beta(h_{(2)(1)}), l_{(1)})}\sharp\beta^2(h_{(2)(2)})\beta(l_{(2)})\\ (18) =&(\alpha^{-1}(a)(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)))\\ &\times [(\beta(h_{(2)(1)(1)}) \cdot 1_A)(h_{(2)(1)(2)}\beta^{-2}(l_{(1)}) \cdot 1_A)]\sharp\beta^2(h_{(2)(2)})\beta(l_{(2)})\\ (5) =&(\alpha^{-1}(a)\underline{((\beta^{-1}(h_{(1)(1)}) \cdot \alpha^{-3}(b))(\beta^{-1}(h_{(1)(2)}) \cdot 1_A))})\\ &\times (h_{(2)(1)}\beta^{-1}(l_{(1)}) \cdot 1_A)\sharp\beta^2(h_{(2)(2)})\beta(l_{(2)})\\ (14) =&(\alpha^{-1}(a)(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)))(h_{(2)(1)}\beta^{-1}(l_{(1)}) \cdot 1_A)\sharp\beta^2(h_{(2)(2)})\beta(l_{(2)})\\ =&\alpha^{-1}(a)(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b)))\sharp h_{(2)}\beta^{-1}(l).\end{aligned}$$

Thus the special partial Hom-crossed product $A\sharp_\omega H$ is called a partial Hom-smash product which is denoted by $A\sharp H$.

4. Equivalence of Partial Hom-crossed Products

In this section, we shall give the criteria in order to decide whether two twisted partial Hom-actions give rise to the same Hom-crossed product. what we shall see is a hom-version of Theorem 1 of [14] for twisted partial Hom-Hopf actions. Assume that the map $\Lambda(h) = h \cdot 1_A$ is central with respect to the convolution product in Hom(H, A), this is, for $f \in$Hom(H, A), we have $\Lambda(h_{(1)})f(h_{(2)}) = f(h_{(1)})\Lambda(h_{(2)})$.

Theorem 4.1. *Assume that $(A\sharp_\omega H, \alpha \otimes \beta)$ and $(A\sharp_\sigma H, \alpha \otimes \beta)$ are both partial Hom-crossed products via $\psi, \Psi : H \otimes A \to A, \psi(h \otimes a) = h \cdot a, \Psi(h \otimes a) = h \bullet a$ and $\omega, \sigma : H \otimes H \to A$. Suppose that there is a Hom-algebra isomorphism*

$$\Phi : A\sharp_\omega H \to A\sharp_\sigma H$$

which is also a left (A, α)-Hom-module and right (H, β)-Hom-comodule map. There exsits linear maps $\mu, \nu \in$Hom(H, A) such that,

(A1) $\mu \circ \beta = \alpha \circ \mu, \nu \circ \beta = \alpha \circ \nu$,
(A2) $\mu * \nu(h) = \beta^{-1}(h) \cdot 1_A$,
(A3) $\mu(h) = (\beta^{-1}(h_{(1)}) \cdot 1_A)\mu(h_{(2)}) = \mu(h_{(1)})(\beta^{-1}(h_{(2)}) \cdot 1_A)$,
(A4) $h \bullet b = [\nu(\beta(h_{(1)(1)}))(h_{(1)(2)} \cdot \alpha^{-2}(b))]\mu(\beta(h_{(2)}))$,
(A5) $\sigma(h, g) = [(\nu(h_{(1)(1)})(h_{(1)(2)(1)} \cdot \nu(\beta^{-1}(g_{(1)(1)}))))\omega(\beta(h_{(1)(2)(2)}), g_{(1)(2)})]\mu(h_{(2)}g_{(2)})$,
(A6) $\Phi(a\sharp_\omega h) = \alpha^{-1}(a)\mu(h_{(1)})\sharp_\sigma\beta(h_{(2)})$.

Conversely, given maps $\mu, \nu \in$Hom(H, A) satisfying (1)-(5), and in addition $\mu(1_H) = \nu(1_H) = 1_A$, the map Φ as presented in (6), is a Hom-algebras isomorphism.

Proof. The left (A, α)-Hom-module structure on the partial Hom-crossed product is given by

$$a \rhd (b\sharp h) = \alpha^{-1}(a)b\sharp\beta(h),$$

and the right (H, β)-Hom-comodule structure is given by

$$\rho(a\sharp h) = (\alpha^{-1}(a)\sharp h_{(1)}) \otimes \beta(h_{(2)}).$$

Let

$$\Phi : A\sharp_\omega H \to A\sharp_\sigma H$$

be the hom-algebra isomorphism which also is a left (A,α)-Hom-module and right (H,β)-Hom-comodule map. Define $\mu,\nu \in$Hom(H,A) as

$$\mu(h) = (id\otimes\varepsilon)\Phi(1_A\sharp_\omega h),\ \nu(h) = (id\otimes\varepsilon)\Phi^{-1}(1_A\sharp_\sigma h).$$

It easily follows that $\alpha\circ\mu = \mu\circ\beta$ and $\alpha\circ\nu = \nu\circ\beta$. Observe that

$$\begin{aligned}\mu(h) = (id\otimes\varepsilon)\Phi(1_A\sharp_\omega h) =&(id\otimes\varepsilon)\Phi((h_{(1)}\cdot 1_A)\sharp_\omega\beta(h_{(2)}))\\ =&(id\otimes\varepsilon)\Phi((h_{(1)}\cdot 1_A)\triangleright(1_A\sharp_\omega h_{(2)}))\\ =&(\beta^{-1}(h_{(1)})\cdot 1_A)\mu(h_{(2)})\\ =&\mu(h_{(1)})(\beta^{-1}(h_{(2)})\cdot 1_A),\end{aligned}$$

where the fourth equation follows from the assumption of Λ. Generally, we have

$$\begin{aligned}(id\otimes\varepsilon)\Phi(a\sharp_\omega h) =&(id\otimes\varepsilon)\Phi(a\triangleright(1_A\sharp_\omega\beta^{-1}(h)))\\ =&\alpha^{-1}(a)(id\otimes\varepsilon)\Phi(1_A\sharp_\omega\beta^{-1}(h)) = \alpha^{-1}(a)\mu(\beta^{-1}(h)).\end{aligned}$$

Set $\Phi(a\sharp_\omega h) = a^A\sharp_\sigma h^H$. Since Φ is a right (H,β)-Hom-comodule map, we have

$$\Phi(\alpha^{-1}(a)\sharp_\omega h_{(1)})\otimes\beta(h_{(2)}) = (\alpha^{-1}(a^A)\sharp_\sigma h^H{}_{(1)})\otimes\beta(h^H{}_{(2)}).$$

Applying $id\otimes\varepsilon\otimes id$ to both sides of the above equality, we gain that

$$\Phi(a\sharp_\omega h) = \alpha^{-1}(a)\mu(h_{(1)})\sharp_\sigma\beta(h_{(2)}).$$

With a totally similar reasoning, we can conclude that

$$\Phi^{-1}(a\sharp_\sigma h) = \alpha^{-1}(a)\nu(h_{(1)})\sharp_\omega\beta(h_{(2)}),$$

and

$$\nu(h) = (\beta^{-1}(h_{(1)})\cdot 1_A)\nu(h_{(2)}) = \nu(h_{(1)})(\beta^{-1}(h_{(2)})\cdot 1_A).$$

Notice that we readily obtain from the above that $\mu(1_H) = 1_A = \nu(1_H)$.

Consider the expression

$$\begin{aligned}(h_{(1)}\cdot 1_A)\sharp_\omega\beta(h_{(2)}) =&1_A\sharp_\omega h = \Phi^{-1}(\Phi(1_A\sharp_\omega h))\\ =&\Phi^{-1}(\alpha(\mu(h_{(1)}))\sharp_\sigma\beta(h_{(2)}))\\ =&\mu(h_{(1)})\nu(\beta(h_{(2)(1)}))\sharp_\omega\beta^2(h_{(2)(2)}).\end{aligned}$$

Applying $id\otimes\varepsilon$ on both sides, we obtain

$$\mu(h_{(1)})\nu(h_{(2)}) = \beta^{-1}(h)\cdot 1_A.$$

Analogously, we can conclude that

$$\nu(h_{(1)})\mu(h_{(2)}) = \beta^{-1}(h)\bullet 1_A.$$

We use the fact that Φ^{-1} preserves the multiplication. Therefore,

$$\Phi^{-1}((a\sharp_\sigma h)(b\sharp_\sigma g)) = \Phi^{-1}(a\sharp_\sigma h)\Phi^{-1}(b\sharp_\sigma g),$$

which gives

$$\begin{aligned}&[\alpha^{-1}(a)((\beta^{-2}(h_{(1)})\bullet\alpha^{-3}(b))(\sigma(\beta^{-1}(h_{(2)(1)}),\beta^{-2}(g_{(1)}))))]\\&\quad\nu(\beta^2(h_{(2)(2)(1)})\beta(g_{(2)(1)}))\sharp_\omega\beta^3(h_{(2)(2)(2)})\beta^2(g_{(2)(2)})\\&=(\alpha^{-1}(a)\nu(h_{(1)}))[(h_{(2)(1)}\cdot(\alpha^{-3}(b)\alpha^{-2}(\nu(g_{(1)}))))\\&\quad(\omega(\beta(h_{(2)(2)(1)}),g_{(2)(1)}))]\sharp_\omega\beta^3(h_{(2)(2)(2)})\beta^2(g_{(2)(2)}).\end{aligned}$$

Applying $id\otimes\varepsilon$ to both sides, we get

$$\begin{aligned}&[\alpha^{-1}(a)((\beta^{-2}(h_{(1)})\bullet\alpha^{-3}(b))\sigma(\beta^{-1}(h_{(2)(1)}),\beta^{-2}(g_{(1)})))]\nu(\beta(h_{(2)(2)})g_{(2)})\\&\quad=[(\alpha^{-2}(a)\nu(\beta^{-1}(h_{(1)})))(h_{(2)(1)}\cdot(\alpha^{-3}(b)\alpha^{-2}(\nu(g_{(1)}))))]\omega(\beta(h_{(2)(2)}),g_{(2)}).\end{aligned}\tag{28}$$

Using this formula for $a=1_A$ and $g=1_H$, we obtain

$$(h_{(1)}\bullet\alpha^{-1}(b))\nu(\beta(h_{(2)}))=\nu(\beta(h_{(1)}))(h_{(2)}\cdot\alpha^{-1}(b)).$$

Multiplying convolutively on the right by $\mu\circ\beta$ and using $\nu(h_{(1)})\mu(h_{(2)})=\beta^{-1}(h)\bullet 1_A$, we have

$$h\bullet b=[\nu(\beta(h_{(1)(1)}))(h_{(1)(2)}\cdot\alpha^{-2}(b))]\mu(\beta(h_{(2)})).$$

On the other hand, putting $a=b=1_A$ in (28), we get

$$\sigma(h_{(1)},g_{(1)})\nu(h_{(2)}g_{(2)})=(\nu(h_{(1)})(h_{(2)(1)}\cdot\nu(\beta^{-1}(g_{(1)}))))\omega(\beta(h_{(2)(2)}),g_{(2)}).$$

Therefore

$$\sigma(h,g)=[(\nu(h_{(1)(1)})(h_{(1)(2)(1)}\cdot\nu(\beta^{-1}(g_{(1)(1)}))))\omega(\beta(h_{(1)(2)(2)}),g_{(1)(2)})]\mu(h_{(2)}g_{(2)}).$$

