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On Sufficiency and Duality for Multiobjective Programming Problems Using Convexificators

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Abstract. In this paper, we consider a multiobjective programming problem with inequality and set constraints. We derive sufficient conditions for the optimality of a feasible point under generalized invexity assumptions in terms of convexificators. We give an example to illustrate that the concept of invexity in terms of convexificators is weaker than invexity in terms of other subdifferentials. We formulate Wolfe and Mond-Weir type duals for the nonsmooth multiobjective programming problem with inequality and set constraints in terms of convexificators. We establish weak, strong, converse, restricted converse and strict converse duality results under the assumptions of invexity and strict invexity using convexificators between the primal and the Wolfe dual. We derive the respective results between the primal and the Mond-Weir dual under the assumptions of generalized pseudoinvexity, strict pseudoinvexity and quasiinvexity in terms of convexificators. We also derive the relationship between a weak vector saddle-point and a weakly efficient solution of the multiobjective programming problem.

1. Introduction

The concepts of Wolfe and Mond-Weir duality given by Wolfe [33] and Mond and Weir [28] respectively are important tools for the search of efficient or weakly efficient solution of a multiobjective optimization problem (see, e.g., [1, 18, 25, 26, 30]). On the other hand, the concept of convexificators was introduced by Demyanov [3] and further studied by Demyanov and Jeyakumar [4] as a convex and compact set to generalize the notion of upper convex and lower concave approximations. By Jeyakumar and Luc [10], a closed but not necessarily convex or bounded convexificator was introduced to allow its application for continuous functions. The idea of convexificators was extended to vector-valued maps by Jeyakumar and Luc [9] and further studies were made by [11–13]. It was shown by Jeyakumar and Luc [10], that for a locally Lipschitz real-valued function many known subdifferentials, like the Clarke subdifferential by Clarke [2], the Michel-Penot subdifferential by Michel and Penot [17], the Mordukhowich subdifferential by

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Mordukhowich and Shao [29] and the Trieman subdifferential by Trieman [31], are convexificators. These known subdifferentials may strictly contain the convex hull of a convexificator of a locally Lipschitz function (see e.g., [32]) and hence the optimality conditions, the calculus rules and the characterizations of the generalized convex functions in terms of convexificators provide sharper results (see, e.g., [5–7, 15, 16]). We see the used of vector variational inequalities in terms of convexificators for vector optimization problems given by [14].

In this paper, we extend the concept introduced by Golestani and Nobakhtian [7] to obtain sufficient optimality and duality results for a multiobjective optimization problem. The outline of this paper is as follows: in Section 2, we give some preliminary definitions and results which will be used in the sequel. In Section 3, we derive sufficient optimality conditions and we illustrate the results using an example for the application of convexificator. In Section 4, we formulate Wolfe type dual to the primal multiobjective programming problem in terms of convexificators and establish weak, strong, converse, restricted converse and strict converse duality results under the assumptions of invex and strict invex functions using convexificators. In Section 5, we present Mond-Weir type dual to the primal multiobjective programming problem in terms of convexificators and derive weak, strong, converse, restricted converse and strict converse duality results under the assumptions of generalized pseudoinvex, strict pseudoinvex and quasiinvex functions using convexificators. In Section 6, we derive saddle-point results for the invex and pseudoinvex assumptions. In Section 7, we conclude the results of this paper and discuss some future research possibilities.

2. Preliminaries

Throughout this paper, $\mathbb{R}^k$ is the usual $k-$dimensional Euclidean space. Let $x := (x_1, \dots, x_k)$ and $y := (y_1, \dots, y_k)$ be two vectors in $\mathbb{R}^k$. Then,

$$x \leqq y \Leftrightarrow x_i \leqq y_i, \forall i = 1, \dots, k,$$

$$x \leq y \Leftrightarrow x_i \leqq y_i, x \neq y, \forall i = 1, \dots, k,$$

$$x < y \Leftrightarrow x_i < y_i, \forall i = 1, \dots, k.$$

Let S be a nonempty subset of $\mathbb{R}^k$. The convex hull of S, the closure of S and the convex cone generated by S containing the origin of $\mathbb{R}^k$ are denoted by coS, clS and $coneS$, respectively. The negative polar cone S^- and the strictly negative polar cone S^s are defined as follows:

$$S^- := \{v \in \mathbb{R}^k : \langle x, v\rangle \leq 0, \forall x \in S\},$$

$$S^s := \{v \in \mathbb{R}^k : \langle x, v\rangle < 0, \forall x \in S\}.$$

The contingent cone $T(S, x)$ and the normal cone $N(S, x)$ at $x \in clS$ are respectively given by

$$T(S, x) := \left\{v \in \mathbb{R}^k : \exists t_n \downarrow 0 \ \ and \ \ v_n \to v \ \text{ such that } \ x + t_n v_n \in S, \forall n\right\},$$

$$N(S, x) := T(S, x)^- = \left\{\xi \in \mathbb{R}^k : \langle \xi, v\rangle \leq 0, \forall v \in T(S, x)\right\}.$$

We recall the following definitions by Jeyakumar and Luc [10] which will be used in the sequel.

Definition 2.1. *Let $h : \mathbb{R}^k \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, $x \in \mathbb{R}^k$ and let $h(x)$ be finite. The lower and upper Dini derivatives of h at x in the direction of $v \in \mathbb{R}^k$ are defined, respectively, as follows:*

$$h^-(x, v) := \liminf_{t \downarrow 0} \frac{h(x + tv) - h(x)}{t}$$

and

$$h^+(x, v) := \limsup_{t \downarrow 0} \frac{h(x + tv) - h(x)}{t}.$$

Definition 2.2. *Let $h : \mathbb{R}^k \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, $x \in \mathbb{R}^k$ and let $h(x)$ be finite.*

(a) *The function h is said to have an upper convexificator $\partial^* h(x) \subset \mathbb{R}^k$ at $x \in \mathbb{R}^k$, iff $\partial^* h(x)$ is closed and for each $v \in \mathbb{R}^k$, one has*
$$h^-(x, v) \leq \sup_{\xi \in \partial^* h(x)} \langle \xi, v \rangle.$$

(b) *The function h is said to have a lower convexificator $\partial_* h(x) \subset \mathbb{R}^k$ at $x \in \mathbb{R}^k$, iff $\partial_* h(x)$ is closed and for each $v \in \mathbb{R}^k$, one has*
$$h^+(x, v) \geq \inf_{\xi \in \partial_* h(x)} \langle \xi, v \rangle.$$

(c) *The function h is said to have an upper regular convexificator $\partial^* h(x) \subset \mathbb{R}^k$ at $x \in \mathbb{R}^k$, iff $\partial^* h(x)$ is closed and for each $v \in \mathbb{R}^k$, one has*
$$h^+(x, v) = \sup_{\xi \in \partial^* h(x)} \langle \xi, v \rangle.$$

(d) *The function h is said to have a lower regular convexificator $\partial_* h(x) \subset \mathbb{R}^k$ at $x \in \mathbb{R}^k$, iff $\partial_* h(x)$ is closed and for each $v \in \mathbb{R}^k$, one has*
$$h^-(x, v) = \inf_{\xi \in \partial_* h(x)} \langle \xi, v \rangle.$$

The following definitions are along the lines by Dutta and Chandra [6] and will be used in the sequel.

