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Hyponormality of Slant Weighted Toeplitz Operators on the Torus

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Abstract. Here we consider a sequence of positive numbers $\beta = \{\beta_k\}_{k\in\mathbb{Z}^n}$ with $\beta_0 = 1$, and assume that there exists $0 < r \leq 1$ such that for each $i = 1, 2, \ldots, n$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, we have $r \leq \frac{\beta_k}{\beta_{k+\epsilon_i}} \leq 1$ if $k_i \geq 0$, and $r \leq \frac{\beta_{k+\epsilon_i}}{\beta_k} \leq 1$ if $k_i < 0$. For such a weight sequence β, we define the weighted sequence space $L^2(\mathbb{T}^n, \beta)$ to be the set of all $f(z) = \sum_{k\in\mathbb{Z}^n} a_k z^k$ for which $\sum_{k\in\mathbb{Z}^n} |a_k|^2 \beta_k^2 < \infty$. Here $\mathbb{T}$ is the unit circle in the complex plane, and for $n \geq 1$, $\mathbb{T}^n$ denotes the n-Torus which is the cartesian product of n copies of $\mathbb{T}$. For $\varphi \in L^\infty(\mathbb{T}^n, \beta)$, we define the slant weighted Toeplitz operator A_φ on $L^2(\mathbb{T}^n, \beta)$ and establish several properties of A_φ. We also prove that A_φ cannot be hyponormal unless $\varphi \equiv 0$.

1. Introduction

Let $\mathbb{T}$ be the unit circle in the complex plane and $L^2(\mathbb{T})$ be the space of all Lebesgue square integrable functions on $\mathbb{T}$. Thus $L^2(\mathbb{T}) = \{f : \mathbb{T} \mapsto \mathbb{C} | f(z) = \sum_{n\in\mathbb{Z}} a_n z^n, a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty\}$. If $e_n(z) := z^n$ for each $n \in \mathbb{Z}$, then $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$. For a bounded function $\varphi \in L^2(\mathbb{T})$, the multiplication operator M_φ on $L^2(\mathbb{T})$ is defined as $M_\varphi f = \varphi f$. In 1995 M. C. Ho [5] defined slant Toeplitz operator A_φ on $L^2(\mathbb{T})$ as $A_\varphi = WM_\varphi$, where W is an operator on $L^2(\mathbb{T})$ defined as $W(e_{2n}) = e_n$ and $We_{2n-1} = 0\, \forall\, n \in \mathbb{Z}$. Since then this class of operators have been widely studied. The spectral properties of slant Toeplitz operators have a connection to the smoothness of wavelets and appear frequently in wavelet analysis. Motivated by the inter disciplinary and multi faceted applications of slant Toeplitz operators, Arora and Kathuria [1] introduced the notion of slant weighted Toeplitz operators. For this they considered the weighted sequence space $L^2(\mathbb{T}, \beta)$ given by $L^2(\mathbb{T}, \beta) = \{f : \mathbb{T} \mapsto \mathbb{C} | f(z) = \sum_{n\in\mathbb{Z}} a_n z^n, a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} |a_n|^2 \beta_n^2 < \infty\}$. The slant weighted Toeplitz operator $A_\varphi^{(\beta)}$ on $L^2(\mathbb{T}, \beta)$ is defined as $A_\varphi^{(\beta)} = WM_\varphi^{(\beta)}$, where $M_\varphi^{(\beta)}$ is the weighted multiplication operator on $L^2(\mathbb{T}, \beta)$. Properties of these operators were further studied in [2–4, 7–9].

In this paper we introduce the slant weighted Toeplitz operators on $L^2(\mathbb{T}^n, \beta)$. For this we consider the unit circle $\mathbb{T}$ in the complex plane $\mathbb{C}$, and for the integer $n \geq 1$, $\mathbb{T}^n$ denotes the n-torus which is the cartesian product of n copies of $\mathbb{T}$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$, we define $z^m := z_1^{m_1} \ldots z_n^{m_n}$ and $|m| := m_1 + \cdots + m_n$. Also for $\lambda \in \mathbb{Z}$, $z^\lambda := z_1^\lambda \ldots z_n^\lambda$, so that $z = z_1 \ldots z_n$. For $i = 1, \ldots, n$ let ϵ_i be the n tuple $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ where $x_j = \delta_{ij}$ for $1 \leq j \leq n$. Consider a sequence of positive numbers $\beta = \{\beta_k\}_{k\in\mathbb{Z}^n}$ with $\beta_0 = 1$, and assume that there exists $0 < r \leq 1$ such that for each $i = 1, 2, \ldots, n$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, we

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have $r \leq \frac{\beta_k}{\beta_{k+\epsilon_i}} \leq 1$ if $k_i \geq 0$, and $r \leq \frac{\beta_{k+\epsilon_i}}{\beta_k} \leq 1$ if $k_i < 0$. Thus, $\beta_k \geq \beta_0 = 1 \, \forall \, k \in \mathbb{Z}^n$, and $r = 1$ iff $\beta_k = \beta_0 \, \forall \, k \in \mathbb{Z}^n$. Under these assumptions, we define $L^2(\mathbb{T}^n, \beta)$ as follows:

$$L^2(\mathbb{T}^n, \beta) = \{f : \mathbb{T}^n \mapsto \mathbb{C} \,|\, f(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k, a_k \in \mathbb{C}, \sum_{k \in \mathbb{Z}^n} |a_k|^2 \beta_k^2 < \infty\}.$$

For $x, y \in L^2(\mathbb{T}^n, \beta)$ define $\langle x, y \rangle = \sum_{k \in \mathbb{Z}^n} x_k \bar{y}_k {\beta_k}^2$, where $x = \sum_k x_k e_k$ and $y = \sum_k y_k e_k$. For each $k \in \mathbb{Z}^n$, let $e_k(z) := z^k$ so that $\{e_k\}_{k \in \mathbb{Z}^n}$ is an orthogonal basis for $L^2(\mathbb{T}^n, \beta)$ with $\|e_k\| = \beta_k \, \forall \, k$. If for each $k \in \mathbb{Z}^n$ we define $f_k = \frac{e_k}{\beta_k}$, then $\{f_k\}$ is an orthonormal basis for $L^2(\mathbb{T}^n, \beta)$. Also for $m, k \in \mathbb{Z}^n$ we have $e_m e_k = e_{m+k}$ and $f_m f_k = \frac{\beta_{m+k}}{\beta_m \beta_k} f_{m+k}$.

Let $L^\infty(\mathbb{T}^n, \beta)$ denote the set of formal Laurent series $\varphi(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$ having the following properties:
(i) $\varphi L^2(\mathbb{T}^n, \beta) \subseteq L^2(\mathbb{T}^n, \beta)$, and
(ii) there exists some $c > 0$ satisfying $\|\varphi f\| \leq c\|f\|$ for each $f \in L^2(\mathbb{T}^n, \beta)$.
For $\varphi \in L^\infty(\mathbb{T}^n, \beta)$, $\|\varphi\|_\infty := \inf\{c > 0 : \|\varphi f\| \leq c\|f\| \text{ for each } f \in L^2(\mathbb{T}^n, \beta)\}$.

We have only considered weights $\{\beta_k\}_{k \in \mathbb{Z}^n}$ for which there exists $0 < r \leq 1$ such that $r \leq \frac{\beta_k}{\beta_{k+\epsilon_i}} \leq 1$ if $k_i \geq 0$, and $r \leq \frac{\beta_{k+\epsilon_i}}{\beta_k} \leq 1$ if $k_i < 0$. For example we include here a particular weight sequence which do not satisfy this condition. For this let us define $\|k\| = \sum_{i=1}^n |k_i|$ for $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, and let $\beta_k := (\|k\|)!$. Then for $k_i > 0$ we have $\frac{\beta_k}{\beta_{k+\epsilon_i}} = \frac{1}{\|k\|+1} \to 0$, as $\|k\| \to \infty$. Also for $k_i < 0$, we have $\frac{\beta_{k+\epsilon_i}}{\beta_k} = \frac{1}{\|k\|} \to 0$, as $\|k\| \to \infty$. So there does not exist $0 < r \leq 1$ satisfying the required condition in this case.