Conversely, let us consider maps $\mu,\nu\in\mathrm{Hom}(H,A)$ satisfying $\mu(1_H)=\nu(1_H)=1_A$ and (A1)-(A5). We shall verify that

$$\Phi:A\sharp_\omega H\to A\sharp_\sigma H,\ a\sharp_\omega h\mapsto\alpha^{-1}(a)\mu(h_{(1)})\sharp_\sigma\beta(h_{(2)})$$

is indeed a Hom-algebra morphism. We see immedialtely that $\Phi(1_A\sharp_\omega 1_H)=1_A\sharp_\sigma 1_H$. For the multiplicativity, we have

$$\begin{aligned}&\Phi(a\sharp_\omega h)\Phi(b\sharp_\omega g)\\&=((\alpha^{-1}(a))\mu(h_{(1)}))[\underline{(h_{(2)(1)}\bullet(\alpha^{-2}(\alpha^{-1}(b)\mu(g_{(1)}))))}\\&\quad\times\underline{\sigma(\beta(h_{(2)(2)(1)}),g_{(2)(1)})}]\sharp_\sigma\beta^3(h_{(2)(2)(2)})\beta^2(g_{(2)(2)})\\&=((\alpha^{-1}(a))\mu(h_{(1)}))\{[(\nu(\beta(h_{(2)(1)(1)(1)}))(h_{(2)(1)(1)(2)}\cdot\alpha^{-4}(\alpha^{-1}(b)\mu(g_{(1)}))))\\&\quad\times\mu(\beta(h_{(2)(1)(2)}))][((\nu(\beta(h_{(2)(2)(1)(1)(1)}))(\beta(h_{(2)(2)(1)(1)(2)(1)})\cdot\nu(\beta^{-1}(g_{(2)(1)(1)(1)}))))\\&\quad\times\omega(\beta^2(h_{(2)(2)(1)(1)(2)(2)}),g_{(2)(1)(1)(2)}))\mu(\beta(h_{(2)(2)(1)(2)})g_{(2)(1)(2)})]\}\sharp_\sigma\beta^3(h_{(2)(2)(2)})\beta^2(g_{(2)(2)})\\(5),(1)\ &=[\alpha^{-1}(a)(\underline{\mu(\beta^2(h_{(1)(1)(1)(1)}))\nu(\beta^2(h_{(1)(1)(1)(2)}))})]\{((h_{(1)(1)(2)}\cdot\alpha^{-3}(\alpha^{-1}(b)\mu(g_{(1)})))\mu(h_{(1)(2)}))\\&\quad\times[((\nu(h_{(2)(1)(1)(1)})(h_{(2)(1)(1)(2)(1)}\cdot\nu(\beta^{-1}(g_{(2)(1)(1)(1)}))))\\&\quad\times\omega(\beta(h_{(2)(1)(1)(2)(2)}),g_{(2)(1)(1)(2)}))\mu(h_{(2)(1)(2)}g_{(2)(1)(2)})]\}\sharp_\sigma\beta^2(h_{(2)(2)})\beta^2(g_{(2)(2)})\\&=(\alpha^{-1}(a)(\beta(h_{(1)(1)(1)})\cdot 1_A))\{((h_{(1)(1)(2)}\cdot\alpha^{-3}(\alpha^{-1}(b)\mu(g_{(1)})))\mu(h_{(1)(2)}))\\&\quad\times[((\nu(h_{(2)(1)(1)(1)})(h_{(2)(1)(1)(2)(1)}\cdot\nu(\beta^{-1}(g_{(2)(1)(1)(1)}))))\\&\quad\times\omega(\beta(h_{(2)(1)(1)(2)(2)}),g_{(2)(1)(1)(2)}))\mu(h_{(2)(1)(2)}g_{(2)(1)(2)})]\}\sharp_\sigma\beta^2(h_{(2)(2)})\beta^2(g_{(2)(2)})\end{aligned}$$

$$
\begin{aligned}
(1),(5) =&[\alpha^{-1}(a)((\beta(h_{(1)(1)(1)(1)})\cdot 1_A)(\beta(h_{(1)(1)(1)(2)})\cdot\alpha^{-3}(b)))]\{[(h_{(1)(1)(2)}\cdot\alpha^{-2}(\mu(g_{(1)})))\mu(h_{(1)(2)})]\\
&\times[((\nu(h_{(2)(1)(1)(1)})(h_{(2)(1)(1)(2)(1)}\cdot\nu(\beta^{-1}(g_{(2)(1)(1)(1)}))))\\
&\times\omega(\beta(h_{(2)(1)(1)(2)(2)}),g_{(2)(1)(1)(2)}))\mu(h_{(2)(1)(2)}g_{(2)(1)(2)})]\}\sharp_\sigma\beta^2(h_{(2)(2)})\beta^2(g_{(2)(2)})\\
(14) =&(\alpha^{-1}(a)(\beta(h_{(1)(1)(1)})\cdot\alpha^{-2}(b)))\{[(h_{(1)(1)(2)}\cdot\alpha^{-2}(\mu(g_{(1)})))\mu(h_{(1)(2)})]\\
&\times[((\nu(h_{(2)(1)(1)(1)})(h_{(2)(1)(1)(2)(1)}\cdot\nu(\beta^{-1}(g_{(2)(1)(1)(1)}))))\\
&\times\omega(\beta(h_{(2)(1)(1)(2)(2)}),g_{(2)(1)(1)(2)}))\mu(h_{(2)(1)(2)}g_{(2)(1)(2)})]\}\sharp_\sigma\beta^2(h_{(2)(2)})\beta^2(g_{(2)(2)})\\
(1),(5) =&(\alpha^{-1}(a)(h_{(1)(1)}\cdot\alpha^{-2}(b)))\{(h_{(1)(2)}\cdot\alpha^{-1}(\mu(g_{(1)})))[(h_{(2)(1)(1)}\cdot\nu(\beta(g_{(2)(1)(1)(1)})))\\
&\times(\omega(h_{(2)(1)(2)(1)},g_{(2)(1)(1)(2)})\mu(h_{(2)(1)(2)(2)}\beta^{-1}(g_{(2)(1)(2)})))]\}\sharp_\sigma\beta^2(h_{(2)(2)})\beta^2(g_{(2)(2)})\\
(1),(5) =&(\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b)))\{[(h_{(2)(1)(1)}\cdot\alpha^{-2}(\mu(g_{(1)})))(h_{(2)(1)(2)}\cdot\nu(\beta(g_{(2)(1)(1)(1)})))]\\
&\times[\omega(\beta(h_{(2)(2)(1)(1)}),\beta(g_{(2)(1)(1)(2)}))\mu(\beta(h_{(2)(2)(1)(2)})g_{(2)(1)(2)})]\}\sharp_\sigma\beta^3(h_{(2)(2)(2)})\beta^2(g_{(2)(2)})\\
(14) =&(\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b)))\{[(h_{(2)(1)}\cdot(\mu(\beta^{-2}(g_{(1)}))\nu(\beta(g_{(2)(1)(1)(1)}))))]\\
&\times[\omega(\beta(h_{(2)(2)(1)(1)}),\beta(g_{(2)(1)(1)(2)}))\mu(\beta(h_{(2)(2)(1)(2)})g_{(2)(1)(2)})]\}\sharp_\sigma\beta^3(h_{(2)(2)(2)})\beta^2(g_{(2)(2)})\\
(5) =&(\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b)))\{[(h_{(2)(1)}\cdot(\mu(\beta(g_{(1)(1)(1)(1)}))\nu(\beta(g_{(1)(1)(1)(2)}))))]\\
&\times[\omega(\beta(h_{(2)(2)(1)(1)}),\beta(g_{(1)(1)(2)}))\mu(\beta(h_{(2)(2)(1)(2)})\beta^{-1}(g_{(1)(2)}))]\}\sharp_\sigma\beta^3(h_{(2)(2)(2)})\beta(g_{(2)})\\
=&(\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b)))\{[(h_{(2)(1)}\cdot(g_{(1)(1)(1)}\cdot 1_A))]\\
&\times[\omega(\beta(h_{(2)(2)(1)(1)}),\beta(g_{(1)(1)(2)}))\mu(\beta(h_{(2)(2)(1)(2)})\beta^{-1}(g_{(1)(2)}))]\}\sharp_\sigma\beta^3(h_{(2)(2)(2)})\beta(g_{(2)})\\
(1),(5) =&(\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b)))\{[(\beta(h_{(2)(1)(1)(1)})\cdot(\beta^{-1}(g_{(1)(1)(1)})\cdot 1_A))\\
&\times\omega(\beta(h_{(2)(1)(1)(2)}),\beta(g_{(1)(1)(2)}))]\mu(\beta(h_{(2)(1)(2)})g_{(1)(2)})\}\sharp_\sigma\beta^2(h_{(2)(2)})\beta(g_{(2)})\\
(17) =&(\alpha^{-1}(a)(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b)))\\
&\times[\omega(\beta(h_{(2)(1)(1)}),g_{(1)(1)})\mu(\beta(h_{(2)(1)(2)})g_{(1)(2)})\}\sharp_\sigma\beta^2(h_{(2)(2)})\beta(g_{(2)})\\
=&[\alpha^{-1}(a)((\beta^{-2}(h_{(1)})\cdot\alpha^{-3}(b))\omega(h_{(2)(1)(1)},\beta^{-1}(g_{(1)(1)})))]\\
&\times\mu(\beta^2(h_{(2)(1)(2)})\beta(g_{(1)(2)}))\sharp_\sigma\beta^2(h_{(2)(2)})\beta(g_{(2)})\\
(5) =&[\alpha^{-1}(a)((\beta^{-2}(h_{(1)})\cdot\alpha^{-3}(b))\omega(\beta^{-1}(h_{(2)(1)}),\beta^{-2}(g_{(1)})))]\\
&\times\mu(\beta^2(h_{(2)(2)(1)})\beta(g_{(2)(1)}))\sharp_\sigma\beta^3(h_{(2)(2)(2)})\beta^2(g_{(2)(2)})\\
=&\Phi((a\sharp_\omega h)(b\sharp_\omega g)).
\end{aligned}
$$

Now, it remains to show that Φ is invertible. Consider the map

$$\Psi : A\sharp_\sigma H \to A\sharp_\omega H,\ \Psi(a\sharp_\sigma h) = \alpha^{-1}(a)\nu(h_{(1)})\sharp_\omega\beta(h_{(2)}).$$

it is easily proved that Ψ and Φ are mutual invertible. $\square$

5. Coquasitriangular Structures on Partial Hom-crossed Products

Let A be a bialgebra and $A\sharp_\omega H$ a partial Hom-crossed product. Define two Hom-linear maps

$$
\begin{aligned}
&\Delta : A\otimes H \to (A\sharp_\omega H)\otimes(A\sharp_\omega H),\\
&\qquad a\otimes h \mapsto (a_{(1)}\sharp h_{(1)})\otimes(a_{(2)}\sharp h_{(2)}),\\
&\varepsilon : A\otimes H \to k,\ \varepsilon(a\otimes h) = \varepsilon_A(a)\varepsilon_H(h).
\end{aligned}
$$

Lemma 5.1. *Δ and ε can induce maps $\widetilde{\Delta}$ and $\widetilde{\varepsilon}$ on $A\sharp_\omega H$, i.e.,*

$$
\begin{aligned}
&\widetilde{\Delta} : A\sharp_\omega H \to (A\sharp_\omega H)\otimes(A\sharp_\omega H), a\sharp h \mapsto (a_{(1)}\sharp h_{(1)})\otimes(a_{(2)}\sharp h_{(2)}),\\
&\widetilde{\varepsilon} : A\sharp_\omega H \to k,\ \widetilde{\varepsilon}(a\sharp h) = \varepsilon_A(a)\varepsilon_H(h).
\end{aligned}
$$

in order to make $(A\sharp H, \widetilde{\Delta}, \widetilde{\varepsilon}, \alpha\otimes\beta)$ *is a monoidal Hom-coalgebra if and only if the following conditions hold,*

$$\varepsilon_A(h\cdot 1_A) = \varepsilon_H(h), \tag{29}$$

$$(h_{(1)}\cdot 1_A)_{(1)}(h_{(2)(1)}\cdot 1_A)\otimes(h_{(1)}\cdot 1_A)_{(2)}(h_{(2)(2)}\cdot 1_A) = (h_{(1)}\cdot 1_A)\otimes(h_{(2)}\cdot 1_A). \tag{30}$$