Definition 2.3. *Let $h : \mathbb{R}^k \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, $x \in \mathbb{R}^k$ and let $h(x)$ be finite.*

(a) *The function h is said to have an upper semi-regular convexificator $\partial^* h(x) \subset \mathbb{R}^k$ at $x \in \mathbb{R}^k$, iff $\partial^* h(x)$ is closed and for each $v \in \mathbb{R}^k$, one has*
$$h^+(x, v) \leq \sup_{\xi \in \partial^* h(x)} \langle \xi, v \rangle.$$

(b) *The function h is said to have a lower semi-regular convexificator $\partial_* h(x) \subset \mathbb{R}^k$ at $x \in \mathbb{R}^k$, iff $\partial_* h(x)$ is closed and for each $v \in \mathbb{R}^k$, one has*
$$h^-(x, v) \geq \inf_{\xi \in \partial_* h(x)} \langle \xi, v \rangle.$$

Remark 2.4. *An upper (respectively, lower) regular convexificator of h at a point is an upper (respectively, lower) semi-regular convexificator of h at the point and every upper (respectively, lower) semi-regular convexificator is an upper (respectively, lower) convexificator.*

Mohan and Neogy [27] introduced the concept of invex sets as follows:

Definition 2.5. *A subset S of $\mathbb{R}^k$ is said to be invex with respect to a vector-valued function $\eta : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$, iff for any $x, y \in S$ and $\lambda \in [0, 1]$, one has*
$$x + \lambda \eta(y, x) \in S.$$

The concept of invexity which was introduced by Hanson [8] may be extended in terms of convexificators as follows:

Definition 2.6. *Let $h : \mathbb{R}^k \to \bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function such that h has an upper semi-regular convexificator at $\bar{x} \in K$, where K is a nonempty subset of $\mathbb{R}^k$, and let $\eta : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ be a vector-valued function:*

(a) *The function h is said to be ∂^*-invex with respect to η at $\bar{x}$ over K, iff for every $x \in K$, one has*
$$h(x) - h(\bar{x}) \geq \langle \xi, \eta(x, \bar{x}) \rangle, \forall \xi \in \partial^* h(\bar{x});$$

(b) *The function h is said to be ∂^*-pseudoinvex with respect to η at $\bar{x}$ over K, iff for every $x \in K$ and $x \neq \bar{x}$, one has*
$$h(x) < h(\bar{x}) \Rightarrow \langle \xi, \eta(x, \bar{x}) \rangle < 0, \forall \xi \in \partial^* h(\bar{x});$$

(c) *The function h is said to be strict ∂^*-pseudoinvex with respect to η at $\bar{x}$ over K, iff for every $x \in K$ and $x \neq \bar{x}$, one has*

$$h(x) \leq h(\bar{x}) \Rightarrow \langle \xi, \eta(x, \bar{x}) \rangle < 0, \forall \xi \in \partial^* h(\bar{x});$$

(d) *The function h is said to be ∂^*-quasiinvex with respect to η at $\bar{x}$ over K, iff for every $x \in K$, one has*

$$h(x) \leq h(\bar{x}) \Rightarrow \langle \xi, \eta(x, \bar{x}) \rangle \leq 0, \forall \xi \in \partial^* h(\bar{x}).$$

We consider the following multiobjective programming problem:

$$(P) \;\; min f(x) := (f_1(x), \ldots, f_m(x))$$

$$s.t \;\; g(x) := (g_1(x), \ldots, g_n(x)) \leqq 0,$$

$$x \in Q,$$

where $f_i : \mathbb{R}^k \to \bar{\mathbb{R}}$ and $g_j : \mathbb{R}^k \to \bar{\mathbb{R}}$ are extended real valued functions for all $i \in I := \{1, \ldots, m\}$ and $j \in J := \{1, \ldots, n\}$ and Q is an invex subset of $\mathbb{R}^k$ with respect to some $\eta : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$.

The following notations will be used in the subsequent analysis.

$$J(\bar{x}) := \{l \in J : g_l(\bar{x}) = 0\};$$

$$F := \cup_{i=1}^m co\partial^* f_i(\bar{x});$$

$$F^i := \cup_{j \in I \setminus \{i\}} co\partial^* f_j(\bar{x});$$

$$G := \cup_{l \in J(\bar{x})} co\partial^* g_l(\bar{x});$$

$$S := \{x \in \mathbb{R}^k : g(x) \leqq 0, x \in Q\}.$$

Recall that a point $\bar{x} \in S$ is said to be a locally efficient solution for (P), iff there exists no $x \in S$ near $\bar{x}$ such that $f(x) \leq f(\bar{x})$. A point $\bar{x} \in S$ is said to be a globally efficient solution for (P), iff there exists no $x \in S$ such that $f(x) \leq f(\bar{x})$. A point $\bar{x} \in S$ is said to be a locally weakly efficient solution for (P), iff there exists no $x \in S$ near $\bar{x}$ such that $f(x) < f(\bar{x})$. A point $\bar{x} \in S$ is said to be a globally weakly efficient solution for (P), iff there exists no $x \in S$ such that $f(x) < f(\bar{x})$.

Consider the following constraint qualification given by Golestani and Nobakhtian [7].

Definition 2.7. *(CQ1) The generalized Mangasarian-Fromovitz constraint qualification is said to be satisfied at $\bar{x}$, iff*

$$(F^i)^s \cap G^s \cap T(Q; \bar{x}) \neq \Phi.$$

Golestani and Nobakhtian [7] derived the following strong Kuhn-Tucker type necessary conditions for (P) in terms of upper semi-regular convexificators.

Theorem 2.8. *Let $\bar{x} \in S$ be a locally weakly efficient solution for (P). Suppose that f_i and g_j are locally Lipschitz functions at $\bar{x}$, and admit bounded upper semi-regular convexificators $\partial^* f_i(\bar{x})$ and $\partial^* g_j(\bar{x})$ for all $i \in I$ and $j \in J$. If (CQ1) holds at $\bar{x}$, then there exists $(\lambda, \mu) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_+$ such that*

$$(i) \qquad 0 \in \sum_{i=1}^m \lambda_i co\partial^* f_i(\bar{x}) + \sum_{j=1}^n \mu_j co\partial^* g_j(\bar{x}) + N(Q, \bar{x}),$$

$$(ii) \qquad \mu_j g_j(\bar{x}) = 0, \forall j = 1, \ldots, n.$$

3. Sufficient Optimality Conditions

In this section, we derive sufficient optimality conditions for (P) under generalized invexity assumptions.

Theorem 3.1. *Let $\bar{x} \in S$ and suppose that f_i are ∂^*−pseudoinvex with respect to η at $\bar{x}$ over S, for all $i \in I$ and g_j are ∂^*−quasiinvex with respect to η at $\bar{x}$ over S, for all $j \in J(\bar{x})$. If there exists $\lambda \geq 0$ and $\mu \geqq 0$ such that*

$$0 \in \sum_{i=1}^{m} \lambda_i co\partial^* f_i(\bar{x}) + \sum_{j=1}^{n} \mu_j co\partial^* g_j(\bar{x}) + N(Q, \bar{x}), \quad and$$

$$\mu_j g_j(\bar{x}) = 0, \forall j = 1, \ldots, n,$$

then $\bar{x}$ is a globally weakly efficient solution for (P).