2. Properties of M_φ

Definition 2.1. *For $\varphi \in L^\infty(\mathbb{T}^n, \beta)$ the Laurent operator M_φ on $L^2(\mathbb{T}^n, \beta)$ is defined as $M_\varphi f = \varphi f \ \forall \ f \in L^2(\mathbb{T}^n, \beta)$. In particular, when $\varphi(z) = z_i$ for $1 \leq i \leq n$, then M_φ is usually denoted as M_{z_i}.*

Theorem 2.2. *For $1 \leq i \leq n$, and $t \in \mathbb{Z}^n$, let $\beta_{t;i} := \frac{\beta_{t+\epsilon_i}}{\beta_t}$. Then we have the following:*

1. $M_{z_i} e_t = e_{t+\epsilon_i}$
2. $M_{z_i} f_t = \beta_{t;i} f_{t+\epsilon_i}$
3. $M_{z_i}^* e_t = \beta_{t-\epsilon_i;i}^2 e_{t-\epsilon_i}$
4. $M_{z_i}^* f_t = \beta_{t-\epsilon_i;i} f_{t-\epsilon_i}$
5. $M_{z_i}^* M_{z_i} e_t = \beta_{t;i}^2 \, e_t$ *and* $M_{z_i}^* M_{z_i} f_t = \beta_{t;i}^2 \, f_t$
6. $M_{z_i} M_{z_i}^* e_t = \beta_{t-\epsilon_i;i}^2 \, e_t$ *and* $M_{z_i} M_{z_i}^* f_t = \beta_{t-\epsilon_i;i}^2 \, f_t$

Proof. For each $i \in \{1, \ldots, n\}$, we have

1. $M_{z_i} e_t(z) = z_i z^t = z^{t+\epsilon_i} = e_{t+\epsilon_i}(z)$. So that $M_{z_i} e_t = e_{t+\epsilon_i} \ \ \forall \ t \in \mathbb{Z}^n$.
2. $M_{z_i} f_t = \frac{1}{\beta_t} M_{z_i} e_t = \frac{\beta_{t+\epsilon_i}}{\beta_t} f_{t+\epsilon_i} = \beta_{t;i} f_{t+\epsilon_i}$.
3. Let $h(z) = \sum_{p \in \mathbb{Z}^n} a_p z^p$, so that $h = \sum_p a_p e_p = \sum_p a_p \beta_p f_p$. Then

$$\langle M_{z_i} h, e_t \rangle = \sum_p a_p \langle M_{z_i} e_p, e_t \rangle = \sum_p a_p \langle e_{p+\epsilon_i}, e_t \rangle = a_{t-\epsilon_i} \beta_t^2 = \langle h, \frac{\beta_t^2}{\beta_{t-\epsilon_i}^2} e_{t-\epsilon_i} \rangle$$

$\implies M_{z_i}^* e_t = \frac{\beta_t^2}{\beta_{t-\epsilon_i}^2} e_{t-\epsilon_i} = \beta_{t-\epsilon_i;i}^2 \, e_{t-\epsilon_i} \ \ \forall \ t \in \mathbb{Z}^n$.

4. $M_{z_i}^* f_t = \frac{1}{\beta_t} M_{z_i}^* e_t = \beta_{t-\epsilon_i;i} f_{t-\epsilon_i}$.

5. $M_{z_i}^* M_{z_i} e_t = M_{z_i}^* e_{t+\epsilon_i} = \beta_{t;i}^2 e_t \;\; \forall\; t \in \mathbb{Z}^n$, and
   $M_{z_i}^* M_{z_i} f_t = \beta_{t;i} M_{z_i}^* f_{t+\epsilon_i} = \beta_{t;i}^2 f_t \;\; \forall\; t \in \mathbb{Z}^n$
6. $M_{z_i} M_{z_i}^* e_t = \beta_{t-\epsilon_i;i}^2 M_{z_i} e_{t-\epsilon_i} = \beta_{t-\epsilon_i;i}^2 e_t \;\forall\; t \in \mathbb{Z}^n$, and
   $M_{z_i} M_{z_i}^* f_t = \beta_{t-\epsilon_i;i} M_{z_i} f_{t-\epsilon_i} = \beta_{t-\epsilon_i;i}^2 f_t \;\forall\; t \in \mathbb{Z}^n$

□

Remark 2.3. *We have* $M_{z_\tau} f_j = \beta_{j;\tau} f_{j+\epsilon_\tau}$ *where* $\beta_{j;\tau} := \frac{\beta_{j+\epsilon_\tau}}{\beta_j} \;\forall\; j \in \mathbb{Z}^n \;\forall\; 1 \le \tau \le n$. *As* $\left\{\beta_{j;\tau}\right\}_{j\in\mathbb{Z}^n}$ *is bounded for each* $1 \le \tau \le n$, *so* M_{z_τ} *is bounded and* $\|M_{z_\tau}\| = \sup_{j\in\mathbb{Z}^n} |\beta_{j;\tau}| \le 1/r$.

Theorem 2.4. *For* $t, k \in \mathbb{Z}^n$, $M_{z^k} f_t = \frac{\beta_{t+k}}{\beta_t} f_{t+k}$.

Proof. Let $k = (k_1, \dots, k_n)$. Then $z^k = z_1^{k_1} \dots z_n^{k_n}$ and $M_{z^k} f_t = M_{z_1^{k_1}} \dots M_{z_n^{k_n}} f_t = \frac{\beta_{t+k}}{\beta_t} f_{t+k}$, since $M_{z_i} M_{z_j} f_t = M_{z_j} M_{z_i} f_t \;\forall\; 1 \le i, j \le n$. □

Theorem 2.5. *If A is a bounded linear operator on* $L^2(\mathbb{T}^n, \beta)$ *that commutes with* $M_{z_i} \;\forall\; 1 \le i \le n$, *then* $A = M_\varphi$ *for* $\varphi \in L^\infty(\mathbb{T}^n, \beta)$.

Proof. Let $\varphi = Ae_0$. Then $\varphi \in L^2$ and $Ae_k = AM_{z^k} e_0 = M_{z^k} A e_0 = z^k \varphi = \varphi e_k$ (since $M_{z_i} A = AM_{z_i} \;\forall\; i \implies M_{z^k} A = AM_{z^k} \;\forall\; k$).
This implies that $Af = \varphi f \;\forall$ polynomials $f \in L^2(\mathbb{T}^n, \beta)$.
For $k \in \mathbb{Z}^n$, define $\psi_k : L^2(\mathbb{T}^n, \beta) \mapsto \mathbb{C}$ as $\psi_k(g) = \beta_k \hat{g}(k)$ where $g(z) = \sum_k \hat{g}(k) z^k$.
We know that if for any two functions $f, g \in L^2(\mathbb{T}^n, \beta)$ we have $\psi_k(f) = \psi_k(g) \;\forall\; k \in \mathbb{Z}^n$ then $f = g$ [6]. Let $g(z) = \sum_k \hat{g}(k) z^k \in L^2(\mathbb{T}^n, \beta)$. Then $Ag \in L^2(\mathbb{T}^n, \beta)$ and $\|Ag\|^2 = \sum_k |\psi_k(Ag)|^2 < \infty$.
Now $Ae_t(z) = \varphi e_t(z) = \varphi(z) z^t = \sum_k \hat{\varphi}(k) z^{k+t} = \sum_k \hat{\varphi}(k-t) z^k$, and so $\psi_k(Ag) = \psi_k(\sum_t \hat{g}(t) Ae_t) = \sum_t \hat{g}(t) \psi_k(Ae_t) = \sum_t \hat{g}(t) \hat{\varphi}(k-t) \beta_k$.
Also $(g\varphi)(z) = g(z)\varphi(z) = \sum_{k\in\mathbb{Z}^n} (\sum_{t\in\mathbb{Z}^n} \hat{g}(t) \hat{\varphi}(k-t)) z^k$ (if $\varphi(z) = \sum_t \hat{\varphi}(t) z^t$).
As $\sum_k |\sum_t \hat{g}(t)\hat{\varphi}(k-t)|^2 \beta_k^2 = \sum_k |\psi_k(Ag)|^2 < \infty$ so $g\varphi \in L^2(\mathbb{T}^n, \beta)$ and $\psi_k(g\varphi) = \sum_{t\in\mathbb{Z}^n} \hat{g}(t)\hat{\varphi}(k-t)\beta_k = \psi_k(Ag)$.
Thus $\varphi g \in L^2(\mathbb{T}^n, \beta)$ and $\|\varphi g\|^2 = \sum_k |\psi_k(\varphi g)|^2 = \sum_k |\psi_k(Ag)|^2 = \|Ag\|^2$.

Therefore $\varphi g = Ag \implies A = M_\varphi$ *for* $\varphi \in L^\infty$. □

Theorem 2.6. *Let A be a bounded linear operator on* $L^2(\mathbb{T}^n, \beta)$. *Then the following are equivalent*

1. $\langle Af_{t+\epsilon_i}, f_{k+\epsilon_i} \rangle = \frac{\beta_{k;i}}{\beta_{t;i}} \langle Af_t, f_k \rangle \;\forall\; t, k \in \mathbb{Z}^n$ *and* $1 \le i \le n$.
2. $AM_{z_i} = M_{z_i} A \;\;\forall\; 1 \le i \le n$.
3. *A is a Laurent operator on* $L^2(\mathbb{T}^n, \beta)$.

Proof. $1 \implies 2$
Suppose $\langle Af_{t+\epsilon_i}, f_{k+\epsilon_i} \rangle = \frac{\beta_{k;i}}{\beta_{t;i}} \langle Af_t, f_k \rangle$. Now $\langle M_{z_i} Af_t, f_k \rangle = \langle Af_t, M_{z_i}^* f_k \rangle = \beta_{k-\epsilon_i;i} \langle Af_t, f_{k-\epsilon_i} \rangle = \beta_{t;i} \langle Af_{t+\epsilon_i}, f_k \rangle = \langle AM_{z_i} f_t, f_k \rangle$
Thus, $AM_{z_i} = M_{z_i} A \;\;\forall\; 1 \le i \le n$.