Proof. Straightforward. □

Proposition 5.2. $A\sharp_\omega H$ *is a Hom-bialgebra via* $\widetilde{\Delta}, \widetilde{\varepsilon}$ *induced by* Δ, ε *defined above if and only if* $\widetilde{\Delta}, \widetilde{\varepsilon}$ *preserve the multiplication if and only if*

$$(h_{(1)}\cdot\alpha^{-1}(b))_{(1)}(h_{(2)(1)}\cdot 1_A)\otimes(h_{(1)}\cdot\alpha^{-1}(b))_{(2)}(h_{(2)(2)}\cdot 1_A) = (h_{(1)}\cdot b_{(1)})\otimes(h_{(2)}\cdot b_{(2)}), \tag{31}$$

$$\beta(h_{(2)(1)})\otimes(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))(h_{(2)(2)}\cdot 1_A) = h_{(1)}\otimes h_{(2)}\cdot\beta^{-1}(b), \tag{32}$$

$$\begin{aligned}\beta(h_{(2)(1)}g_{(2)(1)})\otimes\omega(\beta^{-1}(h_{(1)}),\beta^{-1}(g_{(1)}))(\beta^{-1}(h_{(2)(2)}g_{(2)(2)})\cdot 1_A)\\ = h_{(1)}g_{(1)}\otimes\omega(h_{(2)},g_{(2)}),\end{aligned} \tag{33}$$

$$\begin{aligned}\omega(h_{(1)},g_{(1)})_{(1)}(\beta^{-1}(h_{(2)(1)}g_{(2)(1)})\cdot 1_A)\otimes\omega(h_{(1)},g_{(1)})_{(2)}(\beta^{-1}(h_{(2)(2)}g_{(2)(2)})\cdot 1_A)\\ = \omega(h_{(1)},g_{(1)})\otimes\omega(h_{(2)},g_{(2)}),\end{aligned} \tag{34}$$

$$\varepsilon_A(h\cdot b) = \varepsilon_H(h)\varepsilon_A(b), \varepsilon_A(\omega(h,g)) = \varepsilon_H(h)\varepsilon_H(g), \tag{35}$$

for all $h, g\in H$ *and* $b\in A$.

Proof. It is sufficient to prove that Δ and ε preserve the multiplication. If $\varepsilon((a\sharp h)(b\sharp g)) = \varepsilon(a\sharp h)\varepsilon(b\sharp g)$, then we have

$$\varepsilon_A(a)\varepsilon_A((\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b)))\varepsilon_A(\omega(\beta^{-1}(h_{(2)}),\beta^{-2}(g))) = \varepsilon_A(a)\varepsilon_A(b)\varepsilon_H(h)\varepsilon_H(g). \tag{36}$$

Taking $a = 1_A$ and $g = 1_H$ in (36) yields the first equation of (35), the other follows by considering $a = b = 1_A$ in (36). For all $a, b\in A$ and $h, g\in H$, since

$$\Delta((a\sharp g)(b\sharp h)) = \Delta(a\sharp g)\Delta(b\sharp h),$$

we get

$$\begin{aligned}&[a_{(1)}((\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)}\omega(h_{(2)(1)},\beta^{-1}(g_{(1)}))_{(1)})\sharp\beta^2(h_{(2)(2)(1)})\beta(g_{(2)(1)})]\\ &\quad\otimes[a_{(2)}((\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)}\omega(h_{(2)(1)},\beta^{-1}(g_{(1)}))_{(2)})\sharp\beta^2(h_{(2)(2)(2)})\beta(g_{(2)(2)})]\\ &=a_{(1)}[(\beta^{-1}(h_{(1)(1)})\cdot\alpha^{-2}(b_{(1)}))\omega(h_{(1)(2)(1)},\beta^{-1}(g_{(1)(1)}))]\sharp\beta^2(h_{(1)(2)(2)})\beta(g_{(1)(2)})\\ &\quad\otimes a_{(2)}[(\beta^{-1}(h_{(2)(1)})\cdot\alpha^{-2}(b_{(2)}))\omega(h_{(2)(2)(1)},\beta^{-1}(g_{(2)(1)}))]\sharp\beta^2(h_{(2)(2)(2)})\beta(g_{(2)(2)})\end{aligned} \tag{37}$$

Considering $a = 1_A, g = 1_H$ in (37) yields

$$\begin{aligned}&[(h_{(1)}\cdot\alpha^{-1}(b))_{(1)}(\beta(h_{(2)(1)(1)})\cdot 1_A)\otimes\beta^3(h_{(2)(1)(2)})]\\ &\quad\otimes[(h_{(1)}\cdot\alpha^{-1}(b))_{(2)}(\beta(h_{(2)(2)(1)})\cdot 1_A)\otimes\beta^3(h_{(2)(2)(2)})]\\ &=[\beta(h_{(1)(1)})\cdot b_{(1)}\otimes\beta^2(h_{(1)(2)})]\otimes[\beta(h_{(2)(1)})\cdot b_{(2)}\otimes\beta^2(h_{(2)(2)})].\end{aligned}$$

Applying $id\otimes\varepsilon_H\otimes id\otimes\varepsilon_H$ and $\varepsilon_A\otimes id\otimes id\otimes\varepsilon_H$ to both sides respectively, we obtain (31) and (32). Applying $id\otimes\varepsilon_H\otimes id\otimes\varepsilon_H$ and $\varepsilon_A\otimes id_H\otimes id_A\otimes\varepsilon_H$ to (37) yields (33) and (34).

Conversely, assume that the conditions (31)-(35) hold. ε is a Hom-algebra homorphism following from (35). Next, we shall check that Δ is also a Hom-algebra homorphism. We only prove that Δ preserves the multiplication. In fact, for all $a, b \in A$ and $h, g \in H$, we compute