Proof. Suppose to the contrary that $\bar{x}$ is not the globally weakly efficient solution for (P). Then, there exists a feasible solution x_0 such that

$$f(x_0) < f(\bar{x}).$$

Since f_i is ∂^*−pseudoinvex with respect to η at $\bar{x}$ over S, for all $i \in I$, it follows that

$$\langle \xi_i, \eta(x_0, \bar{x}) \rangle < 0, \forall \xi_i \in \partial^* f_i(\bar{x}), \forall i \in I.$$

By the feasibility of x_0, one has

$$\mu_j g_j(x_0) \leq 0 = u_j g_j(\bar{x}), \forall j \in J(\bar{x}).$$

By the ∂^*−quasiinvexity of g_j with respect to η at $\bar{x}$ over S, for all $j \in J(\bar{x})$, it follows that

$$\langle \zeta_j, \eta(x_0, \bar{x}) \rangle \leq 0, \forall \zeta_j \in \partial^* g_j(\bar{x}), \forall j \in J(\bar{x}).$$

Therefore, for every $\xi_i \in co\partial^* f_i(\bar{x}), i \in I, \zeta_j \in co\partial^* g_j(\bar{x}), j \in J(\bar{x})$ and $v \in N(Q, \bar{x})$, one has

$$\begin{aligned} 0 &> \sum_{i=1}^{m} \lambda_i \langle \xi_i, \eta(x_0, \bar{x}) \rangle + \sum_{j=1}^{n} \mu_j \langle \zeta_j, \eta(x_0, \bar{x}) \rangle \\ &\geq \sum_{i=1}^{m} \lambda_i \langle \xi_i, \eta(x_0, \bar{x}) \rangle + \sum_{j=1}^{n} \mu_j \langle \zeta_j, \eta(x_0, \bar{x}) \rangle + \langle v, \eta(x_0, \bar{x}) \rangle. \end{aligned}$$

This is a contradiction and the proof is complete. □

Similarly, we give sufficient optimality results to a globally efficient solution for (P).

Theorem 3.2. *Let $\bar{x} \in S$ and suppose that f_i are strictly ∂^*−pseudoinvex with respect to η at $\bar{x}$ over S, for all $i \in I$ and g_j are ∂^*−quasiinvex with respect to η at $\bar{x}$ over S, for all $j \in J(\bar{x})$. If there exists $\lambda > 0$ and $\mu \geqq 0$ such that*

$$0 \in \sum_{i=1}^{m} \lambda_i co\partial^* f_i(\bar{x}) + \sum_{j=1}^{n} \mu_j co\partial^* g_j(\bar{x}) + N(Q, \bar{x}), \quad and$$

$$\mu_j g_j(\bar{x}) = 0, \forall j = 1, \ldots, n,$$

then $\bar{x}$ is a globally efficient solution for (P).

The following example illustrates that the concept of invexity in terms of convexificators is weaker than invexity in terms of other subdifferentials.

Example 3.3. *Consider the following nonsmooth function* $f(x_1,\ x_2) = -|x_1| + x_2^2$.
It is easy to see that $f(x)$ *is not convex at* $\bar{x} = (0,\ 0)$. *Now, we have to find out* $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ *s.t.*

$$f(x) - f(\bar{x}) \geq \langle \xi,\ \eta(x,\ \bar{x}) \rangle,\ \forall \xi \in \partial^* f(\bar{x}).$$

The Clarke subdifferential of f *at* $\bar{x}$ *is given by* $co\{(-1,\ 0),\ (1,\ 0)\} = \partial^C f(0,\ 0)$, *whereas the upper semi-regular convexificator of* f *at* $\bar{x}$ *is given by* $\{(-1,\ 0),\ (1,\ 0)\} = \partial^* f(0,\ 0)$. *Since,* $\partial^* f(0,\ 0) \subset \partial^C f(0,\ 0)$, *it is easy to check the invexity in terms of convexificator than in terms of Clarke subdifferential. Then,*

$$-|x_1| + x_2^2 \geq \langle (-1,\ 0),\ (\eta_1,\ \eta_2) \rangle \text{ and } -|x_1| + x_2^2 \geq \langle (1,\ 0),\ (\eta_1,\ \eta_2) \rangle,$$

which implies that

$$|x_1| - x_2^2 \leq \eta_1 \leq -|x_1| + x_2^2.$$

Hence, f *is* ∂^**-invex at* $\bar{x}$ *for any function* $\eta := (\eta_1,\ \eta_2)$ *which satisfy the above inequality. Here we may take* $\eta_1 = (-|x_1| + x_2^2)$ *or* $(|x_1| - x_2^2)$, *whose images are shown in Figure 1 and Figure 2, respectively and* η_2 *is independent of the inequality.*

4. Wolfe Duality

In this section, following the concept of Wolfe [33], we present the Wolfe dual (WD) to (P) as follows:

$$\text{(WD)} \quad \max \quad \Psi(y, \mu) := (\Psi_1(y, \mu), \ldots, \Psi_m(y, \mu))$$

$$\text{s.t. } 0 \in \sum_{i=1}^{m} \lambda_i co\partial^* f_i(y) + \sum_{j=1}^{n} \mu_j co\partial^* g_j(y) + N(Q, y),$$

$$\lambda_i > 0, \forall i = 1, \ldots, m, \sum_{i=1}^{m} \lambda_i = 1,\ \mu_j \geq 0, \forall j = 1, \ldots, n,\ y \in Q,$$

where $\Psi_i(y, \mu) := f_i(y) + \sum_{j=1}^{n} \mu_j g_j(y)$ for all $i \in I$. Let

$$S_W := \{(y, \lambda, \mu) \in \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^n : 0 \in \sum_{i=1}^{m} \lambda_i co\partial^* f_i(y) + \sum_{j=1}^{n} \mu_j co\partial^* g_j(y) + N(Q, y),$$
$$\lambda \in \mathbb{R}^m_{++}, \sum_{i=1}^{m} \lambda_i = 1,\ \mu \in \mathbb{R}^n_{+}, y \in Q\}$$

be the set of all feasible solutions of the (WD). We denote by

$$pr_Q S_W := \{y \in Q : (y, \lambda, \mu) \in S_W\}$$

the projection of the set S_W on Q.

Let J^+ be the set of indices given by

$$J^+ := \{j \in J : \mu_j > 0\}.$$

We establish the weak, strong, converse, restricted converse and strict converse duality theorems for (WD) with respect to (P).

Theorem 4.1 (Weak duality). *Let* $x \in S$ *be a feasible solution for (P),* $(y, \lambda, \mu) \in S_W$ *be a feasible solution for (WD) and* Q *be an invex set with respect to* η. *Suppose that* f_i *and* g_j *are locally Lipschitz functions at* y, *and admit bounded upper semi-regular convexificators* $\partial^* f_i(y)$ *and* $\partial^* g_j(y)$ *for all* $i \in I$ *and* $j \in J$. *Moreover, suppose that* f_i *and* g_j *are* ∂^*-*invex with respect to* η *at* y *on* $S \cup pr_Q S_W$ *for all* $i \in I$ *and* $j \in J$, *respectively. Then,*

$$f(x) \nleq \Psi(y, \mu).$$

Proof. We proceed by contradiction. Suppose that

$$f(x) \leq \Psi(y, \mu),$$

that is,

$$f_i(x) \leq f_i(y) + \sum_{j=1}^{n} \mu_j g_j(y), \forall i \in I,$$

and

$$f_i(x) < f_i(y) + \sum_{j=1}^{n} \mu_j g_j(y), \text{ for at least one } i \in I.$$

Since $\sum_{i=1}^{m} \lambda_i = 1$ and $\lambda \in \mathbb{R}^m_{++}$, it follows that

$$\sum_{i=1}^{m} \lambda_i f_i(x) < \sum_{i=1}^{m} \lambda_i f_i(y) + \sum_{j=1}^{n} \mu_j g_j(y). \tag{1}$$