$2 \implies 3$
This follows from Theorem 2.5

$3 \implies 1$

Let $A = M_\varphi$ where $\varphi(z) = \sum_{m\in\mathbb{Z}^n} \hat{\varphi}(m)z^m = \sum_{m\in\mathbb{Z}^n} \hat{\varphi}(m)e_m(z)$. Then,

$$\begin{aligned}
\langle Af_{t+\epsilon_i}, f_{k+\epsilon_i}\rangle &= \sum_{m\in\mathbb{Z}^n} \frac{\hat{\varphi}(m)}{\beta_{t+\epsilon_i}\beta_{k+\epsilon_i}}\langle e_m e_{t+\epsilon_i}, e_{k+\epsilon_i}\rangle \\
&= \sum_{m\in\mathbb{Z}^n} \frac{\hat{\varphi}(m)}{\beta_{t+\epsilon_i}\beta_{k+\epsilon_i}}\langle e_{t+m+\epsilon_i}, e_{k+\epsilon_i}\rangle = \frac{\hat{\varphi}(k-t)}{\beta_{t+\epsilon_i}\beta_{k+\epsilon_i}}\beta^2_{k+\epsilon_i} \\
&= \frac{\beta_{k+\epsilon_i}}{\beta_{t+\epsilon_i}}\hat{\varphi}(k-t) = \frac{\beta_{k;i}}{\beta_{t;i}}\frac{\beta_k}{\beta_t}\hat{\varphi}(k-t) \\
&= \frac{\beta_{k;i}}{\beta_{t;i}}\langle Af_t, f_k\rangle
\end{aligned}$$

□

3. Slant weighted Toeplitz operator on $L^2(\mathbb{T}^n, \beta)$

Definition 3.1. *$W : L^2(\mathbb{T}^n, \beta) \mapsto L^2(\mathbb{T}^n, \beta)$ is defined as the linear operator with,* $We_k = \begin{cases} e_{\frac{k}{2}}, & \text{if } k \text{ is even ;} \\ 0, & \text{otherwise.} \end{cases}$

Thus $Wf_k = \begin{cases} \frac{\beta_{\frac{k}{2}}}{\beta_k} f_{\frac{k}{2}}, & \text{if } k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$

Definition 3.2. *Let $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. Then we say $k \geq 0$ if $k_i \geq 0\ \forall\, i$. Also, k is said to be even if each k_i is even, otherwise k is said to be odd.*

Theorem 3.3. *W is bounded and $\|W\| \leq 1$*

Proof. As $Wf_k = \begin{cases} \frac{\beta_{\frac{k}{2}}}{\beta_k} f_{\frac{k}{2}}, & \text{if } k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$

So W is bounded and $\|W\| = \sup_{k\in\mathbb{Z}^n} \left|\frac{\beta_k}{\beta_{2k}}\right|$, provided $\left\{\frac{\beta_k}{\beta_{2k}}\right\}_{k\in\mathbb{Z}^n}$ is bounded. For $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, let $\tilde{k}(1) := k$ and $\tilde{k}(i) := (2k_1, \ldots, 2k_{i-1}, k_i, \cdots, k_n)$ for $2 \leq i \leq n$.

Also for $2 \leq i \leq n$, let $\gamma_i := \begin{cases} 1, & \text{if } k_i = 0; \\ \frac{\beta_{\tilde{k}(i)}}{\beta_{\tilde{k}(i)+\epsilon_i}} \cdots \frac{\beta_{\tilde{k}(i)+(k_i-1)\epsilon_i}}{\beta_{\tilde{k}(i)+k_i\epsilon_i}}, & \text{if } k_i > 0; \\ \frac{\beta_{\tilde{k}(i)}}{\beta_{\tilde{k}(i)-\epsilon_i}} \cdots \frac{\beta_{\tilde{k}(i)+(k_i+1)\epsilon_i}}{\beta_{\tilde{k}(i)+k_i\epsilon_i}}, & \text{if } k_i < 0. \end{cases}$

Then $\frac{\beta_k}{\beta_{2k}} = \gamma_1\gamma_2\cdots\gamma_n$, and as $0 < \gamma_i \leq 1\ \forall\ i$, hence $\{\frac{\beta_k}{\beta_{2k}}\}$ is bounded. So W is bounded and $\|W\| \leq 1$. □

Theorem 3.4. *For $p \in \mathbb{Z}^n$, $W^*f_p = \frac{\beta_p}{\beta_{2p}} f_{2p}$ and $W^*e_p = \frac{\beta_p^2}{\beta_{2p}^2} e_{2p}$.*

Proof. Let $p \in \mathbb{Z}^n$. Then for any $k \in \mathbb{Z}^n$, we have

$\langle We_k, e_p\rangle = \begin{cases} \langle e_{\frac{k}{2}}, e_p\rangle, & \text{if } k \text{ is even;} \\ 0, & \text{otherwise.} \end{cases} = \begin{cases} \beta_p^2, & \text{if } k = 2p; \\ 0, & \text{otherwise.} \end{cases}$.

Also, $\langle e_k, e_{2p}\rangle = \begin{cases} \beta_{2p}^2, & \text{if } k = 2p; \\ 0, & \text{otherwise.} \end{cases}$, and so $\langle We_k, e_p\rangle = \frac{\beta_p^2}{\beta_{2p}^2}\langle e_k, e_{2p}\rangle\ \forall\ k \in \mathbb{Z}^n$.

Thus, $\langle Wf, e_p\rangle = \langle f, \frac{\beta_p^2}{\beta_{2p}^2} e_{2p}\rangle\ \forall\ f \in L^2(\mathbb{T}^n, \beta)$ which implies $W^*e_p = \frac{\beta_p^2}{\beta_{2p}^2} e_{2p}$.

Therefore, $W^*f_p = \frac{1}{\beta_p} W^*e_p = \frac{\beta_p}{\beta_{2p}^2} e_{2p} = \frac{\beta_p}{\beta_{2p}} f_{2p}$. □

Corollary 3.5. *For $p \in \mathbb{Z}^n$, $WW^*f_p = \frac{\beta_p^2}{\beta_{2p}^2} f_p$, and* $W^*Wf_p = \begin{cases} \frac{\beta_{\frac{p}{2}}^2}{\beta_p^2} f_p, & \text{if } p \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$.

Definition 3.6. *Let $H^2(\mathbb{T}^n,\beta) = \{f \in L^2(\mathbb{T}^n,\beta) : f(z) = \sum_{k\in\mathbb{Z}^n_+} a_k z^k\}$. Thus $\{f_k\}_{k\in\mathbb{Z}^n_+}$ is an orthonormal basis for $H^2(\mathbb{T}^n,\beta)$. Here Z_+ denotes the set of non negative integers.*

Theorem 3.7. *If P is the projection of $L^2(\mathbb{T}^n,\beta)$ onto $H^2(\mathbb{T}^n,\beta)$, then P reduces W.*

Proof. We have $Pf_k = \begin{cases} f_k, & \text{if } k \in \mathbb{Z}^n_+; \\ 0, & \text{otherwise.} \end{cases}$

Case 1: Let $k \in \mathbb{Z}^n$ and $k \geq 0$. As $Wf_k = \begin{cases} \frac{\beta_{\frac{k}{2}}}{\beta_k} f_{\frac{k}{2}}, & \text{if } k \text{ is even;} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$
so $PWf_k = Wf_k = WPf_k$.

Case 2: Let $k \in \mathbb{Z}^n$ and $k \ngeq 0$. So, $Pf_k = 0 \implies WPf_k = 0 = PWf_k$. Thus, $PW = WP$ and so P reduces W. □

Theorem 3.8. $WM_{z^t}W^* = \begin{cases} \frac{\beta_k^2}{\beta_{2k}^2} M_{z^{\frac{t}{2}}}, & \textit{if } t \textit{ is even;} \\ 0, & \textit{otherwise.} \end{cases}$

Proof. For $k \in \mathbb{Z}^n$,

$$\begin{aligned} WM_{z^t}W^* f_k &= \frac{\beta_k}{\beta_{2k}} WM_{z^t} f_{2k} = \frac{\beta_k}{\beta_{2k}} W \frac{\beta_{2k+t}}{\beta_{2k}} f_{2k+t} \\ &= \frac{\beta_k}{\beta_{2k}^2} \beta_{2k+t} W f_{2k+t} = \begin{cases} \frac{\beta_k}{\beta_{2k}^2} \beta_{k+\frac{t}{2}} f_{k+\frac{t}{2}}, & \text{if } t \text{ is even;} \\ 0, & \text{otherwise.} \end{cases} \\ &= \begin{cases} \frac{\beta_k^2}{\beta_{2k}^2} M_{z^{\frac{t}{2}}}, & \text{if } t \text{ is even;} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

from which the result follows immediately. □

Definition 3.9. *For $\varphi \in L^\infty(\mathbb{T}^n,\beta)$, we define the slant weighted Toeplitz operator $A_\varphi : L^2(\mathbb{T}^n,\beta) \mapsto L^2(\mathbb{T}^n,\beta)$ as $A_\varphi = WM_\varphi$.*

Theorem 3.10. *If A_φ is a slant weighted Toeplitz operator then $M_{z_i}A_\varphi = A_\varphi M_{z_i^2}$ $\forall\, 1 \leq i \leq n$. Equivalently A_φ is slant weighted Toeplitz operator implies that $M_{z^k}A_\varphi = A_\varphi M_{z^{2k}}$ $\forall\ k \in \mathbb{Z}^n$.*

Proof. We have $A_\varphi = WM_\varphi$ for $\varphi \in L^\infty(\mathbb{T}^n,\beta)$. We define $S = \{(k_1,\ldots,k_n) \in \mathbb{Z}^n |\ \text{each } k_i \text{ is either } 0 \text{ or } 1\}$. For $t,\eta \in S$, $t+\eta$ is even iff $t = \eta$.