$$
\begin{aligned}
&\Delta((a\sharp h)(b\sharp l))\\
&=[((\alpha^{-2}(a_{(1)})(\beta^{-2}(h_{(1)})\cdot\alpha^{-3}(b))_{(1)})\omega(h_{(2)(1)},\beta^{-1}(l_{(1)}))_{(1)})\\
&\quad\times((\beta(h_{(2)(2)(1)(1)})l_{(2)(1)(1)})\cdot 1_A)\otimes\beta^{3}(h_{(2)(2)(1)(2)})\beta^{2}(l_{(2)(1)(2)})]\\
&\quad\otimes[((\alpha^{-2}(a_{(2)})(\beta^{-2}(h_{(1)})\cdot\alpha^{-3}(b))_{(2)})\omega(h_{(2)(1)},\beta^{-1}(l_{(1)}))_{(2)})\\
&\quad\times((\beta(h_{(2)(2)(2)(1)})l_{(2)(2)(1)})\cdot 1_A)\otimes\beta^{3}(h_{(2)(2)(2)(2)})\beta^{2}(l_{(2)(2)(2)})]\\
(5),(32)\ &=[((\alpha^{-2}(a_{(1)})(\beta^{-2}(h_{(1)})\cdot\alpha^{-3}(b))_{(1)})\omega(h_{(2)(1)},\beta^{-1}(l_{(1)}))_{(1)})\\
&\quad\times((h_{(2)(2)(1)}\beta^{-1}(l_{(2)(1)}))\cdot 1_A)\otimes\beta^{5}(h_{(2)(2)(2)(1)(2)(1)})\beta^{4}(l_{(2)(2)(1)(2)(1)})]\\
&\quad\otimes[((\alpha^{-2}(a_{(2)})(\beta^{-2}(h_{(1)})\cdot\alpha^{-3}(b))_{(2)})\omega(h_{(2)(1)},\beta^{-1}(l_{(1)}))_{(2)})\\
&\quad\times(((\beta(h_{(2)(2)(2)(1)(1)})l_{(2)(2)(1)(1)})\cdot 1_A)((\beta^{2}(h_{(2)(2)(2)(1)(2)(2)})\beta(l_{(2)(2)(1)(2)(2)}))\cdot 1_A))\\
&\quad\otimes\beta^{3}(h_{(2)(2)(2)(2)})\beta^{2}(l_{(2)(2)(2)})]\\
(5),(1)\ &=[(\alpha^{-1}(a_{(1)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)})(\omega(\beta(h_{(2)(1)(1)}),l_{(1)(1)})_{(1)}\\
&\quad\times((h_{(2)(1)(2)(1)}\beta^{-1}(l_{(1)(2)(1)}))\cdot 1_A))\otimes\beta^{3}(h_{(2)(2)(1)(1)})\beta^{2}(l_{(2)(1)(1)})]\\
&\quad\otimes[(\alpha^{-1}(a_{(2)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)})((\alpha^{-1}(\omega(\beta(h_{(2)(1)(1)}),l_{(1)(1)})_{(2)})\\
&\quad\times((\beta^{-1}(h_{(2)(1)(2)(2)})\beta^{-2}(l_{(1)(2)(2)}))\cdot 1_A))((h_{(2)(2)(1)(2)}\beta^{-1}(l_{(2)(1)(2)}))\cdot 1_A))\\
&\quad\otimes\beta^{2}(h_{(2)(2)(2)})\beta(l_{(2)(2)})]\\
(34)\ &=[(\alpha^{-1}(a_{(1)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)})\omega(\beta(h_{(2)(1)(1)}),l_{(1)(1)})\otimes\beta^{3}(h_{(2)(2)(1)(1)})\beta^{2}(l_{(2)(1)(1)})]\\
&\quad\otimes[(\alpha^{-1}(a_{(2)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)})(\alpha^{-1}(\omega(\beta(h_{(2)(1)(2)}),l_{(1)(2)}))\\
&\quad\times((h_{(2)(2)(1)(2)}\beta^{-1}(l_{(2)(1)(2)}))\cdot 1_A))\otimes\beta^{2}(h_{(2)(2)(2)})\beta(l_{(2)(2)})]\\
(17)\ &=[(\alpha^{-1}(a_{(1)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)})((\beta(h_{(2)(1)(1)(1)})\cdot 1_A)\\
&\quad\times\omega(\beta(h_{(2)(1)(1)(2)}),\beta^{-1}(l_{(1)(1)})))\otimes\beta^{3}(h_{(2)(2)(1)(1)})\beta^{2}(l_{(2)(1)(1)})]\\
&\quad\otimes[(\alpha^{-1}(a_{(2)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)})((h_{(2)(1)(2)(1)}\cdot 1_A)\omega(h_{(2)(1)(2)(2)},\beta^{-2}(l_{(1)(2)})))\\
&\quad\times((h_{(2)(2)(1)(2)}\beta^{-1}(l_{(2)(1)(2)}))\cdot 1_A))\otimes\beta^{2}(h_{(2)(2)(2)})\beta(l_{(2)(2)})]\\
(5)\ &=[(\alpha^{-1}(a_{(1)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)})((\beta(h_{(2)(1)(1)(1)})\cdot 1_A)\\
&\quad\times\omega(\beta^{2}(h_{(2)(1)(1)(2)(1)}),\beta^{-1}(l_{(1)(1)})))\otimes\beta^{3}(h_{(2)(2)(1)(1)})\beta^{2}(l_{(2)(1)(1)})]\\
&\quad\otimes[(\alpha^{-1}(a_{(2)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)})((\beta(h_{(2)(1)(1)(2)(2)})\cdot 1_A)\omega(\beta^{-1}(h_{(2)(1)(2)}),\beta^{-2}(l_{(1)(2)})))\\
&\quad\times((h_{(2)(2)(1)(2)}\beta^{-1}(l_{(2)(1)(2)}))\cdot 1_A))\otimes\beta^{2}(h_{(2)(2)(2)})\beta(l_{(2)(2)})]\\
(32)\ &=[(\alpha^{-1}(a_{(1)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)})((\beta(h_{(2)(1)(1)(1)})\cdot 1_A)\\
&\quad\times\omega(\beta^{3}(h_{(2)(1)(1)(2)(2)(1)}),\beta^{-1}(l_{(1)(1)})))\otimes\beta^{3}(h_{(2)(2)(1)(1)})\beta^{2}(l_{(2)(1)(1)})]\\
&\quad\otimes[(\alpha^{-1}(a_{(2)})(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)})\\
&\quad(((h_{(2)(1)(1)(2)(1)}\cdot 1_A)(\beta(h_{(2)(1)(1)(2)(2)(2)})\cdot 1_A))\omega(\beta^{-1}(h_{(2)(1)(2)}),\beta^{-2}(l_{(1)(2)})))\\
&\quad\times((h_{(2)(2)(1)(2)}\beta^{-1}(l_{(2)(1)(2)}))\cdot 1_A))\otimes\beta^{2}(h_{(2)(2)(2)})\beta(l_{(2)(2)})]\\
(1)\ &=[(\alpha^{-1}(a_{(1)})(((\beta^{-2}(h_{(1)})\cdot\alpha^{-3}(b))_{(1)})(h_{(2)(1)(1)(1)}\cdot 1_A)))\\
&\quad\times\omega(\beta^{4}(h_{(2)(1)(1)(2)(2)(1)}),l_{(1)(1)})\otimes\beta^{3}(h_{(2)(2)(1)(1)})\beta^{2}(l_{(2)(1)(1)})]\otimes[(\alpha^{-1}(a_{(2)})((\beta^{-2}(h_{(1)})\cdot\alpha^{-3}(b))_{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&\beta(h_{(2)(1)(1)(2)(1)}) \cdot 1_A))(((\beta^2(h_{(2)(1)(1)(2)(2)(2)}) \cdot 1_A)\omega(\beta^{-1}(h_{(2)(1)(2)}), \beta^{-2}(l_{(1)(2)})))\\
&\quad \times ((h_{(2)(2)(1)(2)}\beta^{-1}(l_{(2)(1)(2)})) \cdot 1_A)) \otimes \beta^2(h_{(2)(2)(2)})\beta(l_{(2)(2)})]\\
(5) =&[(\alpha^{-1}(a_{(1)})(((\beta(h_{(1)(1)(1)(1)}) \cdot \alpha^{-3}(b))_{(1)})(\beta(h_{(1)(1)(1)(2)(1)}) \cdot 1_A)))\\
&\quad \times \omega(\beta^2(h_{(1)(1)(2)(1)}), l_{(1)(1)}) \otimes \beta^2(h_{(2)(1)(1)})\beta^2(l_{(2)(1)(1)})]\\
&\quad \otimes [(\alpha^{-1}(a_{(2)})((\beta(h_{(1)(1)(1)(1)}) \cdot \alpha^{-3}(b))_{(2)}\\
&(\beta(h_{(1)(1)(1)(2)(2)}) \cdot 1_A)))(((h_{(1)(1)(2)(2)} \cdot 1_A)\omega(\beta^{-2}(h_{(1)(2)}), \beta^{-2}(l_{(1)(2)})))\\
&\quad \times ((\beta^{-1}(h_{(2)(1)(2)})\beta^{-1}(l_{(2)(1)(2)})) \cdot 1_A)) \otimes \beta(h_{(2)(2)})\beta(l_{(2)(2)})]\\
(31) =&[(\alpha^{-1}(a_{(1)})(\beta(h_{(1)(1)(1)(1)}) \cdot \alpha^{-2}(b_{(1)})))\\
&\quad \times \omega(\beta^2(h_{(1)(1)(2)(1)}), l_{(1)(1)}) \otimes \beta^2(h_{(2)(1)(1)})\beta^2(l_{(2)(1)(1)})]\\
&\quad \otimes [(\alpha^{-1}(a_{(2)})(\beta(h_{(1)(1)(1)(2)}) \cdot \alpha^{-2}(b_{(2)})))\\
&(((h_{(1)(1)(2)(2)} \cdot 1_A)\omega(\beta^{-2}(h_{(1)(2)}), \beta^{-2}(l_{(1)(2)})))\\
&\quad \times ((\beta^{-1}(h_{(2)(1)(2)})\beta^{-1}(l_{(2)(1)(2)})) \cdot 1_A)) \otimes \beta(h_{(2)(2)})\beta(l_{(2)(2)})]\\
(5) =&[(\alpha^{-1}(a_{(1)})(\beta^{-1}(h_{(1)(1)}) \cdot \alpha^{-2}(b_{(1)})))\omega(\beta(h_{(1)(2)(1)}), \beta^{-1}(l_{(1)})) \otimes \beta^2(h_{(2)(1)(1)})\beta^2(l_{(2)(1)(1)})]\\
&\quad \otimes [(\alpha^{-1}(a_{(2)})(h_{(1)(2)(2)} \cdot \alpha^{-2}(b_{(2)})))\omega(\beta(h_{(2)(1)(2)}), \beta(l_{(2)(1)(2)})) \otimes \beta(h_{(2)(2)})\beta(l_{(2)(2)})]\\
(17) =&[(\alpha^{-1}(a_{(1)})(\beta^{-1}(h_{(1)(1)}) \cdot \alpha^{-2}(b_{(1)})))\omega(\beta(h_{(1)(2)(1)}), \beta^{-1}(l_{(1)})) \otimes \beta^2(h_{(2)(1)(1)})\beta^2(l_{(2)(1)(1)})]\\
&\quad \otimes [(\alpha^{-1}(a_{(2)})((\beta^{-1}(h_{(1)(2)(2)}) \cdot \alpha^{-3}(b_{(2)}))(h_{(2)(1)(2)(1)} \cdot 1_A)))\\
&\quad \times \omega(\beta^2(h_{(2)(1)(2)(2)}), \beta(l_{(2)(1)(2)})) \otimes \beta(h_{(2)(2)})\beta(l_{(2)(2)})]\\
(5) =&[(\alpha^{-1}(a_{(1)})(\beta^{-2}(h_{(1)}) \cdot \alpha^{-2}(b_{(1)})))\omega(h_{(2)(1)}, \beta^{-1}(l_{(1)})) \otimes \underline{\beta^5(h_{(2)(2)(1)(1)(2)(1)})}\beta^2(l_{(2)(1)(1)})]\\
&\quad \otimes [(\alpha^{-1}(a_{(2)})(\underline{(\beta(h_{(2)(2)(1)(1)(1)}) \cdot \alpha^{-3}(b_{(2)}))(\beta^2(h_{(2)(2)(1)(1)(2)(2)}) \cdot 1_A)})\\
&\quad \times \omega(\beta^2(h_{(2)(2)(1)(2)}), \beta(l_{(2)(1)(2)})) \otimes \beta^2(h_{(2)(2)(2)})\beta(l_{(2)(2)})]\\
(32) =&[(\alpha^{-1}(a_{(1)})(\beta^{-2}(h_{(1)}) \cdot \alpha^{-2}(b_{(1)})))\omega(h_{(2)(1)}, \beta^{-1}(l_{(1)})) \otimes \beta^4(h_{(2)(2)(1)(1)(1)})\beta^2(l_{(2)(1)(1)})]\\
&\quad \otimes [(\alpha^{-1}(a_{(2)})(\beta^2(h_{(2)(2)(1)(1)(2)}) \cdot \alpha^{-2}(b_{(2)})))\\
&\quad \times \omega(\beta^2(h_{(2)(2)(1)(2)}), \beta(l_{(2)(1)(2)})) \otimes \beta^2(h_{(2)(2)(2)})\beta(l_{(2)(2)})]\\
(5) =&(\alpha^{-1}(a_{(1)})(\beta^{-1}(h_{(1)(1)}) \cdot \alpha^{-2}(b_{(1)})))(\omega(h_{(1)(2)(1)}, \beta^{-1}(l_{(1)(1)}))\\
&\quad \times ((h_{(1)(2)(2)(1)}\beta^{-1}(l_{(1)(2)(1)})) \cdot 1_A)) \otimes \beta^3(h_{(1)(2)(2)(2)})\beta^2(l_{(1)(2)(2)})\\
&\quad \otimes (\alpha^{-1}(a_{(2)})(\beta^{-1}(h_{(2)(1)}) \cdot \alpha^{-2}(b_{(2)})))(\omega(h_{(2)(2)(1)}, \beta^{-1}(l_{(2)(1)}))\\
&\quad \times (h_{(2)(2)(2)(1)}\beta^{-1}(l_{(2)(2)(1)}) \cdot 1_A)) \otimes \beta^3(h_{(2)(2)(2)(2)})\beta^2(l_{(2)(2)(2)})\\
=&((\alpha^{-1}(a_{(1)})((\beta^{-2}(h_{(1)(1)}) \cdot \alpha^{-3}(b_{(1)}))\omega(\beta^{-1}(h_{(1)(2)(1)}), \beta^{-2}(l_{(1)(1)}))))\\
&\quad \times (\beta(h_{(1)(2)(2)(1)})l_{(1)(2)(1)} \cdot 1_A)) \otimes \beta^3(h_{(1)(2)(2)(2)})\beta^2(l_{(1)(2)(2)})\\
&\quad \otimes (\alpha^{-1}(a_{(2)})((\beta^{-2}(h_{(2)(1)}) \cdot \alpha^{-3}(b_{(2)}))\omega(\beta^{-1}(h_{(2)(2)(1)}), \beta^{-2}(l_{(2)(1)}))))\\
&\quad \times (\beta(h_{(2)(2)(2)(1)})l_{(2)(2)(1)} \cdot 1_A) \otimes \beta^3(h_{(2)(2)(2)(2)})\beta^2(l_{(2)(2)(2)})\\
=&\Delta(a\sharp h)\Delta(b\sharp l).
\end{aligned}
$$

The proof is completed. □

Before we discuss the coquasitriangular structures on partial Hom-crossed products, we recall the definition of coquasitriangular Hom-Hopf algebra. A coquasitriangular Hom-bialgebra(Hopf) algebra is a monoidal Hom-bialgebra(Hopf) algebra (H, β) endowed with a map $\sigma : H \otimes H \to k \in \widetilde{\mathcal{H}}(\mathcal{M})$ such that

(B1) $\sigma(xy, h) = \sigma(x, h_{(1)})\sigma(y, h_{(2)})$,

(B2) $\sigma(x, hg) = \sigma(x_{(1)}, g)\sigma(x_{(2)}, h)$,
(B3) $\sigma(h_{(1)}, g_{(1)})h_{(2)}g_{(2)} = g_{(1)}h_{(1)}\sigma(h_{(2)}, g_{(2)})$,
(B4) $\sigma(1_H, h) = \sigma(h, 1_H) = \varepsilon_H(h)$,
(B5) $\sigma(\beta(h), \beta(g)) = \sigma(h, g)$,

for all $x, y, h, g \in H$. We call that σ is a Hom-braiding for H.

In what follows, we shall investigate when $A\sharp_\omega H$ forms a coquasitriangular Hom-bialgebra.

Let $\sigma : A\sharp_\omega H \otimes A\sharp_\omega H \to k$ be a linear map. We will use the following notations:

$$\begin{aligned}
p :& A \otimes A \to k, p(a, b) = \sigma(a\sharp 1_H, b\sharp 1_H),\\
\tau :& H \otimes H \to k, \tau(h, g) = \sigma(1_A\sharp h, 1_A\sharp g),\\
\mu :& A \otimes H \to k, \mu(a, h) = \sigma(a\sharp 1_H, 1_A\sharp h),\\
\nu :& H \otimes A \to k, \mu(h, a) = \sigma(1_A\sharp h, a\sharp 1_H).
\end{aligned}$$

Suppose that $(A\sharp_\omega H, \sigma)$ is a coquasitriangular Hom-bialgebra. Then we have the following result.

