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Ordering of k-Uniform Hypertrees by their Distance Spectral Radii

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Abstract. The distance spectral radius of a connected hypergraph is the largest eigenvalue of its distance matrix. In this paper we present a new transformation that decreases distance spectral radius. As applications, if $\Delta \geq \lceil \frac{m+1}{2} \rceil$, we determine the unique k-uniform hypertree of fixed m edges and maximum degree Δ with the minimum distance spectral radius. And we characterize the k-uniform hypertrees on m edges with the fourth, fifth, and sixth smallest distance spectral radius. In addition, we obtain the k-uniform hypertree on m edges with the third largest distance spectral radius.

1. Introduction

A hypergraph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where $V(G)$ is nonempty, and each edge $e \in E(G)$ is a nonempty subset of $V(G)$, see [3]. The size of G is the cardinality of $E(G)$, denoted by $m(G)$. For an integer $k \geq 2$, a hypergraph is k-uniform if all its edges have cardinality k. A (simple) graph is a 2-uniform hypergraph. For two vertices u and v of G, if they are contained in some edge of G, then we say that they are adjacent, or v is a neighbour of u. For $u \in V(G)$, let $N_G(u)$ be the set of neighbours of u in G and $E_G(u)$ be the set of edges containing u in G. The degree of a vertex u in G, denoted by $d_G(u)$, is $|E_G(u)|$. An edge $e = \{w_1, \ldots, w_k\}$ in G is called a pendant edge at w_1 if $d_G(w_1) \geq 2$, $d_G(w_i) = 1$ for $2 \leq i \leq k$.

For $u, v \in V(G)$, a walk from u to v in G is defined to be a sequence of vertices and edges $(v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$ with $v_0 = u$ and $v_p = v$ such that edge e_i contains vertices v_{i-1} and v_i, and $v_{i-1} \neq v_i$ for $i = 1, \ldots, p$. The value p is the length of this walk. A path is a walk with all v_i are distinct and all e_i are distinct. A cycle is a walk containing at least two edges, all e_i are distinct and all v_i are distinct except $v_0 = v_p$. A path $P = (v_0, e_1, v_1, \ldots, v_{p-1}, e_p, v_p)$ in a k-uniform hypergraph G is called a pendant path at v_0, if $d_G(v_0) \geq 2$, $d_G(v_i) = 2$ for $1 \leq i \leq p-1$, $d_G(v) = 1$ for $v \in e_i \backslash \{v_{i-1}, v_i\}$ with $1 \leq i \leq p$, and $d_G(v_p) = 1$. If there is a path from u to v for any $u, v \in V(G)$, then we say that G is connected. A hypertree is a connected hypergraph with no cycles. Note that a k-uniform hypertree with m edges always has $m(k-1)+1$ vertices.

For a k-uniform hypertree G with $V(G) = \{v_1, \ldots, v_n\}$, if $E(G) = \{e_1, \ldots, e_m\}$, where $e_i = \{v_{(i-1)(k-1)+1}, \ldots, v_{(i-1)(k-1)+k}\}$ for $i = 1, \ldots, m$, then G is a k-uniform loose path, denoted by $P_{n,k}$.

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For a k-uniform hypertree G of order n, if there is a partition of the vertex set $V(G)$ into $\{u\} \cup V_1 \cup \cdots \cup V_m$ such that $|V_1| = \cdots = |V_m| = k-1$, and $E(G) = \{\{u\} \cup V_i : 1 \leq i \leq m\}$, then G is a k-uniform hyperstar with center u, denoted by $S_{n,k}$. In particular, $S_{1,k}$ is a hypertree with a single vertex and $S_{k,k}$ is a k-uniform hypertree with a single edge.

Let G be a connected hypergraph on n vertices. For $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u,v)$, is the length of a shortest path connecting them in G. In particular, $d_G(u,u) = 0$. The distance matrix of G is defined as $D(G) = (d_G(u,v))_{u,v \in V(G)}$. Since $D(G)$ is real and symmetric, its eigenvalues are all real. The distance spectral radius of G, denoted by $\rho(G)$, is the largest eigenvalue of $D(G)$. Since $D(G)$ is irreducible, by Perron-Frobenius theorem, $\rho(G)$ is simple and there is a unique unit positive eigenvector corresponding to $\rho(G)$, which is called the distance Perron vector of G, denoted by $x(G)$.

For $X \subseteq V(G)$ with $X \neq \emptyset$, let $G[X]$ be the subhypergraph induced by X, i.e., $G[X]$ has vertex set X and edge set $\{e \subseteq X : e \in E(G)\}$, and let $\sigma_G(X)$ be the sum of the entries of the distance Perron vector of G corresponding to the vertices in X.

The distance matrix is very useful in different fields including the design of communication networks, graph embedding theory as well as molecular stability. Balaban et al. [2] proposed the use of the distance spectral radius of ordinary graphs (2-uniform hypergraphs) as a molecular descriptor, and it was successfully used to make inferences about the extent of branching and boiling points of alkanes, see [2, 8]. Now the distance spectral radius of an ordinary graph has been studied extensively, see [5–7] for classical results, see [4, 9, 14] and survey [1] for recent results. Contrasting the distance spectral properties of graphs, the distance spectral properties of hypergraphs is still in its infancy. Sivasubramanian [13] gave a formula for the inverse of a few q-analogs of the distance matrix of a 3-uniform hypertree. Lin and Zhou [10] studied the distance spectral radius of k-uniform hypergraphs and determined the k-uniform hypertrees with maximum, second maximum, minimum, and second minimum distance spectral radii, respectively. Lin and Zhou [11] determined the unique k-uniform unicyclic hypergraphs of size $m \geq 2$ with minimum and second minimum distance spectral radii, and discussed the possible structure of the k-uniform unicyclic hypergraph(s) of fixed size with maximum distance spectral radius, respectively. Lin et al. [12] studied the distance spectral radius of some particular k-uniform hypertrees.

This paper is organized as follows: In Section 2, we give some preliminary results and present a new transformation that decreases distance spectral radius. With the transformation, if $\Delta \geq \lceil \frac{m+1}{2} \rceil$, we determine the unique k-uniform hypertree of fixed m edges and maximum degree Δ with the minimum distance spectral radius. And we characterize the k-uniform hypertrees on $m \geq 17$ edges with the fourth, fifth, and sixth smallest distance spectral radius in Section 3. In addition, we obtain the k-uniform hypertree on $m \geq 13$ edges with the third largest distance spectral radius in Section 4.