Case 1: Let j be even and $j = 2m$.
So $\varphi(z) = \sum_{k\in\mathbb{Z}^n} a_k z^k = \sum_{t\in S}\sum_{k\in\mathbb{Z}^n} a_{2k+t} z^{2k+t}$, and

$$\begin{aligned} M_{z_i}A_\varphi f_j(z) =& M_{z_i} W(\varphi(z) f_j(z)) \\ =& M_{z_i} W\left(\sum_{t\in S}\sum_{k\in\mathbb{Z}^n} \frac{a_{2k+t}}{\beta_{2m}} z^{2(k+m)+t}\right) \\ =& M_{z_i} \sum_{k\in\mathbb{Z}^n} \frac{a_{2k}}{\beta_{2m}} z^{(k+m)} \quad (\because\ t \text{ is even iff } t = 0) \\ =& z_i \left(\sum_{k\in\mathbb{Z}^n} \frac{a_{2k}}{\beta_{2m}} z^{(k+m)}\right) \end{aligned}$$

and $A_\varphi M_{z_i^2} f_j(z) = WM_\varphi\left(z_i^2 \frac{z^j}{\beta_j}\right) = W\left(\sum_{t\in S}\sum_{k\in\mathbb{Z}^n} \frac{a_{2k+t}}{\beta_{2m}} z_i^2 z^{2(k+m)+t}\right) = \sum_{k\in\mathbb{Z}^n} \frac{a_{2k}}{\beta_{2m}} z_i z^{(k+m)}$
Therefore $M_{z_i} A_\varphi f_j(z) = A_\varphi M_{z_i^2} f_j(z)$ for j even in $\mathbb{Z}^n$.

Case 2: Let $j \in \mathbb{Z}^n$ and j odd. Then $j = 2m + \tau$ where $m \in \mathbb{Z}^n$, $0 \neq \tau \in S$. Then

$$\begin{aligned} M_{z_i} A_\varphi f_j(z) =& M_{z_i} W\left(\sum_{t\in S}\sum_{k\in\mathbb{Z}^n} \frac{a_{2k+t}}{\beta_{2m+\tau}} z^{2(k+m)+t+\tau}\right) \\ =& z_i\left(\sum_{k\in\mathbb{Z}^n} \frac{a_{2k+\tau}}{\beta_{2m+\tau}} z^{(k+m+\tau)}\right) \quad (\because\ t+\tau \text{ is even iff } t = \tau) \end{aligned}$$

and $A_\varphi M_{z_i^2} f_j(z) = W\left(\sum_{t\in S}\sum_{k\in\mathbb{Z}^n} \frac{a_{2k+t}}{\beta_{2m+\tau} z_i^2} z^{2(k+m)+t+\tau}\right) = z_i\left(\sum_{k\in\mathbb{Z}^n} \frac{a_{2k+\tau}}{\beta_{2m+\tau}} z^{(k+m+\tau)}\right)$.
From Case 1 and Case 2 we get, $M_{z_i} A_\varphi = A_\varphi M_{z_i^2}\ \forall\, 1 \le i \le n$. □

Definition 3.11. *For $f \in L^2(\mathbb{T}^n, \beta)$ and $f(z) = \sum_k a_k z^k$, we define $\tilde{f}(z) := \sum_k a_k\left(\frac{\beta_k^2}{\beta_{2k}^2}\right) z^k$, and $f^*(z) = \sum_k a_{2k} \frac{\beta_k^2}{\beta_{2k}^2} z^k$. Also P_e on $L^2(\mathbb{T}^n, \beta)$ is defined as $P_e f(z) = \sum_k a_{2k} z^{2k}$ for $f(z) = \sum_k a_k z^k$.*

Remark 3.12. *As in Theorem 3.3, $\frac{\beta_k}{\beta_{2k}} \le 1\ \forall\ k$ and so $\|\tilde{f}\|^2 = \sum_k |\alpha_k|^2 |\frac{\beta_k}{\beta_{2k}}|^2 \beta_k^2 \le \sum_k |\alpha_k|^2 \beta_k^2 = \|f\|^2$, i.e, $\|\tilde{f}\| \le \|f\|$.*

Theorem 3.13. *For $f \in L^2(\mathbb{T}^n, \beta)$ and $f(z) = \sum_k a_k z^k$, we have the following:*
1. *$W = WP_e$ and $W^* f(z) = \tilde{f}(z^2)$.*
2. *$WW^* f(z) = \tilde{f}(z)$ and $W^* W f(z) = f^*(z^2)$.*
3. *$WM_{z^t} W^* = 0$ for t odd in $\mathbb{Z}^n$.*
4. *$WM_{z^{2t}} W^* f = z^t \tilde{f}$.*
5. *For $f, g \in L^2(\mathbb{T}^n, \beta)$, $W^*(fg) \neq (W^* f)(W^* g)$ unless $\frac{\beta_k^2 \beta_t^2}{\beta_{2(k)}^2 \beta_{2(t)}^2} = \frac{\beta_{k+t}^2}{\beta_{2(k+t)}^2}\ \forall\ k, t$.*
6. *$W((W^* f) \cdot (W^* g)) = \tilde{f} \cdot \tilde{g}$.*

Proof. (1) $Wf(z) = \sum_k a_k W z^k = \sum_k a_{2k} z^k = WP_e f(z)$.
Also $W^* f(z) = \sum_k a_k W^* e_k = \sum_k a_k \frac{\beta_k^2}{\beta_{2k}^2} e_{2k} = \tilde{f}(z^2)$

(2) $WW^* f(z) = W\tilde{f}(z^2) = \tilde{f}(z)$,
and $W^* W f(z) = W^*\left(\sum_k a_{2k} z^k\right) = \sum_k a_{2k} \frac{\beta_k^2}{\beta_{2k}^2} z^{2k} = f^*(z^2)$.

(3) $WM_{z^t} W^* f(z) = Wz^t \tilde{f}(z^2) = 0 \implies WM_z W^* = 0$ for t odd in $\mathbb{Z}^n$.

(4) $WM_{z^{2t}} W^* f(z) = Wz^{2t} \tilde{f}(z^2) = z^t \tilde{f}(z)$ and so $WM_{z^{2t}} W^* f = z^t \tilde{f}\ \ \forall\ f \in L^2(\mathbb{T}^n, \beta)$.

(5) For $f(z) = \sum_k \alpha_k z^k$ and $g(z) = \sum_k \delta_k z^k$ we have $fg = \sum_t \sum_k \alpha_k \delta_t z^{k+t}$ and $(\widetilde{fg})(z) = \sum_t \sum_k \alpha_k \delta_t \frac{\beta_{k+t}^2}{\beta_{2(k+t)}^2} z^{k+t}$. As $\tilde{f}(z)\tilde{g}(z) = \sum_t \sum_k \alpha_k \delta_t \frac{\beta_k^2 \beta_t^2}{\beta_{2(k)}^2 \beta_{2(t)}^2} z^{k+t}$, hence $W^*(fg) \neq (W^* f)(W^* g)$ unless $\frac{\beta_k^2 \beta_t^2}{\beta_{2(k)}^2 \beta_{2(t)}^2} = \frac{\beta_{k+t}^2}{\beta_{2(k+t)}^2}\ \forall\ k, t \in \mathbb{Z}^n$.

(6)

$$\begin{aligned} W(W^* f(z) \cdot W^* g(z)) =& W(\tilde{f}(z^2)\tilde{g}(z^2)) \\ =& W(\sum_t \sum_k \alpha_k \delta_t \frac{\beta_k^2 \beta_t^2}{\beta_{2(k)}^2 \beta_{2(t)}^2} z^{2(k+t)}) \\ =& \tilde{f}(z)\tilde{g}(z) = (WW^* f(z))(WW^* g(z)). \end{aligned}$$

This implies, $W((W^* f) \cdot (W^* g)) = (WW^* f) \cdot (WW^* g) = \tilde{f} \cdot \tilde{g}$. □

Theorem 3.14. *Let $f \in L^2(\mathbb{T}^n, \beta)$. Then $f(z) = \sum_{t \in S} z^t f_t(z^2)$, where $f_t(z) = \sum_k a_{2k+t} z^k$ for $f(z) = \sum_k a_k z^k$.*

Proof. $f(z) = \sum_k a_k z^k = \sum_{t \in S} \sum_{k \in \mathbb{Z}^n} a_{2k+t} z^{2k+t} = \sum_{t \in S} z^t \left(\sum_k a_{2k+t} z^{2k} \right) = \sum_{t \in S} z^t f_t(z^2)$. □

Theorem 3.15. *Let $f, g \in L^2(\mathbb{T}^n, \beta)$ such that one of f and g is in $L^\infty(\mathbb{T}^n, \beta)$. Then $W(fg) = \sum_{t \in S} z^t (W\bar{z}^t f)(W\bar{z}^t g)$.*

Proof. By Theorem 3.14, $f(z) = \sum_{t \in S} z^t f_t(z^2)$ and $g(z) = \sum_{p \in S} z^p g_p(z^2)$.
$\therefore f(z)g(z) = \sum_{t,p \in S} z^{t+p} f_t(z^2) g_p(z^2) = \sum_{t \in S} z^{2t} f_t(z^2) g_t(z^2) + \sum_{t,p \in S, t \neq p} z^{t+p} f_t(z^2) g_p(z^2)$

For $t, p \in S$, $t + p$ is even iff $t = p$. Thus, $W(f(z)g(z)) = W\left(\sum_{t \in S} z^{2t} f_t(z^2) g_t(z^2) \right) = \sum_{t \in S} z^t \left(W f_t(z^2) \right) \left(W g_t(z^2) \right)$.
For $t \in S$, $f(z) = z^t f_t(z^2) + \sum_{p \neq t, p \in S} z^p f_p(z^2) \implies f_t(z^2) = \bar{z}^t f(z) - \sum_{p \neq t, p \in S} z^{p-t} f_p(z^2)$
Therefore $W\left(f_t(z^2) \right) = W\left(\bar{z}^t f(z) \right)$.
Thus, $W(f(z)g(z)) = \sum_{t \in S} z^t (W\bar{z}^t f(z))(W\bar{z}^t g(z))$. □