Proposition 5.3. *With the notations as above, we have*

1. $p(1_A, a) = p(a, 1_A) = \varepsilon_A(a)$, $p(\alpha(a), \alpha(b)) = p(a, b)$,
2. $\tau(1_H, h) = \tau(h, 1_H) = \varepsilon_H(h)$, $\tau(\beta(h), \beta(g)) = \tau(h, g)$,
3. $\mu(1_A, h) = \varepsilon_H(h)$, $\mu(a, 1_H) = \varepsilon_A(a)$, $\mu(\alpha(a), \beta(h)) = \mu(a, h)$,
4. $\nu(1_H, a) = \varepsilon_A(a)$, $\nu(h, 1_A) = \varepsilon_H(h)$, $\nu(\beta(h), \alpha(a)) = \nu(h, a)$.

In order to establish the conditions for a partial Hom-crossed prodcut $A\sharp_\omega H$ to be a coquasitriangular Hom-bialgebra, we shall introduce some definitions what we need.

Definition 5.4. *Let (A, α) and (H, β) be monoidal Hom-bialgebras, $\omega : H\otimes H \to k$ a map in $\widetilde{\mathcal{H}}(\mathcal{M})$ and $p : A\otimes A \to k$ a Hom-braiding for A. A linear map $\mu : A \otimes H \to k$ in $\widetilde{\mathcal{H}}(\mathcal{M})$ is called a right (p, ω) Hom-skew pairing on (A, H), if the following compatibilities are fulfilled, for any $a, b \in A, g, t \in H$,*

(C1) $\mu(ab, t) = \mu(a, t_{(1)})\mu(b, t_{(2)})$,
(C2) $\mu(a_{(1)}, g_{(2)}t_{(2)})p(a_{(2)}, \omega(g_{(1)}, t_{(1)})) = \mu(a_{(1)}, t)\mu(a_{(2)}, g)$,
(C3) $\mu(1_A, h) = \varepsilon_H(h)$, $\mu(a, 1_H) = \varepsilon_A(a)$, $\mu(\alpha(a), \beta(h)) = \mu(a, h)$.

Definition 5.5. *Let (A, α) and (H, β) be monoidal Hom-bialgebras, $\omega : H\otimes H \to k$ a map in $\widetilde{\mathcal{H}}(\mathcal{M})$ and $p : A\otimes A \to k$ a Hom-braiding for A. A linear map $\nu : H \otimes A \to k$ in $\widetilde{\mathcal{H}}(\mathcal{M})$ is called a left (p, ω) Hom-skew pairing on (H, A), if the following compatibilities are fulfilled, for any $a, b \in A, g, h \in H$,*

(D1) $\nu(h, ba) = \nu(h_{(1)}, a)\nu(h_{(2)}, b)$,
(D2) $p(\omega(h_{(1)}, g_{(1)}), a_{(1)})\nu(h_{(2)}g_{(2)}, a_{(2)}) = \nu(h, a_{(1)})\nu(g, a_{(2)})$,
(D3) $\nu(1_H, a) = \varepsilon_A(a)$, $\nu(h, 1_A) = \varepsilon_H(h)$, $\nu(\beta(h), \alpha(a)) = \nu(h, a)$.

Definition 5.6. *Let (A, α) and (H, β) be monoidal Hom-bialgebras, $\omega : H\otimes H \to k$ a map in $\widetilde{\mathcal{H}}(\mathcal{M})$ and $p : A\otimes A \to k$ a Hom-braiding for A, $\mu : A \otimes H \to k$ a right (p, ω) Hom-skew pairing and $\nu : H \otimes A \to k$ a left (p, ω) Hom-skew pairing. A linear map $\tau : H \otimes H \to k$ in $\widetilde{\mathcal{H}}(\mathcal{M})$ is called a (μ, ν) Hom-skew braiding on H, if the following compatibilities are fulfilled, for any $h, g, t \in H$,*

(E1) $\mu(\omega(h_{(1)}, g_{(1)}), t_{(1)})\tau(h_{(2)}g_{(2)}, t_{(2)}) = \tau(h, t_{(1)})\tau(g, t_{(2)})$,
(E2) $\tau(h_{(1)}, g_{(2)}t_{(2)})\nu(h_{(2)}, \omega(g_{(1)}, t_{(1)})) = \tau(h_{(1)}, t)\tau(h_{(2)}, g)$,
(E3) $\tau(h_{(1)}, g_{(1)})h_{(2)}g_{(2)} = g_{(1)}h_{(1)}\tau(h_{(2)}, g_{(2)})$,
(E4) $\tau(1_H, h) = \tau(h, 1_H) = \varepsilon_H(h)$, $\tau(\beta(h), \beta(g)) = \tau(h, g)$.

With the definitions introduced above, we have the following result.

Proposition 5.7. *Let $(A\sharp_\omega H, \sigma)$ be a coquasitriangular Hom-bialgebra. Then*

(F1) the coquasitriangular structure σ can be decomposed into

$$\sigma(a\sharp h, b\sharp g) = \mu(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})\nu(h_{(2)}, b_{(2)}). \tag{38}$$

(F2) (A, p) is a coquasitriangular Hom-bialgebra.
(F3) (A, H, μ) is a right (p, ω) Hom-skew pairing.
(F4) (H, A, ν) is a left (p, ω) Hom-skew pairing.
(F5) (H, τ) is a (μ, ν) Hom-skew braiding.

Proof. First, we prove that relation (38) indeed holds:

$$\begin{aligned}
&\sigma(\alpha(a)\sharp\beta(h), \alpha(b)\sharp\beta(g))\\
&\quad= \sigma((a\sharp 1_H)(1_A\sharp h), (b\sharp 1_H)(1_A\sharp g))\\
&\quad= \sigma(a_{(1)}\sharp 1_H, 1_A\sharp g_{(1)})\sigma(a_{(2)}\sharp 1_H, b_{(1)}\sharp 1_A)\sigma(1_A\sharp h_{(1)}, 1_A\sharp g_{(2)})\sigma(1_A\sharp h_{(2)}, b_{(2)}\sharp 1_H)\\
&\quad= \mu(a_{(1)}, g_{(1)})p(a_{(2)}, b_{(1)})\tau(h_{(1)}, g_{(2)})\nu(h_{(2)}, b_{(2)}).
\end{aligned}$$

(C3), (D3) and (E3) follows from Proposition 5.3. Since σ satisfies (B1), then for all $a, b, c \in A$ and $h, g, t \in H$, we have

$$\begin{aligned}
&\sigma(a[(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))]\sharp\beta^2(h_{(2)(2)})\beta(g_{(2)}), c\sharp t)\\
&\qquad=\sigma(a\sharp h, c_{(1)}\sharp t_{(1)})\sigma(b\sharp g, c_{(2)}\sharp t_{(2)}).
\end{aligned} \tag{39}$$

Considering $h = g = 1_H$ and $c = 1_A$ in (39) yields (C1). Taking $a = b = 1_A$ and $t = 1_H$ in (39), we get (D2). We apply (39) for $a = b = c = 1_A$ and get (E1). Since σ fulfills (B2), we have

$$\begin{aligned}
&\sigma(a\sharp h, b[(\beta^{-1}(g_{(1)}) \cdot \alpha^{-2}(c))\omega(g_{(2)(1)}, \beta^{-1}(t_{(1)}))]\sharp\beta^2(g_{(2)(2)})\beta(t_{(2)}))\\
&\qquad=\sigma(a_{(1)}\sharp h_{(1)}, c\sharp t)\sigma(a_{(2)}\sharp h_{(2)}, b\sharp g).
\end{aligned} \tag{40}$$

If we let $b = c = 1_A$ and $h = 1_H$ in (40), it yields (C2). From (40) applying for $g = t = 1_H$ and $a = 1_A$, we get (D1). Considering $a = b = c = 1_A$ in (40) yields (E2).

By (B3), we have

$$\begin{aligned}
&\sigma(a_{(1)}\sharp h_{(1)}, b_{(1)}\sharp g_{(1)})\\
&\quad\times\{a_{(2)}[(\beta^{-1}(h_{(2)(1)}) \cdot \alpha^{-2}(b_{(2)}))\omega(h_{(2)(2)(1)}, \beta^{-1}(g_{(2)(1)}))]\sharp\beta^2(h_{(2)(2)(2)})\beta(g_{(2)(2)})\}\\
&=\{b_{(1)}[(\beta^{-1}(g_{(1)(1)}) \cdot \alpha^{-2}(a_{(1)}))\omega(g_{(1)(2)(1)}, \beta^{-1}(h_{(1)(1)}))]\sharp\beta^2(g_{(1)(2)(2)})\beta(h_{(1)(2)})\}\\
&\quad\times\sigma(a_{(2)}\sharp h_{(2)}, b_{(2)}\sharp g_{(2)}).
\end{aligned} \tag{41}$$

Taking $a = b = 1_A$ in the above equation and then applying $\varepsilon \otimes id$ to both sides, we gain (E3). □

Proposition 5.8. *Suppose that $A\sharp_\omega H$ has a coquasitriangular structure σ. Then, for all $a, b, c \in A$ and $h, g, t \in H$, we have*

(G1) $\nu(h_{(1)}, b_{(1)})(\beta(h_{(2)(1)}) \cdot \alpha^{-1}(b_{(2)}) \otimes h_{(2)(2)})$
$= (b_{(1)}(h_{(1)(1)} \cdot 1_A) \otimes h_{(1)(2)})\nu(h_{(2)}, b_{(2)})$,
(G2) $\mu(a_{(1)}, g_{(1)})(a_{(2)}(g_{(2)(1)} \cdot 1_A) \otimes g_{(2)(2)}) = (\beta(g_{(1)(1)}) \cdot a_{(1)} \otimes g_{(1)(2)})\mu(a_{(2)}, g_{(2)})$,
(G3) $\tau(h_{(1)}, g_{(1)})\omega(h_{(2)}, g_{(2)}) = \omega(g_{(1)}, h_{(1)})\tau(h_{(2)}, g_{(2)})$,
(G4) $\mu(a_{(1)}, \beta(g_{(2)}))p(a_{(2)}, g_{(1)} \cdot \alpha^{-1}(c)) = p(a_{(1)}, c)\mu(a_{(2)}, g)$,
(G5) $\tau(h_{(1)}, \beta(g_{(2)}))\nu(h_{(2)}, g_{(1)} \cdot \alpha^{-1}(c)) = \nu(h_{(1)}, c)\tau(h_{(2)}, g)$,
(G6) $p(h_{(1)} \cdot \alpha^{-1}(b), c_{(1)})\nu(\beta(h_{(2)}), c_{(2)}) = \nu(h, c_{(1)})p(b, c_{(2)})$,
(G7) $\mu(h_{(1)} \cdot \alpha^{-1}(b), t_{(1)})\tau(\beta(h_{(2)}), t_{(2)}) = \tau(h, t_{(1)})\mu(b, t_{(2)})$.

Proof. If we let $a = 1_A$ and $g = 1_H$ in (41), it follows that (G1) holds. We apply (41) for $b = 1_A$ and $h = 1_H$ and have (G2). Applying $id \otimes \varepsilon$ to (41) for $a = b = 1_A$ yields (G3). The next two compatibilities (G4) and (G5) can be obtained by considering $h = t = 1_H$, $b = 1_A$,respectively, $a = b = 1_A$ and $t = 1_H$ in (40). To this end, relations (G6) and (G7) can be derived from (39) by considering $g = t = 1_H$, $a = 1_A$, respectively, $a = c = 1_A$, $g = 1_H$. □

Next, we shall show that Proposition 5.3, 5.7 and 5.8 establish sufficient conditions for $A\sharp_\omega H$ to be a coquasitriangular bialgebra.