Since f_i and g_j are ∂^*-invex with respect to η at y on $S \cup pr_Q S_W$, therefore

$$\begin{aligned} f_i(x) - f_i(y) &\geq \langle \xi_i, \eta(x, y) \rangle, \forall \xi_i \in \partial^* f_i(y), \forall i \in I, \\ g_j(x) - g_j(y) &\geq \langle \zeta_j, \eta(x, y) \rangle, \forall \zeta_j \in \partial^* g_j(y), \forall j \in J^+. \end{aligned} \tag{2}$$

By the feasibility of x for (P) and the feasibility of (y, λ, μ) for (WD), one has

$$\lambda_i > 0, \forall i \in I, \sum_{i=1}^{m} \lambda_i = 1, \; g_j(x) \leq 0, \mu_j \geq 0, \forall j \in J. \tag{3}$$

From (1),(2) and (3), for every $\xi_i \in co\partial^* f_i(y), i \in I, \zeta_j \in co\partial^* g_j(y), j \in J$ and $v \in N(Q, y)$, one has

$$\begin{aligned} 0 &> \sum_{i=1}^{m} \lambda_i \langle \xi_i, \eta(x, y) \rangle + \sum_{j=1}^{n} \mu_j \langle \zeta_j, \eta(x, y) \rangle \\ &\geq \sum_{i=1}^{m} \lambda_i \langle \xi_i, \eta(x, y) \rangle + \sum_{j=1}^{n} \mu_j \langle \zeta_j, \eta(x, y) \rangle + \langle v, \eta(x, y) \rangle. \end{aligned}$$

This is a contradiction and the proof is complete. □

Theorem 4.2 (Strong duality). *Let $\bar{x} \in S$ be a locally weakly efficient solution for (P). Suppose that f_i and g_j are locally Lipschitz functions at $\bar{x}$, and admit bounded upper semi-regular convexificaotrs $\partial^* f_i(\bar{x})$ and $\partial^* g_j(\bar{x})$ for all $i \in I$ and $j \in J$. If (CQ1) holds at $\bar{x}$, then there exists $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in S_W$ and $\bar{\mu}_j g_j(\bar{x}) = 0$ for all $j \in J$.*

Moreover, if the weak duality between (P) and (WD) in Theorem 4.1 holds, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a globally efficient solution for (WD) and the respective objective values are equal.

Proof. Since $\bar{x}$ satisfies all the conditions of Theorem 2.8, there exists $(\lambda, \mu) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_{+}$ such that

$$0 \in \sum_{i=1}^{m} \lambda_i co\partial^* f_i(\bar{x}) + \sum_{j=1}^{n} \mu_j co\partial^* g_j(\bar{x}) + N(Q, \bar{x}),$$

$$\mu_j g_j(\bar{x}) = 0, \forall j = 1, \ldots, n,$$

which implies that, there exist $\xi_i \in co\partial^* f_i(\bar{x}), i \in I, \zeta_j \in co\partial^* g_j(\bar{x}), j \in J$ and $v \in N(Q, \bar{x})$ such that

$$\sum_{i=1}^{m} \lambda_i \xi_i + \sum_{j=1}^{n} \mu_j \zeta_j + v = 0.$$

Dividing throughout by $\sum_{i=1}^{m} \lambda_i$ and setting

$$\bar{\lambda}_i := \frac{\lambda_i}{\sum_{i=1}^{m} \lambda_i}, \forall i \in I, \bar{\mu}_j := \frac{\mu_j}{\sum_{i=1}^{m} \lambda_i}, \forall j \in J \;\; and \;\; \bar{v} := \left(\frac{1}{\sum_{i=1}^{m} \lambda_i}\right) v,$$

it follows that

$$\sum_{i=1}^{m} \bar{\lambda}_i \xi_i + \sum_{j=1}^{n} \bar{\mu}_j \zeta_j + \bar{v} = 0,$$

and hence

$$0 \in \sum_{i=1}^{m} \bar{\lambda}_i co\partial^* f_i(\bar{x}) + \sum_{j=1}^{n} \bar{\mu}_j co\partial^* g_j(\bar{x}) + N(Q, \bar{x}),$$

$$\bar{\lambda}_i > 0, \forall i = 1, \dots, m, \sum_{i=1}^{m} \bar{\lambda}_i = 1, \;\; \bar{\mu}_j \geq 0, \forall j = 1, \dots, n, \;\; \bar{x} \in Q,$$

which implies that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible solution for (WD) and

$$\bar{\mu}_j g_j(\bar{x}) = 0, \forall j \in J.$$

By the weak duality Theorem 4.1, $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a globally efficient solution for (WD). □

Theorem 4.3 (Converse duality). *Let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a globally weakly efficient (resp. efficient) solution for (WD) and Q be an invex set with respect to η. If the hypothesis of Theorem 4.1 holds at $\bar{y}$ on $S \cup pr_Q S_W$ and $\bar{\mu}_j g_j(\bar{y}) = 0$ for all $j \in J$, then $\bar{y}$ is a globally weakly efficient (resp. efficient) solution for (P).*

Proof. We proceed by contradiction. Suppose that $\bar{y}$ is not a globally weakly efficient (resp. efficient) solution for (P). Then, there exists a feasible solution $\tilde{x} \in S$ such that

$$f(\tilde{x}) < f(\bar{y}) (resp. f(\tilde{x}) \leq f(\bar{y})). \tag{4}$$

Since $f_i, i \in I$ are ∂^*-invex with respect to η at $\bar{y}$ on $S \cup pr_Q S_W$, one has

$$\langle \xi_i, \eta(\tilde{x}, \bar{y}) \rangle < 0, \forall \xi_i \in \partial^* f_i(\bar{y}), \forall i \in I \; (resp. \exists i \in I).$$

By the feasibility of $\tilde{x}$ for (P) and the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (WD) and the assumption in the theorem, one has

$$\bar{\mu}_j g_j(\tilde{x}) \leq 0 = \bar{\mu}_j g_j(\bar{y}), \forall j \in J.$$

By the ∂^*-invexity with respect to η of $g_j, j \in J^+$ at $\bar{y}$ on $S \cup pr_Q S_W$, one has

$$\langle \zeta_j, \eta(\tilde{x}, \bar{y}) \rangle \leq 0, \forall \zeta_j \in \partial^* g_j(\bar{y}), \forall j \in J^+.$$

Therefore, for every $\xi_i \in co\partial^* f_i(\bar{y}), i \in I, \zeta_j \in co\partial^* g_j(\bar{y}), j \in J$ and $v \in N(Q, \bar{y})$, one has

$$\begin{aligned} 0 &> \sum_{i=1}^{m} \lambda_i \langle \xi_i, \eta(\tilde{x}, \bar{y}) \rangle + \sum_{j=1}^{n} \mu_j \langle \zeta_j, \eta(\tilde{x}, \bar{y}) \rangle \\ &\geq \sum_{i=1}^{m} \lambda_i \langle \xi_i, \eta(\tilde{x}, \bar{y}) \rangle + \sum_{j=1}^{n} \mu_j \langle \zeta_j, \eta(\tilde{x}, \bar{y}) \rangle + \langle v, \eta(\tilde{x}, \bar{y}) \rangle. \end{aligned}$$

This is a contradiction and the proof is complete. □

Theorem 4.4 (Restricted converse duality). *Let $\bar{x}$ be any feasible solution for (P) and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be any feasible solution for (WD) such that $f(\bar{x}) = \Psi(\bar{y}, \bar{\mu})$. If the hypothesis of Theorem* 4.1 *holds at $\bar{y}$ on $S \cup pr_Q S_W$, then $\bar{x}$ is a globally efficient solution for (P).*