2. Preliminaries and a new transformation

Let G be a k-uniform hypergraph with $V(G) = \{v_1, \ldots, v_n\}$. A column vector $x = (x_{v_1}, \ldots, x_{v_n})^T \in \mathbb{R}^n$ can be considered as a function defined on $V(G)$ which maps vertex v_i to x_{v_i}, i.e., $x(v_i) = x_{v_i}$ for $i = 1, \ldots, n$. Then

$$x^T D(G) x = \sum_{\{u,v\} \in V(G)} 2 d_G(u,v) x_u x_v,$$

and ρ is a distance eigenvalue with corresponding eigenvector x if and only if $x \neq 0$ and for each $u \in V(G)$,

$$\rho x_u = \sum_{v \in V(G)} d_G(u,v) x_v.$$

The above equation is called the eigenequation of G at u. For a unit column vector $x \in \mathbb{R}^n$ with at least one nonnegative entry, by Rayleigh's principle, we have

$$\rho(G) \geq x^T D(G) x$$

with equality if and only if $x = x(G)$.

Lemma 2.1. *([10]) Let G be a connected hypergraph with η being an automorphism of G and $x = x(G)$. Then $\eta(u) = v$ implies that $x_u = x_v$.*

Let G be a connected k-uniform hypergraph with $m(G) \geq 2$, and let $e = \{w_1, \ldots, w_k\}$ be a pendant edge of G at w_k. For $1 \leq i \leq k-1$, let H_i be a connected k-uniform hypergraph with $v_i \in V(H_i)$. Suppose that $G, H_1, \ldots, H_{k-1}$ are vertex-disjoint. For $0 \leq s \leq k-1$, let $G_{e,s}(H_1, \ldots, H_{k-1})$ be the k-uniform hypergraph obtained by identifying w_i of G and v_i of H_i for $s+1 \leq i \leq k-1$ and identifying w_k of G and v_i of H_i for all i with $1 \leq i \leq s$.

Lemma 2.2. *([10]) Suppose that $m(H_j) \geq 1$ for some j with $1 \leq j \leq k-1$. Then $\rho(G_{e,0}(H_1, \ldots, H_{k-1})) > \rho(G_{e,s}(H_1, \ldots, H_{k-1}))$ for $j \leq s \leq k-1$.*

Let G be a k-uniform hypergraph with $u, v \in V(G)$ and $e_1, \ldots, e_r \in E(G)$ such that $u \notin e_i$ and $v \in e_i$ for $1 \leq i \leq r$. Let $e'_i = (e_i \backslash \{v\}) \cup \{u\}$ for $1 \leq i \leq r$. Suppose that $e'_i \notin E(G)$. Let G' be the hypergraph with $V(G') = V(G)$ and $E(G') = (E(G) \backslash \{e_1, \ldots, e_r\}) \cup \{e'_1, \ldots, e'_r\}$. Then we say that G' is obtained from G by moving edges $e_1, \ldots, e_r$ from v to u.

Theorem 2.3. *Let G be a k-uniform hypergraph with connected induced subhypergraphs $P_{2k-1,k} = (u, e_1, w, e_2, v)$, H_1, H_2 and H_3 such that $H_1 \cap P_{2k-1,k} = \{u\}$, $H_2 \cap P_{2k-1,k} = \{v\}$, $H_3 \cap P_{2k-1,k} = \{w\}$, $H_1 \cap H_2 \cap H_3 = \emptyset$ and $V(G) = V(P_{2k-1,k}) \cup V(H_1) \cup V(H_2) \cup V(H_3)$, where H_1 is a k-uniform hyperstar with center u. Suppose that $k \geq 3$, $m(H_1) \geq 1$ and $m(H_2) \geq 1$. Let G' be a k-uniform hypergraph from G by moving all edges containing v except the edge e_2 from v to u. Then $\rho(G) > \rho(G')$.*

Proof. Let $x = x(G')$. By Lemma 2.1, the entry of x corresponding to each vertex of $V(H_1) \backslash \{u\}$ is the same, which we denote by a, the entry of x corresponding to each vertex of $e_1 \backslash \{u, w\}$ is the same, which we denote by b, and the entry of x corresponding to each vertex of $e_2 \backslash \{w\}$ is the same, which we denote by c.

For G', let $v_a \in V(H_1) \backslash \{u\}$, $v_b \in e_1 \backslash \{u, w\}$, and $v_c \in e_2 \backslash \{w\}$. From the eigenequations of G' at u, v_b, v_c, and v_a, we have

$$\rho(G')x_u = m(H_1)(k-1)a + \sum_{f \in V(H_2) \backslash \{u\}} d_{G'}(f, u)x_f + (k-2)b + 2(k-1)c + \sum_{g \in V(H_3)} d_{G'}(g, u)x_g,$$

$$\rho(G')b = 2m(H_1)(k-1)a + \sum_{f \in V(H_2) \backslash \{u\}} d_{G'}(f, v_b)x_f + (k-3)b + 2(k-1)c + x_u + \sum_{g \in V(H_3)} d_{G'}(g, v_b)x_g,$$

$$\rho(G')c = 3m(H_1)(k-1)a + \sum_{f \in V(H_2) \backslash \{u\}} d_{G'}(f, v_c)x_f + 2(k-2)b + (k-2)c + 2x_u + \sum_{g \in V(H_3)} d_{G'}(g, v_c)x_g,$$

$$\begin{aligned}\rho(G')a =& 2(m(H_1)-1)(k-1)a + (k-2)a + \sum_{f \in V(H_2) \backslash \{u\}} d_{G'}(f, v_a)x_f + 2(k-2)b + 3(k-1)c \\ &+ x_u + \sum_{g \in V(H_3)} d_{G'}(g, v_c)x_g.\end{aligned}$$

Note that for $f \in V(H_2) \backslash \{u\}$, $2d_{G'}(f, v_a) - d_{G'}(f, v_c) > 0$, for $g \in V(H_3)$, $2d_{G'}(g, v_a) - d_{G'}(g, v_c) > 0$. Then we have

$$\begin{aligned}\rho(G')(2a - c) =& m(H_1)(k-1)a - 2ka + \sum_{f \in V(H_2) \backslash \{u\}} [2d_{G'}(f, v_a) - d_{G'}(f, v_c)]x_f \\ &+ 2(k-2)b + 5kc - 4c + \sum_{g \in V(H_3)} [2d_{G'}(g, v_a) - d_{G'}(g, v_c)]x_g \\ &> m(H_1)(k-1)a - 2ka + 2(k-2)b + 5kc - 4c.\end{aligned}$$

Thus

$$(\rho(G') + k)(2a - c) > m(H_1)(k-1)a + 2(k-2)b + 4(k-1)c,$$

which implies $(\rho(G') + k)(2a - c) > 0$. So $2a > c$.