Proposition 5.9. *Let A, H be monoidal Hom-bialgebras, and A also a twisted partial (H,β)-Hom-module algebra. Assume that there exists linear maps $p : A \otimes A \to k, \tau : H \otimes H \to k, \mu : A \otimes H \to k$ and $\nu : H \otimes A \to k$ in $\widetilde{\mathcal{H}}(\mathcal{M})$ such that (A, p) is a coquasitriangular Hom-bialgebra, (A, H, μ) is a right (p, ω) Hom-skew pairing, (H, A, ν) is a left (p, ω) Hom-skew pairing, (H, τ) is a (μ, ν) Hom-skew braiding and the compatibilities (G1)-(G7) are fulfilled. Then $A\sharp_\omega H$ is a coquasitriangular Hom-bialgebra via σ given by (38).*

Proof. Taking $t = g = 1_H$ in (C2),(D2), (E1) and (E2) respectively, one obtains the following useful equations:

$$\mu(a_{(1)}, \beta(g_{(2)}))p(a_{(2)}, g_{(1)} \cdot 1_A) = \mu(\alpha^{-1}(a), g), \tag{42}$$

$$p(h_{(1)} \cdot 1_A, c_{(1)})\nu(\beta(h_{(2)}), c_{(2)}) = \nu(h, \alpha^{-1}(c)), \tag{43}$$

$$\mu(h_{(1)} \cdot 1_A, t_{(1)})\tau(\beta(h_{(2)}), t_{(2)}) = \tau(h, \beta^{-1}(t)), \tag{44}$$

$$\tau(h_{(1)}, \beta(g_{(2)}))\nu(h_{(2)}, g_{(1)} \cdot 1_A) = \tau(\beta^{-1}(h), g). \tag{45}$$

Now, we shall show that σ fulfills the equations (B1)-(B5). The proof is very complicated, thus we only check that (B1) holds, other equalities can been proved similarly.

$$\begin{aligned}
&\sigma((a\sharp h)(b\sharp g), c\sharp t)\\
&=\mu(a_{(1)}[(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))_{(1)}\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))_{(1)}], t_{(1)})\\
&\quad p(a_{(2)}[(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))_{(2)}\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))_{(2)}], c_{(1)})\\
&\quad \tau\underline{(\beta^{2}(h_{(2)(2)(1)})\beta(g_{(2)(1)}), t_{(2)})}\nu\underline{(\beta^{2}(h_{(2)(2)(2)})\beta(g_{(2)(2)}), c_{(2)})}\\
(43,44) &=\mu(a_{(1)}[(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))_{(1)}\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))_{(1)}], t_{(1)})\\
&\quad p(a_{(2)}[(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))_{(2)}\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))_{(2)}], c_{(1)})\\
&\quad \mu((\beta^{2}(h_{(2)(2)(1)(1)})\beta(g_{(2)(1)(1)})) \cdot 1_A, \beta(t_{(2)(1)}))\\
&\quad \tau((\beta^{2}(h_{(2)(2)(1)(2)})\beta(g_{(2)(1)(2)})), t_{(2)(2)})\\
&\quad p((\beta^{2}(h_{(2)(2)(2)(1)})\beta(g_{(2)(2)(1)})) \cdot 1_A, \alpha(c_{(2)(1)}))\\
&\quad \times \nu(\beta^{2}(h_{(2)(2)(2)(2)})\beta(g_{(2)(2)(2)}), c_{(2)(2)})\\
(5) &=\mu(a_{(1)}[(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))_{(1)}\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))_{(1)}], \beta(t_{(1)(1)}))\\
&\quad \mu((\beta^{2}(h_{(2)(2)(1)(1)})\beta(g_{(2)(1)(1)})) \cdot 1_A, \beta(t_{(1)(2)}))\\
&\quad p(a_{(2)}[(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))_{(2)}\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))_{(2)}], \alpha(c_{(1)(1)}))\\
&\quad p((\beta^{2}(h_{(2)(2)(2)(1)})\beta(g_{(2)(2)(1)})) \cdot 1_A, \alpha(c_{(1)(2)}))\\
&\quad \tau((\beta^{2}(h_{(2)(2)(1)(2)})\beta(g_{(2)(1)(2)})), \beta^{-1}(t_{(2)}))\nu(\beta^{2}(h_{(2)(2)(2)(2)})\beta(g_{(2)(2)(2)}), \alpha^{-1}(c_{(2)}))\\
(C1, B1) &=\mu(\{a_{(1)}[(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))_{(1)}\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))_{(1)}]\}\\
&\quad \times [(\beta^{2}(h_{(2)(2)(1)(1)})\beta(g_{(2)(1)(1)})) \cdot 1_A], \beta(t_{(1)}))\\
&\quad p(\{a_{(2)}[(\beta^{-1}(h_{(1)}) \cdot \alpha^{-2}(b))_{(2)}\omega(h_{(2)(1)}, \beta^{-1}(g_{(1)}))_{(2)}]\}[(\beta^{2}(h_{(2)(2)(2)(1)})\beta(g_{(2)(2)(1)})) \cdot 1_A], \alpha(c_{(1)}))\\
&\quad \tau((\beta^{2}(h_{(2)(2)(1)(2)})\beta(g_{(2)(1)(2)})), \beta^{-1}(t_{(2)}))\nu(\beta^{2}(h_{(2)(2)(2)(2)})\beta(g_{(2)(2)(2)}), \alpha^{-1}(c_{(2)}))
\end{aligned}$$

$$
\begin{aligned}
(5) =&\mu(\{a_{(1)}[(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)}\omega(h_{(2)(1)},\beta^{-1}(g_{(1)}))_{(1)}]\}\\
&\times[(\beta^{2}(h_{(2)(2)(1)(1)})\beta(g_{(2)(1)(1)}))\cdot 1_A],\beta(t_{(1)}))\\
&p(\{a_{(2)}[(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)}\omega(h_{(2)(1)},\beta^{-1}(g_{(1)}))_{(2)}]\}\\
&\times[\underline{(\beta^{3}(h_{(2)(2)(1)(2)(2)})\beta^{2}(g_{(2)(1)(2)(2)}))\cdot 1_A}],\alpha(c_{(1)}))\\
&\tau((\underline{\beta^{3}(h_{(2)(2)(1)(2)(1)})\beta^{2}(g_{(2)(1)(2)(1)})}),\beta^{-1}(t_{(2)}))\nu(\beta(h_{(2)(2)(2)})g_{(2)(2)},\alpha^{-1}(c_{(2)}))\\
(32) =&\mu(\{a_{(1)}[(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)}\omega(h_{(2)(1)},\beta^{-1}(g_{(1)}))_{(1)}]\}\\
&\times[(\beta^{2}(h_{(2)(2)(1)(1)})\beta(g_{(2)(1)(1)}))\cdot 1_A],\beta(t_{(1)}))\\
&p(\{a_{(2)}[(\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)}\omega(h_{(2)(1)},\beta^{-1}(g_{(1)}))_{(2)}]\}[((\beta^{2}(h_{(2)(2)(1)(2)(1)})\beta(g_{(2)(1)(2)(1)}))\cdot 1_A)\\
&\times(\beta^{3}(h_{(2)(2)(1)(2)(2)(2)})\beta^{2}(g_{(2)(1)(2)(2)(2)})\cdot 1_A)],\alpha(c_{(1)}))\\
&\tau((\beta^{4}(h_{(2)(2)(1)(2)(2)(1)})\beta^{3}(g_{(2)(1)(2)(2)(1)})),\beta^{-1}(t_{(2)}))\nu(\beta(h_{(2)(2)(2)})g_{(2)(2)},\alpha^{-1}(c_{(2)}))\\
(1) =&\mu(\{\alpha(a_{(1)})[((h_{(1)}\cdot\alpha^{-1}(b))_{(1)})\\
&\times(\omega(h_{(2)(1)},\beta^{-1}(g_{(1)}))_{(1)}((h_{(2)(2)(1)(1)}\beta^{-1}(g_{(2)(1)(1)}))\cdot 1_A))]\},\beta(t_{(1)}))\\
&p(\{(a_{(2)}(h_{(1)}\cdot\alpha^{-1}(b))_{(2)})[(\omega(h_{(2)(1)},\beta^{-1}(g_{(1)}))_{(2)}(\beta(h_{(2)(2)(1)(2)(1)})g_{(2)(1)(2)(1)})\cdot 1_A)\\
&\times(\beta^{3}(h_{(2)(2)(1)(2)(2)(2)})\beta^{2}(g_{(2)(1)(2)(2)(2)})\cdot 1_A)],\alpha(c_{(1)}))\\
&\tau((\beta^{4}(h_{(2)(2)(1)(2)(2)(1)})\beta^{3}(g_{(2)(1)(2)(2)(1)})),\beta^{-1}(t_{(2)}))\nu(\beta(h_{(2)(2)(2)})g_{(2)(2)},\alpha^{-1}(c_{(2)}))\\
(5) =&\mu(\{\alpha(a_{(1)})[((h_{(1)}\cdot\alpha^{-1}(b))_{(1)})\\
&\times\underline{(\omega(\beta^{2}(h_{(2)(1)(1)(1)}),\beta(g_{(1)(1)(1)}))_{(1)}((\beta(h_{(2)(1)(1)(2)(1)})g_{(1)(1)(2)(1)})\cdot 1_A)})]\},\beta(t_{(1)}))\\
&p(\{(a_{(2)}(h_{(1)}\cdot\alpha^{-1}(b))_{(2)})[\underline{(\omega(\beta^{2}(h_{(2)(1)(1)(1)}),\beta(g_{(1)(1)(1)}))_{(2)}((\beta(h_{(2)(1)(1)(2)(2)})g_{(1)(1)(2)(2)})\cdot 1_A})\\
&\times(\beta(h_{(2)(1)(2)(2)})g_{(1)(2)(2)}\cdot 1_A)],\alpha(c_{(1)}))\\
&\tau((\beta^{2}(h_{(2)(1)(2)(1)})\beta(g_{(1)(2)(1)})),\beta^{-1}(t_{(2)}))\nu(h_{(2)(2)}\beta^{-1}(g_{(2)}),\alpha^{-1}(c_{(2)}))\\
(34) =&\mu(\{\alpha(a_{(1)})[((h_{(1)}\cdot\alpha^{-1}(b))_{(1)})(\omega(\beta^{2}(h_{(2)(1)(1)(1)}),\beta(g_{(1)(1)(1)})))]\},\beta(t_{(1)}))\\
&p(\{(a_{(2)}(h_{(1)}\cdot\alpha^{-1}(b))_{(2)})[(\omega(\beta^{2}(h_{(2)(1)(1)(2)}),\beta(g_{(1)(1)(2)})))\\
&\times(\beta(h_{(2)(1)(2)(2)})g_{(1)(2)(2)}\cdot 1_A)],\alpha(c_{(1)})))\\
&\tau((\beta^{2}(h_{(2)(1)(2)(1)})\beta(g_{(1)(2)(1)})),\beta^{-1}(t_{(2)}))\nu(h_{(2)(2)}\beta^{-1}(g_{(2)}),\alpha^{-1}(c_{(2)}))\\
(B1,5) =&\mu(\{\alpha(a_{(1)})[((h_{(1)}\cdot\alpha^{-1}(b))_{(1)})(\omega(\beta(h_{(2)(1)(1)}),g_{(1)(1)}))]\},\beta(t_{(1)}))\\
&p(\alpha^{-1}(a_{(2)}(h_{(1)}\cdot\alpha^{-1}(b))_{(2)})(\omega(\beta(h_{(2)(1)(2)}),g_{(1)(2)})),c_{(1)})\\
&\tau((\beta(h_{(2)(2)(1)})g_{(2)(1)}),\beta^{-1}(t_{(2)}))\\
&\underline{p((\beta^{2}(h_{(2)(2)(2)(1)})\beta(g_{(2)(2)(1)})\cdot 1_A),\alpha(c_{(2)(1)}))\nu(\beta^{2}(h_{(2)(2)(2)(2)})\beta(g_{(2)(2)(2)}),c_{(2)(2)})}\\
(43) =&\mu(\{\alpha(a_{(1)})[((h_{(1)}\cdot\alpha^{-1}(b))_{(1)})\underline{(\omega(\beta(h_{(2)(1)(1)}),g_{(1)(1)}))}]\},\beta(t_{(1)}))\\
&p(\alpha^{-1}(a_{(2)}(h_{(1)}\cdot\alpha^{-1}(b))_{(2)})\underline{(\omega(\beta(h_{(2)(1)(2)}),g_{(1)(2)}))},c_{(1)})\\
&\tau((\beta(h_{(2)(2)(1)})g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta^{2}(h_{(2)(2)(2)})\beta(g_{(2)(2)}),c_{(2)})\\
(17) =&\mu(\{\alpha(a_{(1)})[((h_{(1)}\cdot\alpha^{-1}(b))_{(1)})\\
&\times((\beta(h_{(2)(1)(1)(1)})\cdot 1_A)\omega(\beta(h_{(2)(1)(1)(2)}),\beta^{-1}(g_{(1)(1)})))]\},\beta(t_{(1)}))\\
&p(\alpha^{-1}(a_{(2)}(h_{(1)}\cdot\alpha^{-1}(b))_{(2)})\\
&\times((\beta(h_{(2)(1)(2)(1)})\cdot 1_A)\omega(\beta(h_{(2)(1)(2)(2)}),\beta^{-1}(g_{(1)(2)}))),c_{(1)})\\
&\tau((\beta(h_{(2)(2)(1)})g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta^{2}(h_{(2)(2)(2)})\beta(g_{(2)(2)}),c_{(2)})
\end{aligned}
$$