Theorem 3.16. *WA_φ is a slant weighted Toeplitz operator iff $\varphi = 0$.*

Proof. If $\varphi = 0$ then the result is obvious.
Conversely, let $\varphi \in L^\infty(\mathbb{T}^n, \beta)$, such that WA_φ is a slant weighted Toeplitz operator. By Theorem 3.10, WA_φ is a slant weighted Toeplitz operator implies that $M_{z_i} WA_\varphi = WA_\varphi M_{z_i^2}$ $\forall\, 1 \leq i \leq n$. Using this and Theorem 2.2, we get

$$\langle WA_\varphi f_{k+2\epsilon_j}, f_{t+\epsilon_j} \rangle = \frac{\beta_{t;j}}{\beta_{k+\epsilon_j;j} \beta_{k;j}} \langle WA_\varphi f_k, f_t \rangle \ \forall\ t, k \in \mathbb{Z}^n,\ 1 \leq j \leq n \tag{1}$$

$$\begin{aligned} \text{Now, } \langle WA_\varphi f_{k+2\epsilon_j}, f_{t+\epsilon_j} \rangle &= \frac{\beta_{t+\epsilon_j}}{\beta_{2t+2\epsilon_j}} \langle A_\varphi f_{k+2\epsilon_j}, f_{2t+2\epsilon_j} \rangle \text{ by Theorem 3.4} \\ &= \frac{\beta_{t+\epsilon_j}}{\beta_{2t+2\epsilon_j}} \cdot \frac{\beta_{2t+\epsilon_j;j}}{\beta_{k+\epsilon_j;j} \beta_{k;j}} \langle A_\varphi f_k, f_{2t+\epsilon_j} \rangle \\ &= \frac{\beta_{t+\epsilon_j}}{\beta_{2t+2\epsilon_j}} \cdot \frac{\beta_{2t+\epsilon_j;j} \beta_{2t+\epsilon_j}}{\beta_{k+\epsilon_j;j} \beta_{k;j} \beta_{4t+2\epsilon_j}} \langle M_\varphi f_k, f_{4t+2\epsilon_j} \rangle \\ &= \frac{\beta_{t+\epsilon_j}}{\beta_{k+\epsilon_j;j} \beta_{k;j} \beta_{4t+2\epsilon_j}} \langle M_\varphi f_k, f_{4t+2\epsilon_j} \rangle \end{aligned} \tag{2}$$

$$\text{Also, } \langle WA_\varphi f_k, f_t \rangle = \frac{\beta_t}{\beta_{2t}} \langle WM_\varphi f_k, f_{2t} \rangle = \frac{\beta_t}{\beta_{4t}} \langle M_\varphi f_k, f_{4t} \rangle \tag{3}$$

From Equation 1, 2 and 3 we get $\langle M_\varphi f_k, f_{4t+2\epsilon_j} \rangle = \beta_{4t+\epsilon_j;j} \cdot \beta_{4t;j} \langle M_\varphi f_k, f_{4t} \rangle$.
Equivalently, $\langle M_\varphi e_k, e_{4t+2\epsilon_j} \rangle = \beta^2_{4t+\epsilon_j;j} \cdot \beta^2_{4t;j} \langle M_\varphi e_k, e_{4t} \rangle$.
Let $\varphi(z) = \sum_{q \in \mathbb{Z}^n} a_q z^q$. Then

$$\begin{aligned} \langle M_\varphi e_k, e_{4t+2\epsilon_j} \rangle = \beta^2_{4t+\epsilon_j;j} \cdot \beta^2_{4t;j} \langle M_\varphi e_k, e_{4t} \rangle \ &\text{iff}\ \langle \sum_{q \in \mathbb{Z}^n} a_q z^{q+k}, z^{4t+2\epsilon_j} \rangle = \beta^2_{4t+\epsilon_j;j} \cdot \beta^2_{4t;j} \langle \sum_{q \in \mathbb{Z}^n} a_q z^{q+k}, z^{4t} \rangle \\ &\text{iff}\ \beta^2_{4t+2\epsilon_j} a_{4t+2\epsilon_j-k} = \frac{\beta^2_{4t+2\epsilon_j}}{\beta^2_{4t}} a_{4t-k} \cdot \beta^2_{4t}\ \forall k, t \in \mathbb{Z}^n,\ 1 \leq j \leq n \\ &\text{iff}\ a_{t+2\epsilon_j} = a_t\ \forall t \in \mathbb{Z}^n,\ 1 \leq j \leq n \end{aligned}$$

Thus, for each $t \in \mathbb{Z}^n$ and $1 \leq j \leq n$, we have $a_t = a_{t+2\epsilon_j} = a_{t+4\epsilon_j} = a_{t+6\epsilon_j} = \cdots$
But $|t + 2\lambda\epsilon_j| \to \infty$ as $\lambda \to \infty$, and as $\varphi \in L^\infty(\mathbb{T}^n)$ so $a_{t+2\lambda\epsilon_j} \to 0$ as $n \to \infty$.
Therefore, $a_t = 0\ \forall\ t \in \mathbb{Z}^n \implies \varphi = 0$. □

4. The case when $\{\frac{\beta_{2k}}{\beta_k}\}_k$ is a bounded sequence

In this section we make the added assumption that $\{\frac{\beta_{2k}}{\beta_k}\}_k$ is also bounded which gives us some more interesting results which may not hold otherwise.

Lemma 4.1. *Let $h \in L^2(\mathbb{T}^n,\beta)$ and $\xi(z) = h(z^2)$. Then $\xi \in L^2(\mathbb{T}^n,\beta)$ and $\|h\| \le \|\xi\| \le \lambda\|h\|$ where $\frac{\beta_{2k}}{\beta_k} \le \lambda\ \forall\, k$.*

Proof. Let $h(z) = \sum_{k\in\mathbb{Z}^n} a_k z^k$. Then $\xi(z) = \sum_{k\in\mathbb{Z}^n} a_k z^{2k}$.
Now, $\sum_{k\in\mathbb{Z}^n} |a_k|^2 \beta_{2k}^2 = \sum_{k\in\mathbb{Z}^n} \left(\frac{\beta_{2k}}{\beta_k}\right)^2 |a_k|^2 \beta_k^2 < \infty$, since $\{\frac{\beta_{2k}}{\beta_k}\}_k$ is bounded.
Hence $\xi \in L^2(\mathbb{T}^n,\beta)$ and, $\|\xi\|^2 = \sum_{k\in\mathbb{Z}^n} |a_k|^2 \beta_{2k}^2 \le \sum_{k\in\mathbb{Z}^n} |a_k|^2 \left(\frac{\beta_{2k}}{\beta_k}\right)^2 \beta_k^2 \le \lambda^2 \sum_{k\in\mathbb{Z}^n} |a_k|^2 \beta_k^2 = \lambda^2\|h\|^2$.
As $\frac{\beta_k}{\beta_{2k}} \le 1$, so $\|h\|^2 = \sum_{k\in\mathbb{Z}^n} |a_k|^2 \beta_k^2 \le \sum_{k\in\mathbb{Z}^n} |a_k|^2 \beta_{2k}^2 = \|\xi\|^2$.
Thus the result follows. □

The following result gives the converse part of Theorem 3.10.

Theorem 4.2. *Let A be a bounded linear operator on $L^2(\mathbb{T}^n,\beta)$ such that $M_{z_i}A = AM_{z_i^2}\ \forall\, 1 \le i \le n$. Then A must be a slant weighted Toeplitz operator. Equivalently A is slant weighted Toeplitz operator if $M_{z^k}A = AM_{z^{2k}}\ \ \forall\ k \in \mathbb{Z}^n$.*

Proof. Suppose $M_{z_i}A = AM_{z_i^2}\ \forall\, 1 \le i \le n$. To show that there exists $\varphi \in L^\infty(\mathbb{T}^n,\beta)$ such that $A = WM_\varphi$. We know that $M_{z_i}A = AM_{z_i^2}\ \forall\, 1 \le i \le n$ iff $M_{z^k}A = AM_{z^{2k}}\ \ \forall\ k \in \mathbb{Z}^n$. Let $\varphi(z) = \sum_{t\in S} \varphi_t(z)$ where $\varphi_t(z) := \bar{z}^t\,(Ae_t)\,(z^2)\ \forall\, t \in S$.
Claim: $\varphi \in L^\infty(\mathbb{T}^n,\beta)$.
Let $h \in L^2(\mathbb{T}^n,\beta)$ and $\xi(z) := h(z^2)$. Then by Lemma 4.1, $\xi \in L^2(\mathbb{T}^n,\beta)$ and $\|\xi\| \le \|h\|$. For $t \in S$, we have

$$\begin{aligned} A(z^t\xi(z)) =& A(z^t \sum_{k\in\mathbb{Z}^n} \delta_k z^{2k}), \quad \text{where } \ h(z) = \sum_{k\in\mathbb{Z}^n} \delta_k z^k \\ =& \sum_{k\in\mathbb{Z}^n} \delta_k AM_{z^{2k}} z^t = \sum_{k\in\mathbb{Z}^n} \delta_k M_{z^k} A z^t \\ =& (\sum_{k\in\mathbb{Z}^n} \delta_k z^k) Ae_t(z) = h(z).Ae_t(z) = (h.Ae_t)(z). \end{aligned}$$

$$AM_z^t\xi = h.Ae_t \ \text{ for } \ \xi(z) = h(z^2)S \tag{4}$$

and

$$A(z^t h(z^2)) = (h.Ae_t)(z) \ \ \forall\ t \in S \tag{5}$$

Now, using Equation 4 we get $\|M_{Ae_t}\, h\| = \|Ae_t.h\| = \|AM_z^t.\xi\| \le \|A\|\|\xi\| \le \|A\|\|h\|$.
Therefore M_{Ae_t} is bounded which implies that $Ae_t \in L^\infty(\mathbb{T}^n,\beta)\ \ \forall\ t \in S$.
Thus, $\varphi_t \in L^\infty(\mathbb{T}^n,\beta)\ \ \forall\ t \in S \implies \varphi \in L^\infty(\mathbb{T}^n,\beta)$, and claim is established.