In addition, note that for $f \in V(H_2)\backslash\{u\}$, $d_{G'}(f, v_a) - d_{G'}(f, u) > 0$, for $g \in V(H_3)$, $d_{G'}(g, v_a) - d_{G'}(g, u) > 0$, and $m(H_1) \geq 1$, we have

$$\begin{aligned}\rho(G')(a - x_u) =& m(H_1)(k-1)a - ka + \sum_{f \in V(H_2)\backslash\{u\}} [d_{G'}(f, v_a) - d_{G'}(f, u)]x_f \\ &+(k-2)b + (k-1)c + x_u + \sum_{g \in V(H_3)} [d_{G'}(g, v_a) - d_{G'}(g, u)]x_g \\ \geq& -a + (k-2)b + (k-1)c + x_u,\end{aligned}$$

which implies $(\rho(G') + 1)(a - x_u) > 0$. So $a > x_u$.

Since $m(H_1) \geq 1$ and $m(H_2) \geq 1$,

$$\begin{aligned}&\rho(G')(2x_u + b - c) \\ =& m(H_1)(k-1)a + \sum_{f \in V(H_2)\backslash\{u\}} [2d_{G'}(f, u) + d_{G'}(f, v_b) - d_{G'}(f, v_c)]x_f - x_u \\ &+ 5kc - 4c + (k-3)b + \sum_{g \in V(H_3)} [2d_{G'}(g, u) + d_{G'}(g, v_b) - d_{G'}(g, v_c)]x_g \\ \geq& m(H_1)(k-1)a + (k-3)b + 5kc - 4c - x_u \\ >& a + (k-3)b + (5k-4)c - x_u > 0,\end{aligned}$$

which implies $\rho(G')(2x_u + b - c) > 0$. So $2x_u + b - c > 0$.

As we pass from G to G', the distance between $V(H_2)\backslash\{v\}$ and $V(H_1)$ is decreased by 2, the distance between $V(H_2)\backslash\{v\}$ and $e_1\backslash\{u, w\}$ is decreased by 1, the distance between $V(H_2)\backslash\{v\}$ and $e_2\backslash\{v, w\}$ is increased by 1, $V(H_2)\backslash\{v\}$ and v is increased by 2, and the distance between any other vertex pair remains unchanged. Note that $k \geq 3$, hence,

$$\begin{aligned}\frac{1}{2}(\rho(G) - \rho(G')) \geq& \frac{1}{2}x^T(D(G) - D(G'))x \\ =& \sigma_{G'}(V(H_2)\backslash\{u\})[2\sigma(V(H_1)) + \sigma(e_1\backslash\{u, w\}) - \sigma(e_2\backslash\{v, w\}) - 2x_v] \\ =& \sigma_{G'}(V(H_2)\backslash\{u\})[2m(H_1)(k-1)a + 2x_u + (k-2)b - (k-2)c - 2c] \\ \geq& \sigma_{G'}(V(H_2)\backslash\{u\})[(k-1)(2a - c) + 2x_u + b - c] > 0,\end{aligned}$$

which implies $\rho(G) > \rho(G')$. ■

Let $D_{m,a,b}$ be the k-uniform hypertree obtained from vertex-disjoint k-uniform hyperstar $S_{a(k-1)+1,k}$ with center u and k-uniform hyperstar $S_{b(k-1)+1,k}$ with center v by adding $k-2$ new vertices $w_1, \ldots, w_{k-2}$ and a new edge $\{u, v, w_1, \ldots, w_{k-2}\}$, where $m \geq 3$, $a, b \geq 1$ and $m = a + b + 1$.

For convenience, we call the transformation from $G_{e,0}(H_1, \ldots, H_{k-1})$ to $G_{e,s}(H_1, \ldots, H_{k-1})$ in Lemma 2.2 the α-transformation of $G_{e,s}(H_1, \ldots, H_{k-1})$, and the transformation from G to G' in Theorem 2.3 the β-transformation of G.

Theorem 2.4. *If $\Delta \geq \lceil \frac{m+1}{2} \rceil$, then the $D_{m,\Delta-1,m-\Delta}$ has the minimum distance spectral radius among k-uniform hypertrees with m edges and maximum degree Δ.*

Proof. Let $T \ncong D_{m,\Delta-1,m-\Delta}$ be a k-uniform hypertree with m edges and maximum degree Δ. Since $\Delta \geq \lceil \frac{m+1}{2} \rceil$, T can be transformed into $D_{m,\Delta-1,m-\Delta}$ by α and β-transformations. By Lemma 2.2 and Theorem 2.3, we have $\rho(T) > \rho(D_{m,\Delta-1,m-\Delta})$. ■

3. The first six smallest distance spectral radii of k-uniform hypertrees

Lin and Zhou [10] and Lin et al. [12] have considered to order k-uniform hypertrees by their distance spectral radii, and determined the first three k-uniform hypertrees on m edges with small distance spectral radius.

Lemma 3.1. *([10, 12]) Let $T \notin \{S_{m(k-1)+1,k}, D_{m,m-2,1}, D_{m,m-3,2}\}$ be a k-uniform hypertree with m edges, where $m \geq 5$ and $k \geq 2$. Then*

$$\rho(T) > \rho(D_{m,m-3,2}) > \rho(D_{m,m-2,1}) > \rho(S_{m(k-1)+1,k}).$$

Lemma 3.2. *([10]) Let a and b be two integers with $a \geq b \geq 2$. Then $\rho(D_{m,a,b}) > \rho(D_{m,a+1,b-1})$.*

For $k \geq 2$ and $m \geq 4$, let $E_{m,k}$ be the k-uniform hypertree obtained from $P_{4(k-1)+1,k} = (v_5, e_4, v_4, e_3, v_1, e_1, v_2, e_2, v_3)$ by attaching $m-4$ pendant edges at vertex v_1, where $E_{m,k}$ is depicted in Figure 1.

For $k \geq 3$ and $m \geq 4$, let $F_{m,k}$ be the k-uniform hypertree obtained from $P_{3(k-1)+1,k} = (v_0, e_1, v_1, e_2, v_2, e_3, v_3)$ by attaching $m-3$ pendant edges at a vertex v in $e_2 \backslash \{v_1, v_2\}$, where $F_{m,k}$ is depicted in Figure 1.

For $k \geq 2$ and $m \geq 4$, let $B_{m,k}$ be the k-uniform hypertree obtained from $P_{3(k-1)+1,k} = (v_1, e_1, v_2, e_2, v_3, e_3, v_4)$ by attaching $m-3$ pendant edges at vertex v_1, where $B_{m,k}$ is depicted in Figure 1.

Let $\mathbb{T}_m^{\Delta}$ denote the set of k-uniform hypertrees with m edges and maximum degree Δ. Obviously, $\mathbb{T}_m^m = \{S_{m(k-1)+1,k}\}$, $\mathbb{T}_m^{m-1} = \{D_{m,m-2,1}\}$ and $\mathbb{T}_m^{m-2} = \{D_{m,m-3,2}, B_{m,k}, E_{m,k}, F_{m,k}\}$.