$$\begin{aligned}
(1) =&\mu(\{\alpha(a_{(1)})[(((\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)})(\beta(h_{(2)(1)(1)(1)})\cdot 1_A))\\
&\times\omega(\beta^2(h_{(2)(1)(1)(2)}),g_{(1)(1)})]\},\beta(t_{(1)}))\\
&p(\alpha^{-1}(a_{(2)}((\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)}(\beta(h_{(2)(1)(2)(1)})\cdot 1_A)))\\
&\times\omega(\beta^2(h_{(2)(1)(2)(2)}),g_{(1)(2)}),c_{(1)})\\
&\tau((\beta(h_{(2)(2)(1)})g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta^2(h_{(2)(2)(2)})\beta(g_{(2)(2)}),c_{(2)})\\
(5) =&\mu(\{\alpha(a_{(1)})[(((\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)})(h_{(2)(1)(1)}\cdot 1_A))\\
&\times\omega(\underline{\beta^3(h_{(2)(1)(2)(1)(1)})},g_{(1)(1)})]\},\beta(t_{(1)}))\\
&p(\alpha^{-1}(a_{(2)}((\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)}\underline{(\beta^2(h_{(2)(1)(2)(1)(2)})\cdot 1_A)}))\\
&\times\omega(\beta^2(h_{(2)(1)(2)(2)}),g_{(1)(2)}),c_{(1)})\\
&\tau((\beta(h_{(2)(2)(1)})g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta^2(h_{(2)(2)(2)})\beta(g_{(2)(2)}),c_{(2)})\\
(32) =&\mu(\{\alpha(a_{(1)})[(((\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(1)})(h_{(2)(1)(1)}\cdot 1_A))\\
&\times\omega(\beta^4(h_{(2)(1)(2)(1)(2)(1)}),g_{(1)(1)})]\},\beta(t_{(1)}))\\
&p(\alpha^{-1}(a_{(2)}((\alpha^{-1}((\beta^{-1}(h_{(1)})\cdot\alpha^{-2}(b))_{(2)})(\beta(h_{(2)(1)(2)(1)(1)})\cdot 1_A))\\
&\times(\beta^3(h_{(2)(1)(2)(1)(2)(2)})\cdot 1_A)))\omega(\beta^2(h_{(2)(1)(2)(2)}),g_{(1)(2)}),c_{(1)})\\
&\tau((\beta(h_{(2)(2)(1)})g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta^2(h_{(2)(2)(2)})\beta(g_{(2)(2)}),c_{(2)})\\
(5) =&\mu(\{\alpha(a_{(1)})[\underline{(((\beta^2(h_{(1)(1)(1)(1)})\cdot\alpha^{-2}(b))_{(1)})(\beta^2(h_{(1)(1)(1)(2)(1)})\cdot 1_A)})\\
&\times\omega(\beta^2(h_{(1)(1)(2)(1)}),g_{(1)(1)})]\},\beta(t_{(1)}))\\
&p(\alpha^{-1}(a_{(2)}(\alpha^{-1}\underline{(((\beta^2(h_{(1)(1)(1)(1)})\cdot\alpha^{-2}(b))_{(2)})(\beta^2(h_{(1)(1)(1)(2)(2)})\cdot 1_A)})\\
&\times(\beta(h_{(1)(1)(2)(2)})\cdot 1_A)))\omega(h_{(1)(2)},g_{(1)(2)}),c_{(1)})\\
&\tau((h_{(2)(1)}g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta(h_{(2)(2)})\beta(g_{(2)(2)}),c_{(2)})\\
(34) =&\mu(\{\alpha(a_{(1)})[(\beta^2(h_{(1)(1)(1)(1)})\cdot\alpha^{-1}(b_{(1)}))\omega(\beta^2(h_{(1)(1)(2)(1)}),g_{(1)(1)})]\},\beta(t_{(1)}))\\
&p(\alpha^{-1}(a_{(2)}(\alpha^{-1}(\beta^2(h_{(1)(1)(1)(2)})\cdot\alpha^{-1}(b_{(2)}))(\beta(h_{(1)(1)(2)(2)})\cdot 1_A)))\\
&\omega(h_{(1)(2)},g_{(1)(2)}),c_{(1)})\tau((h_{(2)(1)}g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta(h_{(2)(2)})\beta(g_{(2)(2)}),c_{(2)})\\
(1) =&\mu(\{\alpha(a_{(1)})[(\beta^2(h_{(1)(1)(1)(1)})\cdot\alpha^{-1}(b_{(1)}))\omega(\beta^2(h_{(1)(1)(2)(1)}),g_{(1)(1)})]\},\beta(t_{(1)}))\\
&p(a_{(2)}((\beta(h_{(1)(1)(1)(2)})\cdot\alpha^{-2}(b_{(2)}))\\
&\times((h_{(1)(1)(2)(2)}\cdot 1_A)\omega(\beta^{-2}(h_{(1)(2)}),\beta^{-2}(g_{(1)(2)})))),c_{(1)})\\
&\tau((h_{(2)(1)}g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta(h_{(2)(2)})\beta(g_{(2)(2)}),c_{(2)})\\
(5) =&\mu(\{\alpha(a_{(1)})[(\beta(h_{(1)(1)(1)})\cdot\alpha^{-1}(b_{(1)}))\omega(\beta(h_{(1)(2)(1)}),g_{(1)(1)})]\},\beta(t_{(1)}))\\
&p(a_{(2)}((h_{(1)(1)(2)}\cdot\alpha^{-2}(b_{(2)}))\underline{((h_{(1)(2)(2)(1)}\cdot 1_A)\omega(h_{(1)(2)(2)(2)},\beta^{-2}(g_{(1)(2)}))))},c_{(1)})\\
&\tau((h_{(2)(1)}g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta(h_{(2)(2)})\beta(g_{(2)(2)}),c_{(2)})\\
(17) =&\mu(\{\alpha(a_{(1)})[(\beta(h_{(1)(1)(1)})\cdot\alpha^{-1}(b_{(1)}))\omega(\beta(h_{(1)(2)(1)}),g_{(1)(1)})]\},\beta(t_{(1)}))\\
&p(a_{(2)}((h_{(1)(1)(2)}\cdot\alpha^{-2}(b_{(2)}))\omega(h_{(1)(2)(2)},\beta^{-1}(g_{(1)(2)}))),c_{(1)})\\
&\tau((h_{(2)(1)}g_{(2)(1)}),\beta^{-1}(t_{(2)}))\nu(\beta(h_{(2)(2)})\beta(g_{(2)(2)}),c_{(2)})\\
(C1,B1) =&\mu(\alpha(a_{(1)}),\beta(t_{(1)(1)}))\mu(\beta(h_{(1)(1)(1)})\cdot\alpha^{-1}(b_{(1)}),\beta(t_{(1)(2)(1)}))\\
&\mu(\omega(\beta(h_{(1)(2)(1)}),g_{(1)(1)}),\beta(t_{(1)(2)(2)}))p(a_{(2)},c_{(1)(1)})p(h_{(1)(1)(2)}\cdot\alpha^{-2}(b_{(2)}),c_{(1)(2)(1)})\\
&p(\omega(h_{(1)(2)(2)},\beta^{-1}(g_{(1)(2)})),c_{(1)(2)(2)})p(\beta(h_{(2)(2)(1)})\beta(g_{(2)(2)(1)})\cdot 1_A,\alpha(c_{(2)(1)}))\\
&\nu(\beta(h_{(2)(2)(2)})\beta(g_{(2)(2)(2)}),c_{(2)(2)})\tau((h_{(2)(1)}g_{(2)(1)}),\beta^{-1}(t_{(2)}))
\end{aligned}$$