Proof. We proceed by contradiction. Suppose that $\bar{x}$ is not a globally efficient solution for (P). Then, there exists $\tilde{x} \in S$ such that

$$f(\tilde{x}) \leq f(\bar{x}).$$

By the assumption in the theorem, one has

$$f(\tilde{x}) \leq \Psi(\bar{y}, \bar{\mu}).$$

This is a contradiction to the weak duality between (P) and (WD) and the proof is complete. □

Theorem 4.5 (Strict converse duality). *Let $\bar{x}$ be a locally weakly efficient solution for (P). Suppose that f_i and g_j are locally Lipschitz functions at $\bar{x}$, and admit bounded upper semi-regular convexificators $\partial^* f_i(\bar{x})$ and $\partial^* g_j(\bar{x})$ for all $i \in I$ and $j \in J$ such that (CQ1) is satisfied at $\bar{x}$ and the strong duality between (P) and (WD) as in Theorem 4.2 holds.*

Also, let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a globally efficient solution for (WD). Suppose that f_i and g_j are locally Lipschitz functions at $\bar{y}$, and admit bounded upper semi-regular convexificators $\partial^* f_i(\bar{y})$ and $\partial^* g_j(\bar{y})$ for all $i \in I$ and $j \in J$. Moreover, suppose that $f_i, i \in I$ are strict ∂^*-invex and $g_j, j \in J^+$ are ∂^*-invex with respect to η at $\bar{y}$ on $S \cup pr_Q S_W$, respectively. Then,

$$\bar{x} = \bar{y}.$$

Proof. We proceed by contradiction. Suppose that $\bar{x} \neq \bar{y}$. Then, by the strong duality theorem, there exists Lagrange multipliers $(\tilde{\lambda}, \tilde{\mu}) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $(\bar{x}, \tilde{\lambda}, \tilde{\mu})$ is a globally efficient solution for (WD) and hence

$$f(\bar{x}) = \Psi(\bar{x}, \tilde{\mu}) = \Psi(\bar{y}, \bar{\mu}),$$

that is,

$$f_i(\bar{x}) = f_i(\bar{x}) + \sum_{j=1}^{n} \tilde{\mu}_j g_j(\bar{x}) = f_i(\bar{y}) + \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{y}), \forall i \in I. \tag{5}$$

By the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (WD), there exists $\xi_i \in co\partial^* f_i(\bar{y}), i \in I, \zeta_j \in co\partial^* g_j(\bar{y}), j \in J$ and $v \in N(Q, \bar{y})$ such that

$$\sum_{i=1}^{m} \bar{\lambda}_i \xi_i + \sum_{j=1}^{n} \bar{\mu}_j \zeta_j + v = 0. \tag{6}$$

By the strict ∂^*-invexity with respect to η of $f_i, i \in I$ at $\bar{y}$ on $S \cup pr_Q S_W$, one has

$$f_i(x) - f_i(\bar{y}) > \langle \xi_i, \eta(x, \bar{y}) \rangle, \forall x \in S \cup pr_Q S_W, \forall \xi_i \in \partial^* f_i(\bar{y}), \forall i \in I. \tag{7}$$

By the ∂^*-invexity of $g_j, j \in J^+$ at $\bar{y}$ on $s \cup pr_Q S_W$, it follows that

$$g_j(x) - g_j(\bar{y}) \geq \langle \zeta_j, \eta(x, \bar{y}) \rangle, \forall x \in S \cup pr_Q S_W, \forall \zeta_j \in \partial^* g_j(\bar{y}), \forall j \in J^+,$$

and since $\bar{\mu}_j > 0, \forall j \in J^+$, for every $x \in S \cup pr_Q S_W, \zeta_j \in co\partial^* g_j(\bar{y}), j \in J$, one has

$$\sum_{j=1}^{n} \bar{\mu}_j g_j(x) - \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{y}) \geq \sum_{j=1}^{n} \bar{\mu}_j \langle \zeta_j, \eta(x, \bar{y}) \rangle. \tag{8}$$

By the feasibility of $\bar{x}$ for (P) and since $\bar{\mu}_j \geq 0, j \in J$, one has

$$\sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x}) \leq 0. \tag{9}$$

Combining (5) and (7)-(9), for every $\xi_i \in co\partial^* f_i(\bar{y}), i \in I, \zeta_j \in co\partial^* g_j(\bar{y}), j \in J$, one has

$$\langle \xi_i, \eta(\bar{x}, \bar{y})\rangle + \sum_{j=1}^{n} \bar{\mu}_j \langle \zeta_j, \eta(\bar{x}, \bar{y})\rangle < 0, \forall i \in I.$$

Using the fact that $\bar{\lambda}_i > 0, \forall i \in I$ and $\sum_{i=1}^{m} \bar{\lambda}_i = 1$, the above inequality implies that

$$\begin{aligned} 0 &> \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \eta(\bar{x}, \bar{y})\rangle + \sum_{j=1}^{n} \bar{\mu}_j \langle \zeta_j, \eta(\bar{x}, \bar{y})\rangle \\ &\geq \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \eta(\bar{x}, \bar{y})\rangle + \sum_{j=1}^{n} \bar{\mu}_j \langle \zeta_j, \eta(\bar{x}, \bar{y})\rangle + \langle v, \eta(\bar{x}, \bar{y})\rangle, \end{aligned}$$

for every $\xi_i \in co\partial^* f_i(\bar{y}), i \in I, \zeta_j \in co\partial^* g_j(\bar{y}), j \in J$ and $v \in N(Q, \bar{y})$, a contradiction to (6) and hence the result. □

5. Mond-Weir Duality

In this section, following the concept of Mond and Weir [28], we present the Mond-Weir dual (MWD) to (P) as follows:

$$\begin{aligned} &\text{(MWD)} \quad \max \quad f(y) := (f_1(y), \ldots, f_m(y)) \\ &\text{s.t. } 0 \in \sum_{i=1}^{m} \lambda_i co\partial^* f_i(y) + \sum_{j=1}^{n} \mu_j co\partial^* g_j(y) + N(Q, y), \\ &\lambda_i > 0, \forall i = 1, \ldots, m, \sum_{i=1}^{m} \lambda_i = 1, \ \mu_j \geq 0, \mu_j g_j(y) \geq 0, \forall j = 1, \ldots, n, \ y \in Q. \end{aligned}$$

Let

$$\begin{aligned} S_{MW} := \{(y, \lambda, \mu) \in \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^n : 0 \in \sum_{i=1}^{m} \lambda_i co\partial^* f_i(y) + \sum_{j=1}^{n} \mu_j co\partial^* g_j(y) + N(Q, y), \\ \lambda \in \mathbb{R}^m_{++}, \sum_{i=1}^{m} \lambda_i = 1, \ \mu \in \mathbb{R}^n_+, \mu_j g_j(y) \geq 0, j = 1, \ldots, n, y \in Q\} \end{aligned}$$

be the set of all feasible solutions of the (MWD). We denote by

$$pr_Q S_{MW} := \{y \in Q : (y, \lambda, \mu) \in S_{MW}\}$$

the projection of the set S_{MW} on Q.

We establish the weak, strong, converse, restricted converse and strict converse duality theorems for (MWD) with respect to (P).