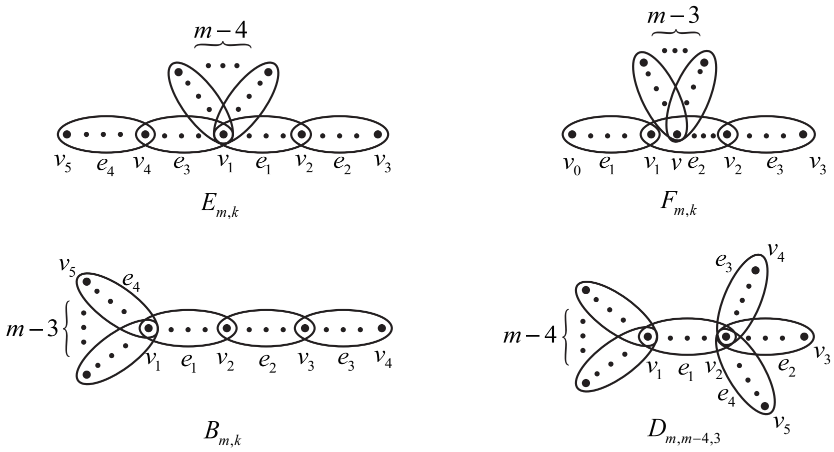

Figure 1: k-uniform hypertrees $B_{m,k}, E_{m,k}, F_{m,k}$ and $D_{m,m-4,3}$

Lemma 3.3. *([12]) For $k \geq 3$ and $m \geq 4$, $\rho(E_{m,k}) > \rho(F_{m,k})$.*

By Lemma 2.2, we have the following result.

Lemma 3.4. *For $k \geq 3$ and $m \geq 5$, $\rho(F_{m,k}) > \rho(D_{m,m-3,2})$.*

Theorem 3.5. *Let $T \in \mathbb{T}_m^{\Delta}$ and $T \ncong D_{m,m-4,3}$, where $m \geq 7$ and $\Delta \leq m-3$. Then $\rho(T) > \rho(D_{m,m-4,3})$.*

Proof. We distinguish the following two cases.

Case 1. If $\Delta \geq 4$, then by α and β-transformation, T can be transformed into $D_{m,m-\Delta,\Delta-1}$. By Lemma 2.2 and 3.2, Theorem 2.3, we have $\rho(T) > \rho(D_{m,m-\Delta,\Delta-1}) > \rho(D_{m,m-4,3})$.

Case 2. If $\Delta \leq 3$, then by using one or two times α-transformation, T can be transformed into T' with maximum degree 4 or 5($\leq m-3$). Thus we have $\rho(T) > \rho(T') > \rho(D_{m,m-4,3})$ from Case 1. ■

Lemma 3.6. *If $m \geq 7$, then $\rho(B_{m,k}) > \rho(D_{m,m-4,3})$.*

Proof. Since $D_{m,m-4,3}$ can be obtained from $B_{m,k}$ by moving e_4 from v_1 to v_2 and moving e_3 from v_3 to v_2. As we pass from $B_{m,k}$ to $D_{m,m-4,3}$, the distance between $e_4 \backslash \{v_1\}$ and v_1 is increased by 1, the distance between $e_4 \backslash \{v_1\}$ and $E(v_1) \backslash \{e_1, e_4\}$ is increased by 1, the distance between $e_4 \backslash \{v_1\}$ and v_2 is decreased by 1, the distance between $e_4 \backslash \{v_1\}$ and v_3 is decreased by 1, the distance between $e_3 \backslash \{v_3\}$ and v_1 is decreased by 1, the distance between $e_3 \backslash \{v_3\}$ and $E(v_1) \backslash \{e_1, e_4\}$ is decreased by 1, the distance between $e_3 \backslash \{v_3\}$ and v_3 is increased by 1, the distance between $e_3\backslash\{v_3\}$ and v_2 is decreased by 1, and the distance between any other vertex pair remains unchanged or decreased.

Let $x = x(D_{m,m-4,3})$. By Lemma 2.1, the entry of x corresponding to each vertex of $e_4\backslash\{v_2\}$, $e_3\backslash\{v_2\}$ and $e_2\backslash\{v_2\}$ is the same, which we denote by a, the entry of x corresponding to each vertex of $E(v_1)\backslash\{e_1\}$ is the same, which we denote by b. Thus

$$\begin{aligned}\frac{1}{2}(\rho(B_{m,k}) - \rho(D_{m,m-4,3})) \geq& \frac{1}{2}x^T(D(B_{m,k}) - D(D_{m,m-4,3}))x\\ \geq&(k-1)a[-x_{v_1} - (m-4)(k-1)b + a\\ &+x_{v_1} + (m-4)(k-1)b - a + 2x_{v_2}] > 0,\end{aligned}$$

which implies $\rho(B_{m,k}) > \rho(D_{m,m-4,3})$. ■

Lemma 3.7. *If $m \geq 17$, then $\rho(D_{m,m-4,3}) > \rho(E_{m,k})$.*

Proof. Since $E_{m,k}$ can be obtained from $D_{m,m-4,3}$ by moving e_3 from v_2 to v_1 and moving e_4 from v_2 to v_4. As we pass from $D_{m,m-4,3}$ to $E_{m,k}$, the distance between $e_3\backslash\{v_2\}$ and v_1 is decreased by 1, the distance between $e_3\backslash\{v_2\}$ and $E(v_1)\backslash\{e_1\}$ is decreased by 1, the distance between $e_3\backslash\{v_2\}$ and v_2 is increased by 1, the distance between $e_3\backslash\{v_2\}$ and $e_2\backslash\{v_2\}$ is increased by 1, the distance between $e_4\backslash\{v_2\}$ and v_4 is decreased by 1, the distance between $e_4\backslash\{v_2\}$ and $e_1\backslash\{v_1, v_2\}$ is increased by 1, the distance between $e_4\backslash\{v_2\}$ and v_2 is increased by 2, the distance between $e_4\backslash\{v_2\}$ and $e_2\backslash\{v_2\}$ is increased by 2, and the distance between any other vertex pair remains unchanged.

Let $x = x(E_{m,k})$. By Lemma 2.1, the entry of x corresponding to each vertex of $e_4\backslash\{v_4\}$ and $e_2\backslash\{v_2\}$ is the same, which we denote by c, the entry of x corresponding to each vertex of $e_3\backslash\{v_1, v_4\}$ and $e_1\backslash\{v_1, v_2\}$ is the same, which we denote by b, the entry of x corresponding to each vertex of $E(v_1)\backslash\{e_1, e_3\}$ is the same, which we denote by a, and $x_{v_2} = x_{v_4}$.