$$
\begin{aligned}
(5) =&\mu(\alpha(a_{(1)}),\beta(t_{(1)(1)}))\mu(\beta(h_{(1)(1)(1)})\cdot\alpha^{-1}(b_{(1)}),\beta(t_{(1)(2)(1)}))\\
&\mu(\omega(\beta(h_{(1)(2)(1)}),g_{(1)(1)}),\beta(t_{(1)(2)(2)}))p(a_{(2)},c_{(1)(1)})p(h_{(1)(1)(2)}\cdot\alpha^{-2}(b_{(2)}),\alpha^{-1}(c_{(1)(2)}))\\
&p(\omega(h_{(1)(2)(2)},\beta^{-1}(g_{(1)(2)})),c_{(2)(1)(1)})p(\beta(h_{(2)(2)(1)})\beta(g_{(2)(2)(1)})\cdot 1_A,\alpha^2(c_{(2)(1)(2)}))\\
&\nu(\beta(h_{(2)(2)(2)})\beta(g_{(2)(2)(2)}),c_{(2)(2)})\tau((h_{(2)(1)}g_{(2)(1)}),\beta^{-1}(t_{(2)}))\\
(B1) =&\mu(\alpha(a_{(1)}),\beta(t_{(1)(1)}))\mu(\beta(h_{(1)(1)(1)})\cdot\alpha^{-1}(b_{(1)}),\beta(t_{(1)(2)(1)}))\\
&\mu(\omega(\beta(h_{(1)(2)(1)}),g_{(1)(1)}),\beta(t_{(1)(2)(2)}))p(a_{(2)},c_{(1)(1)})p(h_{(1)(1)(2)}\cdot\alpha^{-2}(b_{(2)}),\alpha^{-1}(c_{(1)(2)}))\\
&p(\omega(h_{(1)(2)(2)},\beta^{-1}(g_{(1)(2)}))(\beta^{-1}(h_{(2)(2)(1)})\beta^{-1}(g_{(2)(2)(1)})\cdot 1_A)],c_{(2)(1)})\\
&\nu(\beta(h_{(2)(2)(2)})\beta(g_{(2)(2)(2)}),c_{(2)(2)})\tau((h_{(2)(1)}g_{(2)(1)}),\beta^{-1}(t_{(2)}))\\
(5) =&\mu(\alpha(a_{(1)}),\beta(t_{(1)(1)}))\mu(h_{(1)(1)}\cdot\alpha^{-1}(b_{(1)}),\beta(t_{(1)(2)(1)}))\\
&\mu(\omega(\beta(h_{(2)(1)(1)}),\beta^{-1}(g_{(1)})),\beta(t_{(1)(2)(2)}))p(a_{(2)},c_{(1)(1)})p(\beta^{-1}(h_{(1)(2)})\cdot\alpha^{-2}(b_{(2)}),\beta^{-1}(c_{(1)(2)}))\\
&p(\omega(\beta(h_{(2)(2)(1)}),\beta(g_{(2)(2)(1)})),c_{(2)(1)})\\
&\nu(\beta(h_{(2)(2)(2)})\beta(g_{(2)(2)(2)}),c_{(2)(2)})\tau((\beta(h_{(2)(1)(2)})g_{(2)(1)}),\beta^{-1}(t_{(2)}))\\
(D2) =&\mu(\alpha(a_{(1)}),\beta(t_{(1)(1)}))\mu(h_{(1)(1)}\cdot\alpha^{-1}(b_{(1)}),\beta(t_{(1)(2)(1)}))\\
&\mu(\omega(\beta(h_{(2)(1)(1)}),\beta^{-1}(g_{(1)})),\beta(t_{(1)(2)(2)}))p(a_{(2)},c_{(1)(1)})p(\beta^{-1}(h_{(1)(2)})\cdot\alpha^{-2}(b_{(2)}),\beta^{-1}(c_{(1)(2)}))\\
&\nu(\beta(h_{(2)(2)}),c_{(2)(1)})\nu(\beta(g_{(2)(2)}),c_{(2)(2)})\tau((\beta(h_{(2)(1)(2)})g_{(2)(1)}),\beta^{-1}(t_{(2)}))\\
(5) =&\mu(\alpha(a_{(1)}),t_{(1)})\mu(h_{(1)(1)}\cdot\alpha^{-1}(b_{(1)}),t_{(2)(1)})\underline{\mu(\omega(\beta(h_{(2)(1)(1)}),g_{(1)(1)}),\beta(t_{(2)(2)(1)}))}\\
&p(a_{(2)},c_{(1)(1)})p(\beta^{-1}(h_{(1)(2)})\cdot\alpha^{-2}(b_{(2)}),\beta^{-1}(c_{(1)(2)}))\\
&\nu(\beta(h_{(2)(2)}),c_{(2)(1)})\nu(g_{(2)},c_{(2)(2)})\underline{\tau((\beta(h_{(2)(1)(2)})g_{(1)(2)}),\beta(t_{(2)(2)(2)}))}\\
(E1) =&\mu(\alpha(a_{(1)}),t_{(1)})\mu(h_{(1)(1)}\cdot\alpha^{-1}(b_{(1)}),t_{(2)(1)})\tau(\beta(h_{(2)(1)}),\beta(t_{(2)(2)(1)}))\\
&p(a_{(2)},c_{(1)(1)})p(\beta^{-1}(h_{(1)(2)})\cdot\alpha^{-2}(b_{(2)}),\beta^{-1}(c_{(1)(2)}))\\
&\nu(\beta(h_{(2)(2)}),c_{(2)(1)})\nu(g_{(2)},c_{(2)(2)})\tau(g_{(1)},\beta(t_{(2)(2)(2)}))\\
(43) =&\mu(\alpha(a_{(1)}),t_{(1)})\mu(h_{(1)(1)}\cdot\alpha^{-1}(b_{(1)}),t_{(2)(1)})\tau(\beta(h_{(2)(1)}),\beta(t_{(2)(2)(1)}))\\
&p(a_{(2)},c_{(1)(1)})p(\beta^{-1}(h_{(1)(2)})\cdot\alpha^{-2}(b_{(2)}),\alpha^{-1}(c_{(1)(2)}))\\
&p(\beta(h_{(2)(2)(1)})\cdot 1_A,\alpha(c_{(2)(1)(1)}))\nu(\beta(h_{(2)(2)(2)}),c_{(2)(1)(2)})\nu(g_{(2)},c_{(2)(2)})\tau(g_{(1)},\beta(t_{(2)(2)(2)}))\\
(5) =&\mu(\alpha(a_{(1)}),t_{(1)})\mu(\beta^{-1}(h_{(1)})\cdot\alpha^{-1}(b_{(1)}),t_{(2)(1)})\tau(\beta^2(h_{(2)(1)(1)}),\beta(t_{(2)(2)(1)}))\\
&p(a_{(2)},\beta^{-1}(c_{(1)}))p((\beta(h_{(2)(1)(2)})\cdot\alpha^{-1}(b_{(2)})),\alpha(c_{(2)(1)(1)}))\\
&\nu(h_{(2)(2)}),c_{(2)(1)(2)})\nu(g_{(2)},c_{(2)(2)})\tau(g_{(1)},\beta(t_{(2)(2)(2)}))\\
=&\mu(\alpha(a_{(1)}),t_{(1)})\mu(\beta^{-1}(h_{(1)})\cdot\alpha^{-1}(b_{(1)}),t_{(2)(1)})\tau(\beta(h_{(2)(1)}),\beta(t_{(2)(2)(1)}))\\
&p(a_{(2)},\beta^{-1}(c_{(1)}))p((h_{(2)(2)(1)}\cdot\alpha^{-2}(b_{(2)})),c_{(2)(1)(1)})\\
&\nu(\beta(h_{(2)(2)(2)}),c_{(2)(1)(2)})\nu(g_{(2)},c_{(2)(2)})\tau(g_{(1)},\beta(t_{(2)(2)(2)}))\\
=&\mu(\alpha(a_{(1)}),t_{(1)})\mu(\beta^{-1}(h_{(1)})\cdot\alpha^{-1}(b_{(1)}),t_{(2)(1)})\tau(\beta(h_{(2)(1)}),\beta(t_{(2)(2)(1)}))\\
&p(a_{(2)},\beta^{-1}(c_{(1)}))p(\alpha^{-1}(b_{(2)}),c_{(2)(1)(2)})\nu(h_{(2)(2)},c_{(2)(1)(1)})\nu(g_{(2)},c_{(2)(2)})\tau(g_{(1)},\beta(t_{(2)(2)(2)}))\\
=&\mu(\alpha(a_{(1)}),t_{(1)})\mu(b_{(1)},\beta(t_{(2)(1)(2)}))\tau(h_{(1)},\beta(t_{(2)(1)(1)}))\\
&p(a_{(2)},\beta^{-1}(c_{(1)}))p(\alpha^{-1}(b_{(2)}),c_{(2)(1)(2)})\nu(\beta^{-1}(h_{(2)}),c_{(2)(1)(1)})\nu(g_{(2)},c_{(2)(2)})\tau(g_{(1)},t_{(2)(2)})\\
=&\sigma(a\sharp h,c_{(1)}\sharp t_{(1)})\sigma(b\sharp g,c_{(2)}\sharp t_{(2)})
\end{aligned}
$$

The proof is completed. □

If ω is trivial, partial Hom-crossed products reduced to partial Hom-smash products. By Proposition 5.9, we can get the coquasitriangular structures on partial Hom-smash products.

References

[1] Y. Doi, Equivalent crossed products for a Hopf algebra, Comm. Alg., 17(1989), 3053–3085.
[2] R. Exel, Circle actions on C^*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences, Journal of Functional Analysis, 122(1994), 361-401.
[3] R. Exel, Partial dynamical systems, Fell bundles and applications, arxiv:1511.04565.
[4] M. Dokuchaev, R. Exel, Associativity of crossed products by partial actions, enveloping actions and partial representations, Trans. Amer. Math. Soc., 357(2005), 1931-1952.
[5] M. Dokuchaev, R. Exel, J. J. Simon, Crossed products by twisted partial actions and graded algebras, J. Algebra, 320(2008), 3278-3310.
[6] M. Dokuchaev, R. Exel, J. J. Simon, Globalization of twisted partial actions, Trans. Amer. Math. Soc., 362(2010), 4137-4160.
[7] M. Dokuchaev, M. Ferrero, A. Paques, Partial actions and Galois theory, Journal of Pure and Applied Algebra, 208(2007), 77-87.
[8] D. Bagio, J. Lazzarin, A. Paques, Crossed products by twisted partial actions: separability, semisimplicity and Frobenius properties, Comm. Alg., 38(2010), 496-508.
[9] S. Caenepeel, K. Janssen, Partial (co)actions of Hopf algebras and partial Hopf-Galois theory, Comm. Alg., 36(2008), 2923-2946.
[10] E. R. Alvares, M. M. S. Alves, E. Batista, Partial Hopf module categories, J. Pure Appl. Algebra, 217 (2013), 1517-1534.
[11] M.M.S. Alves, E. Batista, Enveloping Actions for Partial Hopf Actions, Comm. Alg., 38 (2010), 2872-2902.
[12] M. M. S. Alves, E. Batista, Partial Hopf actions, partial invariants and a Morita context, J. Algebra Discrete Math., 3 (2009), 1-19.
[13] M. M. S. Alves, E. Batista, Globalization theorems for partial Hopf (co)actions and some of their applications, Contemp. Math., 537 (2011), 13-30.
[14] M. M. S. Alves, E. Batista, M. Dokuchaev, A. Paques, Twisted partial actions of Hopf algebras, Israel J. Math., 197 (2013), 263-308.
[15] M. M. S. Alves, E. Batista, M. Dokuchaev, A. Paques, Twisted partial actions of Hopf algebras, J. Australian Math. Soc. (2016), available on CJO2016. doi:10.1017/S1446788715000774.
[16] M. M. S. Alves, E. Batista, J. Vercruysse, Partial representations of Hopf algebras, J. Algebra, 426 (2015), 137-187.
[17] S. Caenepeel and I. Goyvaerts, Monoidal Hom-Hopf algebras, Comm. Alg., 39 (2011), 2216-2240.
[18] Y. Y. Chen, L. Y. Zhang, The category of Yetter-Drinfeld Hom-modules and the quantum Hom-Yang-Baxter equation, J. Math. Phys., 55 (2014), 031702.
[19] Q. G. Chen, B. Yang, Cleft coextension for symmetric twisted partial coactions on coalgebras, Osaka J. Math., 56(2)(2019), 301-322.
[20] Q. G. Chen, D. G. Wang, X. D. Kang, Twisted partial coactions of Hopf algebras, Front. Math. China, 12(1)(2017), 63-86.
[21] A. Makhlouf, S. Silvestrov, Hom-algebra structures, J. Gen. Lie Theory Appl., 2(2) (2008), 51-64.
[22] A. Makhlouf, S. Silvestrov, Hom-algebras and Hom-coalgebras, J. Algebra Appl., 9(4) (2010), 553-589.
[23] A. Makhlouf, F. Panaite, Yetter-Drinfeld modules for Hom-bialgebras, J. Math. Phys., 55 (2014), 013501.
[24] D. Yau, Hom-algebras and homology, J. Lie Theory, 19 (2009), 409-421.
[25] D. Yau, Hom-quantum groups I: Quasitriangular Hom-bialgebras, J. Phys. A., 45 (2012), 065203.
[26] D. Yau, Hom-Yang-Baxter-equation, Hom-Lie algebras and quasitriangular bialgebras, J. Phys. A., 42 (2009), 165202.
[27] V. Drinfeld, Quantum groups, Proc. Int. Congr. Math. Berkeley., 1(1987), 798-820.
[28] M. Jimbo, A q-difference analogue of $U(gl(N+1))$ and the Yang-Baxter equation, Lett. Math. Phys., 10(1985), 63-69.
[29] R. Larson, J. Towber, Two dual classes of bialgebras related to the concepts of quantum groups and quantum Lie algebras. Comm. Alg., 19(1991), 3295-3345.
[30] S. Majid, Quantum Groups and Quantum Probability, In Quantum Probability and Related Topics. Vol. 6. River Edge, NJ: World Scientific, 1991, pp. 333-358.
[31] S. Majid, Foundations of Quantum Group Theory, Cambridge University Press, Cambridge (1995).
[32] E. K. Sklyanin, Some algebraic structures connected with the Yang-Baxter equation, Funct. Anal. Appl., 16(1982), 263-270.
[33] E. Batista, Partial actions: what they are and why we care, Bull. Belg. Math. Soc. Simon Stevin, 24 (2017), 35-71.
[34] E. Batista, J. Vercruysse, Dual constructions for partial actions of Hopf algebras, J. Pure Appl. Algebra, 220 (2016), 518-559.