Theorem 5.1 (Weak duality). *Let x and $(y, \lambda, \mu) \in S_{MW}$ be any feasible solutions for (P) and (MWD), respectively, and let Q be an invex set with respect to η. Suppose that f_i and g_j are locally Lipschitz functions at y, and admit bounded upper semi-regular convexificators $\partial^* f_i(y)$ and $\partial^* g_j(y)$ for all $i \in I$ and $j \in J$, respectively. Moreover, suppose that one of the following conditions holds:*

(a) *$f_i, i \in I$ are ∂^*-pseudoinvex and $g_j, j \in J^+$ are ∂^*-quasiinvex at y with respect to η on $S \cup pr_Q S_{MW}$, respectively. Then,*

$$f(x) \nless f(y).$$

(b) *$\sum_{i=1}^{m} \lambda_i f_i(\cdot)$ is ∂^*-pseudoinvex and $\sum_{j \in J^+} \mu_j g_j(\cdot)$ is ∂^*-quasiinvex with respect to η at y on $S \cup pr_Q S_{MW}$. Then,*

$$f(x) \nleq f(y).$$

Proof. (a) We proceed by contradiction. Suppose that

$$f(x) < f(y),$$

that is,

$$f_i(x) < f_i(y), \ \forall i \in I. \tag{10}$$

By the feasibility of x for (P) and the feasibility of (y, λ, μ) for (MWD), one has

$$\mu_j g_j(x) \leq \mu_j g_j(y), \forall j \in J.$$

Since, $\mu_j \geq 0$, for all $j \in J$ and $\mu_j > 0$ for $j \in J^+$ then,

$$g_j(x) \leq g_j(y), \forall j \in J^+.$$

By the ∂^*-pseudoinvexity of f_i for all $i \in I$ and by the ∂^*-quasiinvexity of g_j for all $j \in J^+$ with respect to η at y on $S \cup pr_Q S_{MW}$, one has

$$\langle \xi_i, \eta(x, y) \rangle < 0, \ \forall \xi_i \in \partial^* f_i(y), \ \forall i \in I,$$

$$\langle \zeta_j, \eta(x, y) \rangle \leq 0, \forall \zeta_j \in \partial^* g_j(y), \forall j \in J^+.$$

Therefore, for every $\xi_i \in co\partial^* f_i(y), i \in I, \zeta_j \in co\partial^* g_j(y), j \in J$ and $v \in N(Q, y)$, one has

$$\begin{aligned} 0 &> \sum_{i=1}^{m} \lambda_i \langle \xi_i, \eta(x, y) \rangle + \sum_{j=1}^{n} \mu_j \langle \zeta_j, \eta(x, y) \rangle \\ &\geq \sum_{i=1}^{m} \lambda_i \langle \xi_i, \eta(x, y) \rangle + \sum_{j=1}^{n} \mu_j \langle \zeta_j, \eta(x, y) \rangle + \langle v, \eta(x, y) \rangle. \end{aligned}$$

This is a contradiction and the proof of part (a) is complete.
(b) We proceed by contradiction. Suppose that

$$f(x) \leq f(y),$$

that is,

$$f_i(x) \leq f_i(y), \ \forall i \in I,$$

$$f_i(x) < f_i(y), \ \text{for at least one } i \in I.$$

By the feasibility of x for (P), the feasibility of (y, λ, μ) for (MWD), one has

$$\sum_{i=1}^{m} \lambda_i f_i(x) < \sum_{i=1}^{m} \lambda_i f_i(y)$$

and

$$\sum_{j \in J^+} \mu_j g_j(x) \leq \sum_{j \in J^+} \mu_j g_j(y).$$

Since $f_i, i \in I$ and $g_j, j \in J^+$ are locally Lipschitz functions at y and $\lambda_i > 0, i \in I, \mu_j > 0, j \in J^+$, therefore $\sum_{i=1}^{m} \lambda_i f_i(\cdot)$ and $\sum_{j \in J^+} \mu_j g_j(\cdot)$ are also locally Lipschitz functions at y, and admit bounded upper semi-regular convexificator $\sum_{i=1}^{m} \lambda_i \partial^* f_i(y)$ and $\sum_{j \in J^+} \mu_j \partial^* g_j(y)$, respectively. Hence, by ∂^*-pseudoinvexity of $\sum_{i=1}^{m} \lambda_i f_i(\cdot)$ and ∂^*- quasiinvexity of $\sum_{j \in J^+} \mu_j g_j(\cdot)$ at y on $S \cup pr_Q S_{MW}$, one has

$$\left\langle \sum_{i=1}^{m} \lambda_i \xi_i, \eta(x, y) \right\rangle < 0, \forall \xi_i \in \partial^* f_i(y), i \in I,$$

and

$$\left\langle \sum_{j\in J^+} \mu_j\zeta_j, \eta(x,y) \right\rangle \leq 0, \forall \zeta_j \in \partial^* g_j(y), j \in J^+.$$

Therefore, for every $\xi_i \in co\partial^* f_i(y), i \in I, \zeta_j \in co\partial^* g_j(y), j \in J$ and $v \in N(Q,y)$, one has

$$\begin{aligned} 0 &> \left\langle \sum_{i=1}^{m} \lambda_i\xi_i + \sum_{j=1}^{n} \mu_j\zeta_j, \eta(x,y) \right\rangle \\ &\geq \left\langle \sum_{i=1}^{m} \lambda_i\xi_i + \sum_{j=1}^{n} \mu_j\zeta_j + v, \eta(x,y) \right\rangle. \end{aligned}$$

This is a contradiction and the proof is complete. □

Theorem 5.2 (Strong duality). *Let $\bar{x} \in S$ be a locally weakly efficient solution for (P). Suppose that f_i and g_j are locally Lipschitz functions at $\bar{x}$, and admit bounded upper semi-regular convexificaotrs $\partial^* f_i(\bar{x})$ and $\partial^* g_j(\bar{x})$ for all $i \in I$ and $j \in J$. If (CQ1) holds at $\bar{x}$, then there exists $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu}) \in S_{MW}$ is a feasible solution for (MWD).*

Moreover, if the weak duality between (P) and (MWD) in Theorem 5.1 holds, then $\bar{x}$ is a globally weakly efficient (resp. efficient) solution for (MWD).

Proof. Since $\bar{x}$ satisfies all the conditions of Theorem 2.8, there exists $(\tilde{\lambda}, \tilde{\mu}) \in \mathbb{R}^m_{++} \times \mathbb{R}^n_{+}$ such that

$$0 \in \sum_{i=1}^{m} \tilde{\lambda}_i co\partial^* f_i(\bar{x}) + \sum_{j=1}^{n} \tilde{\mu}_j co\partial^* g_j(\bar{x}) + N(Q,\bar{x}),$$

$$\tilde{\mu}_j g_j(\bar{x}) = 0, \forall j = 1,\dots,n,$$

which implies that, there exists $\xi_i \in co\partial^* f_i(\bar{x}), i \in I, \zeta_j \in co\partial^* g_j(\bar{x}), j \in J$ and $v \in N(Q,\bar{x})$ such that

$$\sum_{i=1}^{m} \tilde{\lambda}_i\xi_i + \sum_{j=1}^{n} \tilde{\mu}_j\zeta_j + v = 0.$$