Let $v_a \in E(v_1)\backslash\{e_1, e_3\}$, $v_b \in e_1\backslash\{v_1, v_2\}$, and $v_c \in e_2\backslash\{v_2\}$. From the eigenequations of $E_{m,k}$ at v_2, v_b, v_c, and v_a, we have

$$\rho(E_{m,k})x_{v_2} = x_{v_1} + 2(m-4)(k-1)a + 3(k-2)b + 4(k-1)c + 2x_{v_2},$$

$$\rho(E_{m,k})b = x_{v_1} + 2(m-4)(k-1)a + 2(k-2)b + (k-3)b + 5(k-1)c + 3x_{v_2},$$

$$\rho(E_{m,k})c = 2x_{v_1} + 3(m-4)(k-1)a + 5(k-2)b + (k-2)c + 4(k-1)c + 4x_{v_2},$$

$$\rho(E_{m,k})a = x_{v_1} + 2(m-5)(k-1)a + (k-2)a + 4(k-2)b + 6(k-1)c + 4x_{v_2}.$$

Then

$$\rho(E_{m,k})(b - x_{v_2}) = -b + (k-1)c + x_{v_2},$$

which implies $(\rho(E_{m,k}) + 1)(b - x_{v_2}) = (k-1)c > 0$. So $b > x_{v_2}$.

Since $b > x_{v_2}$,

$$\begin{aligned}\rho(E_{m,k})(a - x_{v_2}) =& - ka + (k-2)b + 2(k-1)c + 2x_{v_2}\\ >& - ka + 2(k-1)c + kx_{v_2},\end{aligned}$$

which implies $(\rho(E_{m,k}) + k)(a - x_{v_2}) = 2(k-1)c > 0$. So $a > x_{v_2}$.

Similarly, we have

$$\rho(E_{m,k})(2x_{v_2} - c) = (m-4)(k-1)a + (k-2)b + 3(k-1)c + c,$$

which implies $\rho(E_{m,k})(2x_{v_2} - c) > 0$. So $2x_{v_2} > c$.

Since $m \geq 17$, $b > x_{v_2}$, $a > x_{v_2}$ and $2x_{v_2} > c$, we have

$$\begin{aligned}\frac{1}{2}(\rho(D_{m,m-4,3}) - \rho(E_{m,k})) \geq & \frac{1}{2}x^T(D(D_{m,m-4,3}) - D(E_{m,k}))x \\ \geq & [(k-2)b + x_{v_2}][x_{v_1} + (m-4)(k-1)a - x_{v_2} - (k-1)c] \\ & + x_{v_2}(k-1)c - (k-1)c[(k-2)b + 2x_{v_2} + 2(k-1)c] \\ = & [(k-2)b + x_{v_2}][x_{v_1} + (m-4)(k-1)a - x_{v_2}] \\ & - (k-1)c[2(k-2)b + 2x_{v_2} + 2(k-1)c] \\ \geq & [(k-2)b + x_{v_2}][x_{v_1} + 13(k-1)a - x_{v_2}] \\ & - 2(k-1)x_{v_2}[2(k-2)b + 2x_{v_2} + 4(k-1)x_{v_2}] \\ \geq & [(k-2)b + x_{v_2}][x_{v_1} + 12(k-1)x_{v_2} + (k-2)x_{v_2}] \\ & - 2(k-1)x_{v_2}[2(k-2)b + 2x_{v_2} + 4(k-1)x_{v_2}] \\ > & 8(k-2)(k-1)bx_{v_2} + 8(k-1)x_{v_2}^2 - 8(k-1)^2x_{v_2}^2 + (k-2)x_{v_2}^2 \\ > & (k-2)x_{v_2}^2 > 0,\end{aligned}$$

which implies $\rho(D_{m,m-4,3}) > \rho(E_{m,k})$. ■

Theorem 3.8. *Let $T \notin \{S_{m(k-1)+1,k}, D_{m,m-2,1}, D_{m,m-3,2}, F_{m,k}, E_{m,k}, D_{m,m-4,3}\}$ be a k-uniform hypertree with m edges, where $m \geq 17$ and $k \geq 3$. Then*

$$\rho(T) > \rho(D_{m,m-4,3}) > \rho(E_{m,k}) > \rho(F_{m,k}) > \rho(D_{m,m-3,2}) > \rho(D_{m,m-2,1}) > \rho(S_{m(k-1)+1,k}).$$

Proof. Since $T \notin \{S_{m(k-1)+1,k}, D_{m,m-2,1}, D_{m,m-3,2}, F_{m,k}, E_{m,k}, D_{m,m-4,3}\}$, $T \cong B_{m,k}$ or $T \in \mathbb{T}_m^{\Delta}$, where $\Delta \leq m-3$. By Lemmas 3.1, 3.3, 3.4, 3.6 and 3.7, Theorem 3.5, we can obtain the result. ■

4. The third largest distance spectral radius of k-uniform hypertrees

Let G be a connected k-uniform hypergraph with $m(G) \geq 1$. For $u \in V(G)$, and positive integers p and q, let $G_u(p,q)$ be a k-uniform hypergraph obtained from G by attaching two pendant paths of lengths p and q at u, respectively, and let $G_u(p,0)$ be a k-uniform hypergraph obtained from G by attaching a pendant path of length p at u.

Lemma 4.1. *([10]) Let G be a connected k-uniform hypergraph with $m(G) \geq 1$ and $u \in V(G)$. For integers $p \geq q \geq 1$, $\rho(G_u(p,q)) < \rho(G_u(p+1,q-1))$.*

Let G be a connected k-uniform hypergraph with $u, v \in e \in E(G)$. For positive integers p and q, let $G_{u,v}(p,q)$ be a k-uniform hypergraph obtained from G by attaching a pendant path of length p at u and a pendant path of length q at v, and let $G_{u,v}(p,0)$ be a k-uniform hypergraph obtained from G by attaching a pendant path of length p at u.

Lemma 4.2. *([10]) Let G be a connected k-uniform hypergraph with $m(G) \geq 2$ and $u, v \in e \in E(G)$. Suppose that $G - e$ consists of k components. For integers $p, q \geq 1$, $\rho(G_{u,v}(p,q)) < \rho(G_{u,v}(p+1,q-1))$ or $\rho(G_{u,v}(p,q)) < \rho(G_{u,v}(p-1,q+1))$.*

For positive integers Δ and m with $1 \leq \Delta \leq m$, let $B_{m,k}^{\Delta}$ be the k-uniform hypertree obtained from vertex-disjoint hyperstar $S_{(\Delta-1)(k-1)+1,k}$ with center u and loose path $P_{m(k-1)+1-(\Delta-1)(k-1),k}$ with an end vertex v by identifying u and v. In particular, $B_{m,k}^{\Delta} \cong P_{m(k-1)+1,k}$ if $\Delta = 1, 2$.