Let $f \in L^2(\mathbb{T}^n, \beta)$. So by Theorem 3.14, $f(z) = \sum_{t \in S} z^t f_t(z^2)$.

$$\begin{aligned}
\text{Therefore, } A_\varphi f(z) = WM_\varphi f(z) &= W(\varphi(z) f(z)) \\
&= \sum_{t \in S} z^t \left(W \overline{z}^t \varphi(z)\right)\left(W \overline{z}^t f(z)\right) \text{ by Theorem 3.15} \\
&= \sum_{t \in S} z^t \left(W \sum_k z^{-(t+k)} (Ae_k)(z^2)\right)\left(W \sum_k z^{-t+k} f_k(z^2)\right) \\
&= \sum_{t \in S} z^t \left(\overline{z}^t (Ae_t)(z)\right)(f_t(z)) \\
&= \sum_{t \in S} ((Ae_t) \cdot f_t)(z) \ \ (\text{since } |z| = 1) \\
&= \sum_{t \in S} A\left(z^t f_t(z^2)\right), \ \ \text{by Equation 5} \\
&= Af(z)
\end{aligned}$$

Thus, $A_\varphi f = Af \ \ \forall \ f \in L^2(\mathbb{T}^n, \beta)$, which implies $A = A_\varphi$. □

Corollary 4.3. *$M_{z_i} W = W M_{z_i^2}$ $1 \le i \le n$ and so W is a slant Weighted Toeplitz operator with $W = A_\varphi$ where $\varphi(z) = 1$.*

Corollary 4.4. *For $\varphi, \psi \in L^\infty(\mathbb{T}^n, \beta)$, the following must hold:*

1. *$A_\varphi + A_\psi$ is a slant weighted Toeplitz operator and $A_\varphi + A_\psi = A_{\varphi+\psi}$.*
2. *$M_\varphi A_\psi$ is a slant weighted Toeplitz operator and $M_{\varphi(z)} A_{\psi(z)} = A_{\varphi(z^2)\psi(z)}$ for all $z \in \mathbb{T}^n$.*
3. *$M_\varphi A_\psi = A_\psi M_\varphi$ if and only if $\varphi(z^2)\psi(z) = \varphi(z)\psi(z)$ for all $z \in \mathbb{T}^n$.*

Proof. Since, A_φ, A_ψ are slant weighted Toeplitz operators, so by Theorem 3.10 we have $M_{z_i} A_\varphi = A_\varphi M_{z_i^2}$ and $M_{z_i} A_\psi = A_\psi M_{z_i^2}$ $\forall \, 1 \le i \le n$. From here the result follows immediately by applying Theorem 4.2. □

Corollary 4.5. *For $\varphi, \psi \in L^\infty(\mathbb{T}^n, \beta)$, $A_\varphi A_\psi$ is a slant weighted Toeplitz operator if and only if $A_\varphi A_\psi = 0$.*

Proof. Using Corollary 4.4(2), we get $A_\varphi A_\psi = W A_{\varphi(z^2)\psi(z)}$. Also, by Theorem 3.16, $W A_{\varphi(z^2)\psi(z)}$ is a slant weighted Toeplitz operator if and only if $\varphi(z^2)\psi(z) = 0 \ \forall \, z \in \mathbb{T}^n$. Thus, $A_\varphi A_\psi$ is a slant weighted Toeplitz operator if and only if $A_\varphi A_\psi = 0$. □

Theorem 4.6. *Let $\{\frac{\beta_{2k}}{\beta_k}\}_k$ be bounded. A bounded linear operator A on $L^2(\mathbb{T}^n, \beta)$ is a slant weighted Toeplitz operator iff $\langle A f_{k+2\epsilon_j}, f_{t+\epsilon_j} \rangle = \frac{\beta_{t;j}}{\beta_{k+\epsilon_j;j}\beta_{k;j}} \langle A f_k, f_t \rangle \ \forall \ t, k \in \mathbb{Z}^n, \ 1 \le j \le n$*

Proof. By Theorems 3.10 and 4.2 we have
A is a slant weighted Toeplitz operator
iff $M_{z_j} A = A M_{z_j^2}$ $\forall \, 1 \le j \le n$
iff $\langle M_{z_j} A f_k, f_t \rangle = \langle A M_{z_j^2} f_k, f_t \rangle$, $\forall \, k, t \in \mathbb{Z}^n$, $1 \le j \le n$
iff $\langle A f_k, \beta_{t-\epsilon_j;j} f_{t-\epsilon_j} \rangle = \langle A M_{z_j} (\beta_{k;j} f_{k+\epsilon_j}), f_t \rangle$ by Theorem 2.2
iff $\beta_{t-\epsilon_j;j} \langle A f_k, f_{t-\epsilon_j} \rangle = \beta_{k;j} \langle A(\beta_{k+\epsilon_j;j} f_{k+2\epsilon_j}), f_t \rangle$, $\forall \, k, t \in \mathbb{Z}^n$, $1 \le j \le n$.
Replacing $t - \epsilon_j$ with t in the above relation, we get
$\beta_{t;j} \langle A f_k, f_t \rangle = \beta_{k+\epsilon_j;j} \beta_{k;j} \langle A f_{k+2\epsilon_j}, f_{t+\epsilon_j} \rangle$, $\forall \, k, t \in \mathbb{Z}^n$, $1 \le j \le n$
iff $\langle A f_{k+2\epsilon_j}, f_{t+\epsilon_j} \rangle = \frac{\beta_{t;j}}{\beta_{k+\epsilon_j;j}\beta_{k;j}} \langle A f_k, f_t \rangle$, $\forall \, k, t \in \mathbb{Z}^n$, $1 \le j \le n$
□

5. The hyponormal slant weighted Toeplitz operator A_φ

Definition 5.1. *Let $f \in L^2(\mathbb{T}^n, \beta)$ with $f(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$. Also let $S_f := \{k \in \mathbb{Z}^n : a_k \neq 0\}$, and for $i = 1, 2, \cdots, n$, define $m_i := \inf\{k_i : k = (k_1, \cdots, k_n) \in S_f\}$ and $M_i := \sup\{k_i : k = (k_1, \cdots, k_n) \in S_f\}$.*
If for each i both m_i and M_i exist finitely, then f is said to be a trigonometric polynomial in z.

Definition 5.2. *Let $f \in L^2(\mathbb{T}^n, \beta)$ be a trigonometric polynomial with $f(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$ and $S_f := \{k \in \mathbb{Z}^n : \; a_k \neq 0\}$. Let $\mathfrak{I}_f := \{(p, t) : \; p, t \in S_f, \; p \neq t\}$. For $(p, t) \in \mathfrak{I}_f$ let $u_0 := t$ and for $j \in \mathbb{N}$, let $u_j := \frac{p + u_{j-1}}{2}$. We define order of (p, t), denoted as $o(p, t)$, to be the non-negative integer η such that $p + u_\eta$ is odd and $p + u_j$ is even $\forall\; 0 \leq j < \eta$. Moreover, we define $[p : t] = \{u_j : 0 \leq j \leq o(p, t)\}$. So for $u_j \in [p : t]$ with $1 \leq j \leq o(p, t)$, if $u_j = (u_1^{(j)}, \ldots, u_n^{(j)})$, then $u_i^{(j)} = \frac{p_i + u_i^{(j-1)}}{2} = \frac{\sum_{\tau=0}^{j-1} 2^\tau p_i + t_i}{2^j}$ $\forall\, i = 1, \ldots, n$.*

Remark 5.3. *For a trigonometric polynomial $f \in L^2(\mathbb{T}^n, \beta)$ with $\mathfrak{I}_f \neq \Phi$, if $(p, t) \in \mathfrak{I}_f$ and $0 < o(p, t) = \eta$, then there may exist $0 < j \leq \eta$ such that $u_j \notin S_f$. Thus for $(p, t) \in \mathfrak{I}_f$ it is not necessary that $[p : t] \subset S_f$.*

In view of the above remark we propose the following definition.