Dividing throughout by $\sum_{i=1}^{m} \tilde{\lambda}_i$ and setting

$$\bar{\lambda}_i := \frac{\tilde{\lambda}_i}{\sum_{i=1}^{m} \tilde{\lambda}_i}, \forall i \in I, \bar{\mu}_j := \frac{\tilde{\mu}_j}{\sum_{i=1}^{m} \tilde{\lambda}_i}, \forall j \in J \ \text{ and } \ \bar{v} := \left(\frac{1}{\sum_{i=1}^{m} \tilde{\lambda}_i}\right)v,$$

it follows that

$$\sum_{i=1}^{m} \bar{\lambda}_i\xi_i + \sum_{j=1}^{n} \bar{\mu}_j\zeta_j + \bar{v} = 0,$$

and hence

$$0 \in \sum_{i=1}^{m} \bar{\lambda}_i co\partial^* f_i(\bar{x}) + \sum_{j=1}^{n} \bar{\mu}_j co\partial^* g_j(\bar{x}) + N(Q,\bar{x}),$$

$$\bar{\lambda}_i > 0, \forall i = 1,\dots,m, \sum_{i=1}^{m} \bar{\lambda}_i = 1, \ \bar{\mu}_j \geq 0, \bar{\mu}_j g_j(\bar{x}) = 0, \forall j = 1,\dots,n, \ \bar{x} \in Q,$$

which implies that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible solution for (MWD). By the weak duality Theorem 5.1, it follows that $\bar{x}$ is a globally weakly efficient (resp. efficient) solution for (MWD). □

Theorem 5.3 (Converse duality). *Let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a globally weakly efficient (resp. efficient) solution for (MWD). If the hypothesis of Theorem 5.1 holds at $\bar{y}$ on $S \cup pr_Q S_{MW}$, then $\bar{y}$ is a globally weakly efficient (resp. efficient) solution for (P).*

Proof. (a) We proceed by contradiction. Suppose that $\bar{y}$ is not a globally weakly efficient solution for (P). Then, there exists a feasible solution $\tilde{x} \in S$ such that

$$f(\tilde{x}) < f(\bar{y}). \tag{11}$$

Since $f_i, i \in I$ are ∂^*−pseudoinvex with respect to η at $\bar{y}$ on $S \cup pr_Q S_{MW}$, one has

$$\langle \xi_i, \eta(\tilde{x}, \bar{y}) \rangle < 0, \forall \xi_i \in \partial^* f_i(\bar{y}), \forall i \in I.$$

By the feasibility of $\tilde{x}$ for (P) and the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (MWD) and the assumption in the theorem, one has

$$\bar{\mu}_j g_j(\tilde{x}) \leq 0 = \bar{\mu}_j g_j(\bar{y}), \forall j \in J.$$

By the ∂^*−quasiinvexity with respect to η of $g_j, j \in J^+$ at $\bar{y}$ on $S \cup pr_Q S_{MW}$, one has

$$\langle \zeta_j, \eta(\tilde{x}, \bar{y}) \rangle \leq 0, \forall \zeta_j \in \partial^* g_j(\bar{y}), \forall j \in J^+.$$

Therefore, for every $\xi_i \in co\partial^* f_i(\bar{y}), i \in I, \zeta_j \in co\partial^* g_j(\bar{y}), j \in J$ and $v \in N(Q, \bar{y})$, one has

$$\begin{aligned} 0 &> \left\langle \sum_{i=1}^{m} \bar{\lambda}_i \xi_i + \sum_{j=1}^{m} \bar{\mu}_j \zeta_j, \eta(\tilde{x}, \bar{y}) \right\rangle \\ &\geq \left\langle \sum_{i=1}^{m} \bar{\lambda}_i \xi_i + \sum_{j=1}^{m} \bar{\mu}_j \zeta_j + v, \eta(\tilde{x}, \bar{y}) \right\rangle. \end{aligned}$$

This is a contradiction and the proof of part (a) is complete.
(b) We proceed by contradiction. Suppose that $\bar{y}$ is not a globally efficient solution for (P). Then, there exists a feasible solution $\tilde{x} \in S$ such that

$$f(\tilde{x}) \leq f(\bar{y}). \tag{12}$$

By the feasibility of $\tilde{x}$ for (P), the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (MWD), the inequality (12) and the assumptions in the theorem, one has

$$\sum_{i=1}^{m} \bar{\lambda}_i f_i(\tilde{x}) < \sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{y})$$

and

$$\sum_{j \in J^+} \bar{\mu}_j g_j(\tilde{x}) \leq \sum_{j \in J^+} \bar{\mu}_j g_j(\bar{y}).$$

Since $f_i, i \in I$ and $g_j, j \in J^+$ are locally Lipschitz functions at $\bar{y}$ and $\bar{\lambda}_i > 0, i \in I, \bar{\mu}_j > 0, j \in J^+$, therefore $\sum_{i=1}^{m} \bar{\lambda}_i f_i(\cdot)$ and $\sum_{j \in J^+} \bar{\mu}_j g_j(\cdot)$ are also a locally Lipschitz functions at $\bar{y}$, and admits bounded upper semi-regular convexificator $\sum_{i=1}^{m} \bar{\lambda}_i \partial^* f_i(\bar{y})$ and $\sum_{j \in J^+} \bar{\mu}_j \partial^* g_j(\bar{y})$, respectively. Now, by the ∂^*−pseudoinvexity of $\sum_{i=1}^{m} \bar{\lambda}_i f_i(\cdot)$ and ∂^*−quasiinvexity of $\sum_{j \in J^+} \bar{\mu}_j g_j(\cdot)$ at $\bar{y}$ on $S \cup pr_Q S_{MW}$, one has

$$\left\langle \sum_{i=1}^{m} \bar{\lambda}_i \xi_i, \eta(\tilde{x}, \bar{y}) \right\rangle < 0, \forall \xi_i \in \partial^* f_i(\bar{y}), i \in I,$$

and

$$\left\langle \sum_{j \in J^+} \bar{\mu}_j \zeta_j, \eta(\tilde{x}, \bar{y}) \right\rangle \leq 0, \forall \zeta_j \in \partial^* g_j(\bar{y}), j \in J^+.$$

Therefore, for every $\xi_i \in co\partial^* f_i(\bar{y}), i \in I, \zeta_j \in co\partial^* g_j(\bar{y}), j \in J$ and $v \in N(Q, \bar{y})$, one has

$$\begin{aligned} 0 &> \left\langle \sum_{i=1}^{m} \bar{\lambda}_i \xi_i + \sum_{j=1}^{n} \bar{\mu}_j \zeta_j, \eta(\tilde{x}, \bar{y}) \right\rangle \\ &\geq \left\langle \sum_{i=1}^{m} \bar{\lambda}_i \xi_i + \sum_{j=1}^{n} \bar{\mu}_j \zeta_j + v, \eta(\tilde{x}, \bar{y}) \right\rangle. \end{aligned}$$

This is a contradiction and the proof of part (b) is complete. □

Theorem 5.4 (Restricted converse duality). *Let $\bar{x}$ be any feasible solution for (P) and $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be any feasible solution for (MWD) such that $f(\bar{x}) = f(\bar{y})$. If the hypothesis of Theorem 5.1 holds at $\bar{y}$ on $S \cup pr_Q S_{MW}$, then $\bar{x}$ is a globally weakly efficient (resp. efficient) solution for (P).*

Proof. We proceed by contradiction. Suppose that $\bar{x}$ is not a globally weakly efficient (resp. efficient) solution for (P). Then, there exists $\tilde{x} \in S$ such that

$$f(\tilde{x}) \ < \ (\leq) \ f(\bar{x}).$$

By the assumption in the theorem, one has

$$f(\tilde{x}) \ < \ (\leq) \ f(\bar{y}).$$

This is a contradiction to the weak duality between (P) and (MWD) and the proof is complete. □

Theorem 5.5 (Strict converse duality). *Let $\bar{x}$ be a locally weakly efficient solution for (P). Suppose that f_i and g_j are locally Lipschitz functions at $\bar{x}$, and admit bounded upper semi-regular convexificators $\partial^* f_i(\bar{x})$ and $\partial^* g_j(\bar{x})$ for all $i \in I$ and $j \in J$ such that (CQ1) is satisfied at $\bar{x}$ and the strong duality between (P) and (MWD) as in Theorem 5.2 holds.*