Lemma 4.3. *([10]) Let T be a k-uniform hypertree with m edges and maximum degree Δ, where $1 \leq \Delta \leq m$. Then $\rho(T) \leq \rho(B_{m,k}^{\Delta})$ with equality if and only if $T \cong B_{m,k}^{\Delta}$.*

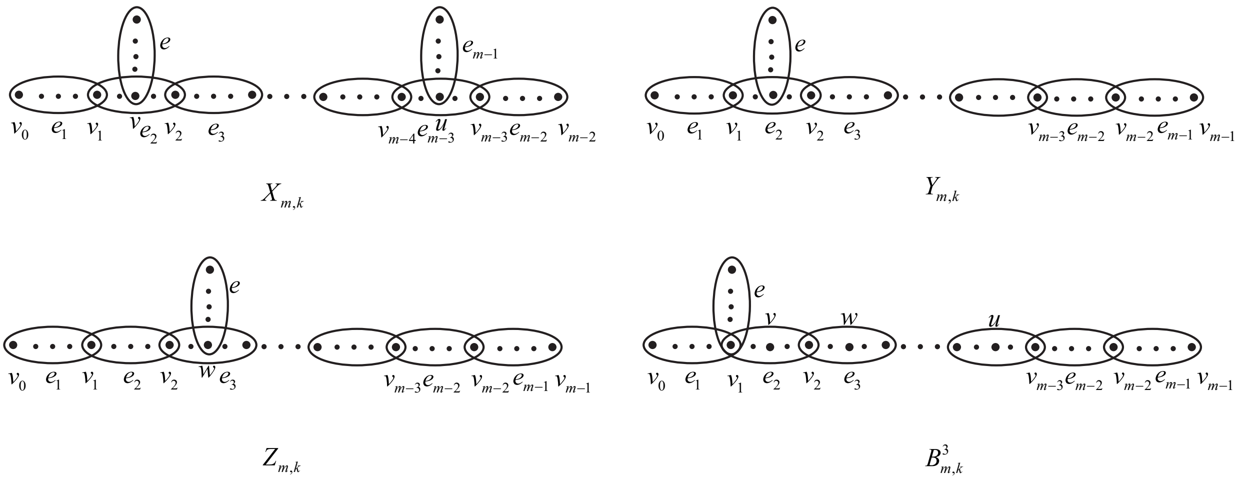

Figure 2: k-uniform hypertrees $X_{m,k}, Y_{m,k}, Z_{m,k}$ and $B^3_{m,k}$

For $k \geq 3$ and $m \geq 7$, let $X_{m,k}$ be the k-uniform hypertree obtained from $P_{m(k-1)+1-2(k-1),k} = (v_0, e_1, v_1, e_2, v_2, e_3, v_3, \ldots, v_{m-4}, e_{m-3}, v_{m-3}, e_{m-2}, v_{m-2})$ by attaching a pendant edge e at a vertex v in $e_2\backslash\{v_1, v_2\}$ and attaching a pendant edge e_{m-1} at a vertex u in $e_{m-3}\backslash\{v_{m-3}, v_{m-4}\}$, where $X_{m,k}$ is depicted in Figure 2.

For $k \geq 3$ and $m \geq 4$, let $Y_{m,k}$ be the k-uniform hypertree obtained from $P_{m(k-1)+1-(k-1),k} = (v_0, e_1, v_1, e_2, v_2, e_3, v_3, \ldots, v_{m-3}, e_{m-2}, v_{m-2}, e_{m-1}, v_{m-1})$ by attaching a pendant edge e at a vertex v in $e_2\backslash\{v_1, v_2\}$, where $Y_{m,k}$ is depicted in Figure 2.

For $k \geq 3$ and $m \geq 6$, let $Z_{m,k}$ be the k-uniform hypertree obtained from $P_{m(k-1)+1-(k-1),k} = (v_0, e_1, v_1, e_2, v_2, e_3, v_3, \ldots, v_{m-3}, e_{m-2}, v_{m-2}, e_{m-1}, v_{m-1})$ by attaching a pendant edge e at a vertex w in $e_3\backslash\{v_2, v_3\}$, where $Z_{m,k}$ is depicted in Figure 2.

Lemma 4.4. *For $k \geq 3$ and $m \geq 7$, $\rho(B^3_{m,k}) > \rho(X_{m,k})$.*

Proof. Since $X_{m,k}$ can be obtained from $B^3_{m,k}$ by moving edge e from v_1 to a vertex $v \in e_2\backslash\{v_1, v_2\}$ and moving edge e_{m-1} from v_{m-2} to a vertex $u \in e_{m-3}\backslash\{v_{m-4}, v_{m-3}\}$. As we pass from $B^3_{m,k}$ to $X_{m,k}$, the distance between $e\backslash\{v_1\}$ and e_1 is increased by 1, the distance between $e\backslash\{v_1\}$ and v is decreased by 1, the distance between $e\backslash\{v_1\}$ and $e_{m-1}\backslash\{v_{m-2}\}$ is decreased by 1, the distance between $e_{m-1}\backslash\{v_{m-2}\}$ and u is decreased by 2, the distance between $e_{m-1}\backslash\{v_{m-2}\}$ and v_{m-2} is increased by 2, the distance between $e_{m-1}\backslash\{v_{m-2}\}$ and $e_{m-2}\backslash\{v_{m-3}, v_{m-2}\}$ is increased by 1, the distance between $e_{m-1}\backslash\{v_{m-2}\}$ and v_{m-3} is unchanged, the distance between $e_{m-1}\backslash\{v_{m-2}\}$ and any other vertices is decreased by 1, and the distance between any other vertex pair remains unchanged.

Let $x = x(X_{m,k})$. By Lemma 2.1, the entry of x corresponding to each vertex of $e\backslash\{v\}$, $e_{m-1}\backslash\{u\}$, $e_1\backslash\{v_1\}$, $e_{m-2}\backslash\{v_{m-3}\}$ is the same, which we denote by a, and $x_{v_1} = x_v = x_u = x_{v_{m-3}} = b$.

From the eigenequations of $X_{m,k}$ at v_0 and v, we have

$$\rho(X_{m,k})a = (k-2)a + 3b + 2x_{v_2} + 2\sum_{w \in e_2\backslash\{v_1, v, v_2\}} x_w + 3(k-1)a + \sum_{w' \in V(X_{m,k})\backslash\{e_1, e_2, e\}} d_{X_{m,k}}(v_0, w')x_{w'},$$

$$\rho(X_{m,k})b = 2(k-1)a + b + x_{v_2} + \sum_{w \in e_2\backslash\{v_1, v, v_2\}} x_w + (k-1)a + \sum_{w' \in V(X_{m,k})\backslash\{e_1, e_2, e\}} d_{X_{m,k}}(v, w')x_{w'}.$$

Obviously, we have $\rho(X_{m,k})(a-b) > 0$, which implies $a > b$. Thus $\rho(X_{m,k})(2b-a) > 0$, which implies $2b > a$.