Definition 5.4. *Let $f \in L^2(\mathbb{T}^n, \beta)$ be a trigonometric polynomial with $f(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$ and $S_f := \{k \in \mathbb{Z}^n : \; a_k \neq 0\}$. Then*
$$\tilde{\mathfrak{I}}_f := \begin{cases} \cup_{(p,t) \in \mathfrak{I}_f} [p : t] \cup S_f, & \text{if } \mathfrak{I}_f \neq \Phi; \\ S_f, & \text{otherwise.} \end{cases}$$

Remark 5.5. *For $f \in L^2(\mathbb{T}^n, \beta)$ and $\mathfrak{I}_f \neq \Phi$, we have $S_f \subseteq \tilde{\mathfrak{I}}_f$ because for $p, t \in S_f$ with $p \neq t$, we get $t \in [p : t]$ and $p \in [t : p]$.*

For easy reference we list below a few notations to be used in subsequent results:
For non zero $f \in L^2(\mathbb{T}^n, \beta)$ with $f(z) = \sum_{k \in \mathbb{Z}^n} a_k z^k$ we have:

1. $S_f := \{k \in \mathbb{Z}^n : a_k \neq 0\}$.
2. $\mathfrak{I}_f := \{(p, t) : \; p, t \in S_f, \; p \neq t\}$. If $f(z) = a_p z^p$ with $a_p \neq 0$, then $\mathfrak{I}_f = \Phi$ and $S_f = \{p\}$.
3. $\tilde{\mathfrak{I}}_f = \begin{cases} \cup_{(p,t) \in \mathfrak{I}_f} [p : t] \cup S_f, & \text{if } \mathfrak{I}_f \neq \Phi; \\ S_f, & \text{otherwise.} \end{cases}$
4. $m_f := \inf\{|k| : \; k \in S_f\}$ and $M_f := \sup\{|k| : \; k \in S_f\}$. Recall that for $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, $|k| := k_1 + \cdots + k_n$.
5. For $p \in S_f$, $J_p := \{k \in \tilde{\mathfrak{I}}_f : \; |p| \leq |k| \leq M_f\}$ and $J^p := \{k \in \tilde{\mathfrak{I}}_f : \; m_f \leq |k| < |p|\}$.

Theorem 5.6. *Let $f \in L^2(\mathbb{T}^n, \beta)$ be a trigonometric polynomial with $f(z) = \sum_{t \in \mathbb{Z}^n} a_t z^t$. Then for each $k = (k_1, k_2, \cdots, k_n) \in \tilde{\mathfrak{I}}_f$ we have $m_f \leq |k| \leq M_f$, and $m_i \leq k_i \leq M_i$ $\forall\; 1 \leq i \leq n$.*

Proof. Let $k = (k_1, k_2, \cdots, k_n) \in \tilde{\mathfrak{I}}_f$. If $k \in S_f$, then $m_i \leq k_i \leq M_i$ $\forall\; i$ and $m_f \leq |k| \leq M_f$. If $k \notin S_f$, then there exists $(p, t) \in \mathfrak{I}_f$ such that $k \in [p : t]$.
Let $p = (p_1, \cdots, p_n)$ and $t = (t_1, \cdots, t_n)$. Then $[k, t] = \{u_j : \; 0 \leq j \leq \eta\}$ where $\eta = o(p, t)$, $u_0 = t$ and $u_j = \frac{p + u_{j-1}}{2}$ for $1 \leq j \leq \eta$. If $u_j = (u_1^{(j)}, \ldots, u_n^{(j)})$ then for $1 \leq j \leq \eta$ we have $u_i^{(j)} = \frac{p_i + u_i^{(j-1)}}{2}$ $\forall\, 1 \leq i \leq n$.

Claim: For $0 \leq j \leq \eta$, $m_f \leq |u_j| \leq M_f$ and $m_i \leq u_i^{(j)} \leq M_i$ $\forall\, 1 \leq i \leq n$.
As $u_0 = t \in S_f$ so the claim holds trivially for $j = 0$.
Again, $u_1 = \frac{p+t}{2}$ implies $|u_1| = \frac{|p|+|t|}{2}$, and as $m_f \leq |p|, |t| \leq M_f$, so $m_f \leq |u_1| \leq M_f$. Also, $m_i \leq p_i, t_i \leq M_i$ $\forall\; i$ implies $m_i \leq u_i^{(1)} = \frac{p_i + t_i}{2} \leq M_i$. Thus the claim holds for $j = 1$.
Applying induction to $j \geq 2$ we see that $m_f \leq |p|$, $|u_{j-1}| \leq M_f$ implies $m_f \leq |u_j| \leq M_f$, and $m_i \leq p_i, u_i^{(j-1)} \leq M_i\, \forall\, i$ implies $m_i \leq u_i^{(j)} \leq M_i\, \forall\, i$.
Thus the claim is established.
Now $k \in [p : t]$ implies there exists $0 \leq j \leq r$ such that $k = u_j$ which in turn implies that $m_f \leq |k| \leq M_f$ and $m_i \leq k_i \leq M_i$ $\forall\; 1 \leq i \leq n$. $\square$

Corollary 5.7. *For trigonometric polynomial $f \in L^2(\mathbb{T}^n,\beta)$, $S_f, \mathfrak{I}_f$ and $\tilde{\mathfrak{I}}_f$ are finite sets.*

Proof. Let $\mathcal{R}_i = \{k_i : k = (k_1, \cdots, k_n) \in \tilde{\mathfrak{I}}_f\}$. Then $\mathcal{R}_i \subset \mathbb{Z}$ and $m_i \leq \lambda \leq M_i \ \forall \ \lambda \in \mathcal{R}_i$. Thus, $\mathcal{R}_i$ is a finite set. This is true for each $i = 1, 2, \cdots, n$.
$\therefore \tilde{\mathfrak{I}}_f = \mathcal{R}_1 \times \mathcal{R}_2 \times \cdots \times \mathcal{R}_n$ is a finite set. As $S_f \subset \tilde{\mathfrak{I}}_f$, so S_f is also finite. Also, $\mathfrak{I}_f \subset S_f \times S_f$ and so $\mathfrak{I}_f$ is finite. □

A bounded linear operator T on a Hilbert space H is said to be hyponormal iff $T^*T - TT^* \geq 0$. So for a hyponormal operator T we must necessarily have $\langle (T^*T - TT^*)f, f\rangle \geq 0 \ \forall \ f \in H$. Here we will show that for a trigonometric polynomial $\varphi \in L^\infty(\mathbb{T}^n,\beta)$, A_φ is hyponormal iff $\varphi = 0$. For this we will consider the orthonormal basis $\{f_k\}_{k\in\mathbb{Z}^n}$ of $L^2(\mathbb{T}^n,\beta)$ and for each $k \in \mathbb{Z}^n$, define $d_k = \langle \left(A^*_\varphi A_\varphi - A_\varphi A^*_\varphi\right) f_k, f_k\rangle$. We will show that for $\varphi \neq 0$, there must exists $k \in \mathbb{Z}^n$ such that $d_k < 0$, implying that A_φ is not hyponormal.

Lemma 5.8. *Let $\varphi \in L^\infty(\mathbb{T}^n,\beta)$ with $\varphi(z) = \sum_{k\in\mathbb{Z}^n} a_k z^k$ and for $t \in \mathbb{Z}^n$, let $d_t = \langle \left(A^*_\varphi A_\varphi - A_\varphi A^*_\varphi\right) f_t, f_t\rangle$. Then $d_t = \sum_{p\in\mathbb{Z}^n} C_p^{(t)} |a_p|^2$ where*

$$C_p^{(t)} = \begin{cases} \frac{\beta^2_{\frac{t+p}{2}}}{\beta_t^2} - \frac{\beta_t^2}{\beta_{2t-p}^2}, & \text{if } t+p \text{ is even;} \\ -\frac{\beta_t^2}{\beta_{2t-p}^2}, & \text{if } t+p \text{ is odd.} \end{cases}$$

Proof. We have
$$A_\varphi f_t(z) = WM_\varphi f_t(z) = W\varphi(z)\frac{z^t}{\beta_t} = W\left(\sum_k a_k \frac{z^{k+t}}{\beta_t}\right) = W\left(\sum_k a_{k-t}\frac{z^k}{\beta_t}\right) = \sum_k a_{2k-t}\frac{\beta_k}{\beta_t} f_k$$
and
$$\langle A_\varphi f_s, f_t\rangle = \langle \sum_k a_{2k-s}\frac{\beta_k}{\beta_s} f_k, f_t\rangle = a_{2k-s}\frac{\beta_t}{\beta_s} = \langle f_s, \sum_k \bar{a}_{2t-k}\frac{\beta_t}{\beta_k} f_k\rangle.$$
So $A^*_\varphi f_t = \sum_k \bar{a}_{2t-k}\frac{\beta_t}{\beta_k} f_k$.

$$\text{Thus, } d_t = \|A_\varphi f_t\|^2 - \|A^*_\varphi f_t\|^2 = \sum_k |a_{2k-t}|^2 \frac{\beta_k^2}{\beta_t^2} - \sum_k |a_{2t-k}|^2 \frac{\beta_t^2}{\beta_k^2} = \sum_{p\in\mathbb{Z}^n} C_p^{(t)} |a_p|^2,$$

$$\text{where } C_p^{(t)} = \begin{cases} \frac{\beta^2_{\frac{t+p}{2}}}{\beta_t^2} - \frac{\beta_t^2}{\beta_{2t-p}^2}, & \text{if } t+p \text{ is even;} \\ -\frac{\beta_t^2}{\beta_{2t-p}^2}, & \text{if } t+p \text{ is odd.} \end{cases}$$ □

Remark 5.9. *From the above result we observe the following:*

1. *If $p = t$ then $C_p^{(t)} = 0$*
2. *If for $t \in \mathbb{Z}^n$ we have $p \in \mathbb{Z}^n$ such that $p + t$ is even and $\beta_{\frac{t+p}{2}}\beta_{2t-p} = \beta_t^2$, then $C_p^{(t)} = 0$.*