Hence,

$$\begin{aligned}\frac{1}{2}(\rho(B^3_{m,k})-\rho(X_{m,k}))\geq&\frac{1}{2}(D(B^3_{m,k})-D(X_{m,k}))\\ >&(k-1)a[-(k-1)a-b+b+(k-1)a]\\ +&(k-1)a[2b-2a-(k-2)a+(k-1)a]\\ >&(k-1)a(2b-a)>0,\end{aligned}$$

which implies $\rho(B^3_{m,k})>\rho(X_{m,k})$. ∎

Lemma 4.5. *For $m\geq 13$ and $k\geq 3$, $\rho(B^3_{m,k})>\rho(Z_{m,k})$.*

Proof. Since $Z_{m,k}$ can be obtained from $B^3_{m,k}$ by moving edge e from v_1 to a vertex $w\in e_3\backslash\{v_2,v_3\}$. As we pass from $B^3_{m,k}$ to $Z_{m,k}$, the distance between $e\backslash\{v_1\}$ and e_1 is increased by 2, the distance between $e\backslash\{v_1\}$ and $e_2\backslash\{v_1,v_2\}$ is increased by 1, the distance between $e\backslash\{v_1\}$ and w is decreased by 2, the distance between $e\backslash\{v_1\}$ and v_2 is unchanged, and the distance between $e\backslash\{v_1\}$ and any other vertex is decreased by 1.

Let $x=x(Z_{m,k})$. By Lemma 2.1, the entry of x corresponding to each vertex of $e\backslash\{w\}$ is the same, which we denote by α, the entry of x corresponding to each vertex of $e_i\backslash\{v_{i-1},v_i\}$ ($i=1,2$ or $i\geq 4$) is the same, which we denote by y_i, the entry of x corresponding to each vertex of $e_3\backslash\{v_2,v_3,w\}$ is the same, which we denote by a, and $x_{v_0}=y_1$.

From the eigenequations of $Z_{m,k}$ at v_0, v_1, w, v_3 and v_i ($i\geq 4$), we have

$$\begin{aligned}\rho(Z_{m,k})x_{v_0}=&(k-2)y_1+x_{v_1}+2(k-2)y_2+2x_{v_2}+3(k-3)a+3x_w+4(k-1)\alpha\\ &+\sum_{i=3}^{m-1}ix_{v_i}+(k-2)\sum_{i=4}^{m-1}iy_i,\end{aligned}$$

$$\begin{aligned}\rho(Z_{m,k})x_{v_1}=&x_{v_0}+(k-2)y_1+(k-2)y_2+x_{v_2}+2(k-3)a+2x_w+3(k-1)\alpha\\ &+\sum_{i=3}^{m-1}(i-1)x_{v_i}+(k-2)\sum_{i=4}^{m-1}(i-1)y_i,\end{aligned}$$

$$\begin{aligned}\rho(Z_{m,k})x_w=&3x_{v_0}+3(k-2)y_1+2x_{v_1}+2(k-2)y_2+x_{v_2}+(k-3)a+(k-1)\alpha\\ &+\sum_{i=3}^{m-1}(i-2)x_{v_i}+(k-2)\sum_{i=4}^{m-1}(i-2)y_i,\end{aligned}$$

$$\begin{aligned}\rho(Z_{m,k})x_{v_3}=&3x_{v_0}+3(k-2)y_1+2x_{v_1}+2(k-2)y_2+x_{v_2}+x_w+(k-3)a+2(k-1)\alpha\\ &+\sum_{i=4}^{m-1}(i-3)x_{v_i}+(k-2)\sum_{i=4}^{m-1}(i-3)y_i,\end{aligned}$$

$$\begin{aligned}\rho(Z_{m,k})x_{v_i}=&ix_{v_0}+i(k-2)y_1+(i-1)x_{v_1}+(i-1)(k-2)y_2+(i-2)x_{v_2}+(i-2)x_w\\ &+(i-2)(k-3)a+(i-1)(k-1)\alpha+\sum_{j=3}^{i}(i-j)x_{v_j}+\sum_{j=i+1}^{m-1}(j-i)x_{v_j}\\ &+(k-2)\sum_{j=4}^{i}(i-j+1)y_j+(k-2)\sum_{j=i+1}^{m-1}(j-i)y_j.\end{aligned}$$

Let $f(i) = 2i^2 - i(2m+8) + m^2 - m + 2$. For $5 \le i \le m-1$ and $m \ge 13$, $f(i)$ has minimum value when $i = \frac{m+4}{2}$. By calculation, we have $f(\frac{m+4}{2}) > 0$ for $m \ge 13$. Hence,

$$\begin{aligned}
&\rho(Z_{m,k})(\sum_{i=3}^{m-1} x_{v_i} + 2x_w - 2x_{v_0} - 2x_{v_1}) \\
>&2\sum_{i=3}^{m-1}(i-2)x_{v_i} + \sum_{i=4}^{m-1}(i-3)x_{v_i} + \sum_{i=4}^{m-1}[\sum_{j=3}^{i}(i-j)x_{v_j} + \sum_{j=i+1}^{m-1}(j-i)x_{v_j}] \\
&+ (k-2)[2\sum_{i=4}^{m-1}(i-2)y_i + \sum_{i=4}^{m-1}(i-3)y_i + \sum_{i=4}^{m-1}(\sum_{j=4}^{i}(i-j+1)y_j + \sum_{j=i+1}^{m-1}(j-i)y_j)] \\
&- \sum_{i=3}^{m-1}(4i-2)x_{v_i} - (k-2)\sum_{i=4}^{m-1}(4i-2)y_i \\
>&[\frac{(m-3)(m-4)}{2} + 2 - 10]x_{v_3} + [\frac{(m-4)(m-5)}{2} + 5 - 14]x_{v_4} \\
&+ \sum_{i=5}^{m-1}[\frac{(m-i-1)(m-i) + (i-3)(i-4)}{2} + 3i - 7 - (4i-2)]x_{v_i} \\
&+ [\frac{(m-3)(m-4)}{2} + 5 - 14]y_4 + [\frac{(m-4)(m-5)}{2} + 9 - 18]y_5 \\
&+ \sum_{i=6}^{m-1}[\frac{(m-i+1)(m-i) + (i-3)(i-4)}{2} + 3i - 7 - (4i-2)]y_i > 0,
\end{aligned}$$

which implies $\sum_{i=3}^{m-1} x_{v_i} + 2x_w - 2x_{v_0} - 2x_{v_1} > 0$.

Similarly, we can obtain

$$(k-2)[\sum_{i=4}^{m-1} y_i - 2y_1 - y_2] > 0.$$

Hence,

$$\begin{aligned}
\frac{1}{2}(\rho(B^3_{m,k}) - \rho(Z_{m,k})) \ge&\frac{1}{2}(D(B^3_{m,k}) - D(Z_{m,k})) \\
\ge&(k-1)\alpha[-2x_{v_0} - 2(k-2)y_1 - 2x_{v_1} - (k-2)y_2 + (k-3)a + 2x_w \\
&+ \sum_{i=3}^{m-1} x_{v_i} + (k-2)\sum_{i=4}^{m-1} y_i] > 0,
\end{aligned}$$

which implies $\rho(B^3_{m,k}) > \rho(Z_{m,k})$. ∎

Theorem 4.6. *For $m \ge 13$ and $k \ge 3$, let T be a k-uniform hypertree with m edges. Suppose that $T \notin \{P_{m(k-1)+1,k}, Y_{m,k}\}$, then $\rho(T) \le \rho(B^3_{m,k})$ with equality if and only if $T \cong B^3_{m,k}$.*

Proof. Let $T \notin \{P_{m(k-1)+1,k}, Y_{m,k}\}$ be a k-uniform hypertree on m edges with maximum distance spectral radius. Let Δ be the maximum degree of T. Obviously, $\Delta \ge 2$.

If $\Delta \ge 4$, then by Lemma 4.3, we have $T \cong B^{\Delta}_{m,k}$. For $k \ge 3$, $B^{\Delta-1}_{m,k} \ncong P_{m(k-1)+1,k}, Y_{m,k}$. By Lemma 4.1, we have $\rho(T) = \rho(B^{\Delta}_{m,k}) < \rho(B^{\Delta-1}_{m,k})$, a contradiction. So $\Delta = 2$ or 3.

Case 1. For $\Delta = 3$. By Lemma 4.3, we have $T \cong B^3_{m,k}$.

Case 2. For $\Delta = 2$. Since $T \ncong P_{m(k-1)+1,k}$, there is at least one edge with at least three vertices of degree 2 in T. Suppose that there are at least two such edges. Let w be a vertex of degree 1 in T. Choose an edge $e = \{w_1, \dots, w_k\}$ in T with at least three vertices of degree 2 such that $d_T(w, w_1)$ is as large as possible, where $d_T(w, w_1) = d_T(w, w_i) - 1$ for $2 \le i \le k$. Then there are two pendant paths at different vertices of e, say P at w_i and Q at w_j, where $1 \le i < j \le k$. Let p and q with $p \ge q \ge 1$ be the lengths of P and Q, respectively. Then $T \cong H_{w_i,w_j}(p, q)$ with $H = T[V(T)\backslash(V(P \cup Q)\backslash\{w_i, w_j\})]$. Note that $T' = H_{w_i,w_j}(p + 1, q - 1)$ is a k-uniform hypertree that is not isomorphic to $P_{m(k-1)+1,k}$. If T' is also not isomorphic to $Y_{m,k}$, then by Lemma 4.2, we have $\rho(T) < \rho(T')$, a contradiction. If T' is isomorphic to $Y_{m,k}$, then T is isomorphic to the k-uniform hypertree obtained from $P_{m(k-1)+1-2(k-1),k} = (u_0, e_1, u_1, e_2, u_2, \dots, e_{m-3}, u_{m-3}, e_{m-2}, u_{m-2})$ by attaching a pendant edge e' at a vertex v in $e_2\backslash\{u_1, u_2\}$ and attaching a pendant edge e'' at a vertex u in $e_i\backslash\{u_{i-1}, u_i\}$, where $3 \le i \le m - 3$. If $3 \le i \le m - 4$, then $\rho(T) < \rho(Z_{m,k})$, a contradiction. If $i = m - 3$, by Lemma 4.4, then $\rho(T) < \rho(B^3_{m,k})$, a contradiction. Thus e is the unique edge with at least three vertices of degree 2.

Suppose that there are four vertices w_1, w_2, w_3, w_4 of degree 2 in e. Let Q_i be the pendant path of length l_i at w_i, where $l_i \ge 1$ for $i = 1, 2$. Without loss of generality, suppose that $l_1 \ge l_2$. Let $G = T[V(T)\backslash(V(Q_1 \cup Q_2)\backslash\{w_1, w_2\})]$, then $T \cong G_{w_1,w_2}(l_1, l_2)$. Note that $T^* = G_{w_1,w_2}(l_1 + 1, l_2 - 1)$ is a k-uniform hypertree that is not isomorphic to $P_{m(k-1)+1,k}$. If T^* is also not isomorphic to $Y_{m,k}$, then by Lemma 4.2, we have $\rho(T) < \rho(T^*)$, a contradiction. If T^* is isomorphic to $Y_{m,k}$, then T is isomorphic to the k-uniform hypertree obtained from $P_{m(k-1)+1-2(k-1),k} = (u_0, e_1, u_1, e_2, u_2, \dots, u_{m-3}, e_{m-2}, u_{m-2})$ by attaching pendant edges e' and e'' at y and z in $e_2\backslash\{u_1, u_2\}$, respectively, where $y \ne z$. Note that $T \cong H_{y,z}(1, 1)$ with $H = T[V(T)\backslash((e' \cup e'')\backslash\{y, z\})]$. Let $T^{**} = H_{y,z}(2, 0)$. Note that $T^{**} \cong Z_{m,k}$. Then by Lemma 4.2, we have $\rho(T^{**}) > \rho(T)$, a contradiction. Thus there are exactly three vertices of degree 2 in e, say w_1, w_2, w_3.

Let Q_i be the pendant path at w_i with length l_i, where $i = 1, 2, 3$ and $l_i \ge 1$. Without loss of generality, suppose that $l_1 \ge l_2 \ge l_3 \ge 2$. Let $G = T[V(T)\backslash(V(Q_1 \cup Q_2)\backslash\{w_1, w_2\})]$, then $T \cong G_{w_1,w_2}(l_1, l_2)$. Note that $T^* = G_{w_1,w_2}(l_1 + 1, l_2 - 1)$ is a k-uniform hypertree that is not isomorphic to $P_{m(k-1)+1,k}$ and $Y_{m,k}$. Then by Lemma 4.2, we have $\rho(T^*) > \rho(T)$, a contradiction. Thus there is at least one of Q_1, Q_2, Q_3 with length 1.

As above, T is a k-uniform hypergraph obtained from $P_{m(k-1)+1-k+1,k} = (u_0, e_1, u_1, e_2, u_2, \dots, u_{m-2}, e_{m-1}, u_{m-1})$ by attaching a pendant edge at a vertex of $e_i\backslash\{u_{i-1}, u_i\}$ with $3 \le i \le m - 3$. Then by Lemma 4.2, we have $\rho(Z_{m,k}) \ge \rho(T)$. Thus $T \cong Z_{m,k}$ for $\Delta = 2$.

By Lemma 4.5, we have $\rho(B^3_{m,k}) > \rho(Z_{m,k})$. Thus $T \cong B^3_{m,k}$. ■

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