Lemma 5.10. *Let $f \in L^2(\mathbb{T}^n,\beta)$ be a trigonometric polynomial. Then for $p \in S_f$, $\sum_{t\in J_p} C_p^{(t)} \leq 0$, where equality holds iff $S_f = \{p\}$*

Proof. By Corollary 5.7, $\tilde{\mathfrak{I}}_f$ is a finite set, and so J_p is also a finite set. If $J_p = \{p\}$ then by Remark 5.9(1), $\sum_{t\in J_p} C_p^{(t)} = C_p^{(p)} = 0$. Suppose there exists $k \in J_p$, $k \neq p$.
Let $u_0 = k$ and for $j \in \mathbb{N}$, let $u_j = \frac{p+u_{j-1}}{2}$. Let order of (p, k) be the smallest non-negative integer η such that

$p + u_\eta$ is odd and let $[p : k] := \{u_j : \ 0 \le j \le \eta\}$.
Recall that $J_p = \{k \in \mathfrak{I}_f : |p| \le |k| \le M_f\}$. As $k \in J_p$, so $|p| \le |k| \le M_f$. Hence, if $k + p$ even, then $\frac{k+p}{2} \in J_p$ because $|p| \le |\frac{p+k}{2}| = \frac{|p|+|k|}{2} \le M_f$. By a similar argument each $u_j \in J_p$, and so $[p : k] \subset J_p$.
Claim: $\sum_{t\in[p:k]} C_p^{(t)} < 0$.
If $\eta = 0$ then $[p : k] = \{k\}$ and $\sum_{t\in[p:k]} C_p^{(t)} = C_p^{(k)} = -\frac{\beta_k^2}{\beta_{2k-p}^2} < 0$.
If $\eta > 0$ then

$$C_p^{(u_0)} = \frac{\beta_{u_1}^2}{\beta_{u_0}} - \frac{\beta_k^2}{\beta_{2k-p}^2}$$
$$C_p^{(u_j)} = \frac{\beta_{u_{j+1}}^2}{\beta_{u_j}} - \frac{\beta_{u_j}^2}{\beta_{u_{j-1}}^2} \text{ for } 0 < j < r$$
$$\text{and } C_p^{(u_\eta)} = -\frac{\beta_{u_\eta}^2}{\beta_{u_{\eta-1}}^2}$$
$$\sum_{t\in[p:k]} C_p^{(t)} = \sum_{j=0}^{\eta} C_p^{(u_j)} = -\frac{\beta_k^2}{\beta_{2k-p}^2},$$

and the claim is established.
Since, J_p is a finite set, so we can choose a finite number of distinct terms $k(1), \cdots, k(\tau)$ in J_p, such that

1. $k(j) \ne p \ \forall \ 1 \le j \le \tau$.
2. $J_p = \cup_{j=1}^{\tau}[p : k(j)]$
3. For $i \ne j$, $k(i) \notin [p : k(j)]$

Thus, $\sum_{t\in J_p} C_p^{(t)} = \sum_{j=1}^{\tau} \sum_{t\in[p:k(j)]} C_p^{(t)} < 0$. □

Lemma 5.11. *Let $f \in L^2(\mathbb{T}^n, \beta)$ be a trigonometric polynomial. If $J^p \ne \emptyset$ for $p \in S_f$, then $\sum_{t\in J^p} C_p^{(t)} < 0$.*

Proof. By Corollary 5.7, $\mathfrak{I}_f$ is a finite set, and so J^p is also finite. As in Lemma 5.10, we can show that for each $k \in J^p$, $[p : k] \subset J^p$ and $\sum_{t\in[p:k]} C_p^{(t)} < 0$. Also as J^p is a finite set so we can choose distinct elements $k(1), \cdots, k(\tau)$ in J^p such that $J^p = \cup_{j=1}^{\tau}[p : k(j)]$, and for $i \ne j$, $k(i) \notin [p : k(j)]$. Thus, $\sum_{t\in J^p} C_p^{(t)} = \sum_{j=1}^{\tau} \sum_{t\in[p:k(j)]} C_p^{(t)} < 0$. □

Lemma 5.12. *If $f \in L^2(\mathbb{T}^n, \beta)$ is a trigonometric polynomial, then there exists $p \in S_f$ such that $|p| = m_f$.*

Proof. By Corollary 5.7, S_f is a finite set and so there exists $p \in S_f$ such that $|p| = \inf\{|k| : k \in S_f\} = m_f$. □

Theorem 5.13. *Let $\varphi \in L^2(\mathbb{T}^n, \beta)$ be a non-zero trigonometric polynomial and $\mathfrak{I}_\varphi = \emptyset$. Then there exists $t \in \mathbb{Z}^n$ such that $d_t = \langle \left(A_\varphi^* A_\varphi - A_\varphi A_\varphi^*\right) f_t, f_t\rangle < 0$.*

Proof. As $\mathfrak{I}_\varphi = \emptyset$, so $S_f = \{p\}$ and $\varphi(z) = a_p z^p$, $a_p \ne 0$. Choose $t \in \mathbb{Z}^n$ such that $p + t$ is odd. Then $d_t = \sum_{q\in\mathbb{Z}^n} C_q^{(t)}|a_q|^2 = C_p^{(t)}|a_p|^2 = -\frac{\beta_t^2}{\beta_{2t-p}^2} < 0$. □

Theorem 5.14. *Let $\varphi \in L^2(\mathbb{T}^n, \beta)$ be a trigonometric polynomial and $\mathfrak{I}_\varphi \ne \emptyset$. If $p \in S_\varphi$ such that $|p| = m_\varphi$, then $\sum_{t\in J_p} d_t < 0$, where $d_t = \langle \left(A_\varphi^* A_\varphi - A_\varphi A_\varphi^*\right) f_t, f_t\rangle < 0$.*

Proof. By Lemma 5.8, there exists $p \in S_\varphi$ such that $|p| = m_\varphi$. Further, $\mathfrak{I}_\varphi \ne \emptyset$ implies that there exists $t \in S_\varphi$ such that $t \ne p$.As $|p| = m_\varphi \le |t| \le M_\varphi$, so $t \in J_p$. Thus J_p can not be singleton, and by Lemma 5.10, we have $\sum_{t\in J_p} C_p^{(t)} < 0$.

Let $k \in \tilde{\mathfrak{I}}_\varphi$. Then by Theorem 5.6, $m_\varphi \leq |k| \leq M_\varphi$ which implies that $k \in J_p$ because $|p| = m_\varphi$. Thus, $J_p = \tilde{\mathfrak{I}}_\varphi$.

$$\begin{aligned}\text{Therefore } \sum_{t\in J_p} d_t &= \sum_{t\in J_p}\Big(\sum_{q\in\mathbb{Z}^n C_q^{(t)}} |a_q|^2\Big)\\ &= \sum_{q\in S_\varphi}\Big(\sum_{t\in J_p} C_q^{(t)} t\Big)|a_q|^2 \ \ (\text{since } \ a_q = 0 \ \text{ for } \ q \notin S_\varphi)\\ &= \sum_{t\in\mathcal{S}_p} C_p^{(t)}|a_p|^2 + \sum_{q\in S_\varphi, q\neq p}\Big(\sum_{t\in\tilde{\mathfrak{I}}_\varphi} C_q^{(t)}\Big)|a_q|^2\end{aligned}$$

Claim: $\sum_{t\in\tilde{\mathfrak{I}}_\varphi} C_q^{(t)} \leq 0$ for $p \in S_\varphi, q \neq p$. As $|p| = m_\varphi$ so $|q| \geq |p|$.

1. If $|q| = |p|$ then $J_q = \{k \in \tilde{\mathfrak{I}}_\varphi : |q| \leq |k| \leq M_\varphi\} = J_p = \tilde{\mathfrak{I}}_\varphi$, and so by Lemma 5.10, $\sum_{t\in\tilde{\mathfrak{I}}_\varphi} C_q^{(t)} = \sum_{t\in J_q} C_q^{(t)} \leq 0$.
2. If $|q| > |p|$ then $\tilde{\mathfrak{I}}_\varphi = J_q \cup J^q$ where $J_q \cap J^q = \emptyset$ and $p \in J^q, q \in J_q$.
   Therefore $\sum_{t\in\tilde{\mathfrak{I}}_\varphi} C_q^{(t)} = \sum_{t\in J_q} C_q^{(t)} + \sum_{t\in J^q} C_q^{(t)} < 0$, by Lemma 5.10 and 5.12.

Thus, $\sum_{t\in\tilde{\mathfrak{I}}_\varphi} C_q^{(t)} \leq 0 \ \forall \ q \in S_\varphi, \ q \neq p$. Also $\sum_{t\in J_p} C_p^{(t)} < 0$ by Lemma 5.10.
Hence $\sum_{t\in J_p} d_t < 0$. □

Theorem 5.15. *Let $\varphi \in L^2(\mathbb{T}^n, \beta)$ be a trigonometric polynomial. If $\varphi \not\equiv 0$, then A_φ can not be hyponormal.*

Proof. If $\mathfrak{I}_\varphi = \emptyset$, then by Theorem 5.13 there exists $t \in \mathbb{Z}^n$ such that $d_t < 0$ and so A_φ can not be hyponormal. If $\mathfrak{I}_\varphi \neq \emptyset$, then by Theorem 5.14, $\sum_{t\in J_p} d_t < 0$ where $p \in S_\varphi$ such that $|p| = m_\varphi\}$.
Thus there must exist $t \in J_p$ such that $d_t < 0$, and so A_φ can not be hyponormal. □

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