Also, let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a globally weakly efficient (resp. efficient) solution for (MWD). Suppose that f_i and g_j are locally Lipschitz functions at $\bar{y}$, and admit bounded upper semi-regular convexificators $\partial^* f_i(\bar{y})$ and $\partial^* g_j(\bar{y})$ for all $i \in I$ and $j \in J$. Moreover, suppose that one of the following conditions holds:

(a) *$f_i, i \in I$ are strict ∂^*-pseudoinvex and $g_j, j \in J^+$ are ∂^*-quasiinvex with respect to η at $\bar{y}$ on $S \cup pr_Q S_{MW}$, respectively;*
(b) *$\sum_{i=1}^{m} \bar{\lambda}_i f_i(\cdot)$ is strictly ∂^*-pseudoinvex and $\sum_{j \in j^+} \bar{\mu}_j g_j(\cdot)$ is ∂^*-quasiinvex at $\bar{y}$ on $S \cup pr_Q S_{MW}$.*
Then,

$$\bar{x} = \bar{y}.$$

Proof. We proceed by contradiction. Suppose that $\bar{x} \neq \bar{y}$.
(a) By the strong duality theorem, there exists Lagrange multipliers $(\tilde{\lambda}, \tilde{\mu}) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $(\bar{x}, \tilde{\lambda}, \tilde{\mu})$ is a globally weakly efficient solution for (MWD). Hence,

$$f(\bar{x}) = f(\bar{y}),$$

that is,

$$f_i(\bar{x}) = f_i(\bar{y}), \forall i \in I. \tag{13}$$

By the strict ∂^*-pseudoinvexity of $f_i, i \in I$ at $\bar{y}$ on $S \cup pr_Q S_{MW}$, one has

$$\langle \xi_i, \eta(\bar{x}, \bar{y}) \rangle < 0, \forall \xi_i \in \partial^* f_i(\bar{y}), \forall i \in I.$$

By the feasibility of $\bar{x}$ for (P) and the $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (MWD), one has

$$\bar{\mu}_j g_j(\bar{x}) \leq 0 \leq \bar{\mu}_j g_j(\bar{y}), \forall j \in J.$$

By the ∂^*−quasiinvexity of $g_j, j \in J^+$ at $\bar{y}$ on $S \cup pr_Q S_{MW}$, if follows that

$$\langle \zeta_j, \eta(\bar{x}, \bar{y}) \rangle \leq 0, \forall \zeta_j \in \partial^* g_j(\bar{y}), \forall j \in J^+.$$

Therefore, for every $\xi_i \in co\partial^* f_i(\bar{y}), i \in I, \zeta_j \in co\partial^* g_j(\bar{y}), j \in J$ and $v \in N(Q, \bar{y})$, one has

$$\begin{aligned} 0 &> \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \eta(\bar{x}, \bar{y}) \rangle + \sum_{j=1}^{n} \bar{\mu}_j \langle \zeta_j, \eta(\bar{x}, \bar{y}) \rangle \\ &\geq \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \eta(\bar{x}, \bar{y}) \rangle + \sum_{j=1}^{n} \bar{\mu}_j \langle \zeta_j, \eta(\bar{x}, \bar{y}) \rangle + \langle v, \eta(\bar{x}, \bar{y}) \rangle. \end{aligned}$$

This is a contradiction and the proof is complete.
(*b*) By the strong duality theorem, there exists Lagrange multipliers $(\tilde{\lambda}, \tilde{\mu}) \in \mathbb{R}^m \times \mathbb{R}^n$ such that $(\bar{x}, \tilde{\lambda}, \tilde{\mu})$ is a globally efficient solution for (MWD). Hence,

$$f(\bar{x}) = f(\bar{y}),$$

that is,

$$f_i(\bar{x}) = f_i(\bar{y}), \forall i \in I. \tag{14}$$

By the feasibility of $\bar{x}$ for (P), the feasibility of $(\bar{y}, \bar{\lambda}, \bar{\mu})$ for (MWD) and the equalities (14), one has

$$\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) = \sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{y}),$$

and

$$\sum_{j \in J^+} \bar{\mu}_j g_j(\bar{x}) \leq 0 \leq \sum_{j \in J^+} \bar{\mu}_j g_j(\bar{y}).$$

By the assumptions in the theorem, $\sum_{i=1}^m \bar{\lambda}_i f_i(\cdot)$ and $\sum_{j \in J^+} \bar{\mu}_j g_j(\cdot)$ are locally Lipschitz functions at $\bar{y}$, and admit bounded upper semi-regular convexificators $\sum_{i=1}^m \bar{\lambda}_i \partial^* f_i(\cdot)$ and $\sum_{j \in J^+} \bar{\mu}_j \partial^* g_j(\cdot)$, respectively. By the strict ∂^*−pseudoinvexity of $\sum_{i=1}^m \bar{\lambda}_i f_i(\cdot)$ and ∂^*−quasiinvexity of $\sum_{j \in J^+} \bar{\mu}_j g_j(\cdot)$ at $\bar{y}$ on $S \cup pr_Q S_{MW}$, one has

$$\left\langle \sum_{i=1}^{m} \bar{\lambda}_i \xi_i, \eta(\bar{x}, \bar{y}) \right\rangle < 0, \forall \xi_i \in \partial^* f_i(\bar{y}), i \in I,$$

and

$$\left\langle \sum_{j \in J^+} \bar{\mu}_j \zeta_j, \eta(\bar{x}, \bar{y}) \right\rangle \leq 0, \forall \zeta_j \in \partial^* g_j(\bar{y}), j \in J^+.$$

Therefore, for every $\xi_i \in co\partial^* f_i(\bar{y}), i \in I, \zeta_j \in co\partial^* g_j(\bar{y}), j \in J$ and $v \in N(Q, \bar{y})$, one has

$$\begin{aligned} 0 &> \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \eta(\bar{x}, \bar{y}) \rangle + \sum_{j=1}^{n} \bar{\mu}_j \langle \zeta_j, \eta(\bar{x}, \bar{y}) \rangle \\ &\geq \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \eta(\bar{x}, \bar{y}) \rangle + \sum_{j=1}^{n} \bar{\mu}_j \langle \zeta_j, \eta(\bar{x}, \bar{y}) \rangle + \langle v, \eta(\bar{x}, \bar{y}) \rangle. \end{aligned}$$

This is a contradiction and the proof is complete. □

6. Nonsmooth Saddle-Point Analysis

In this section, we derive some nonsmooth weak vector saddle-point theorems for (P) under ∂^*-invexity and generalized ∂^*-invexity assumptions. The Lagrangian function of (P) is

$$L(x,\ \mu) = f(x) + \mu^T g(x)e,$$

where $x \in Q,\ \mu \in \mathbb{R}^n_+$ and $e = (1,\ \dots,\ 1) \in \mathbb{R}^n$.

Definition 6.1. *A point $(\bar{x},\ \bar{\mu}) \in Q \times \mathbb{R}^n_+$ is said to be a weak vector saddle-point of (P) if*

$$L(x,\ \bar{\mu}) \nless L(\bar{x},\ \bar{\mu}),\ \ \forall x \in Q,$$

$$L(\bar{x},\ \bar{\mu}) \nless L(\bar{x},\ \mu),\ \ \forall \mu \in \mathbb{R}^n_+.$$

6.1. Saddle-point analysis under ∂^*-invexity

In this section, we establish the relationship between a weak vector saddle-point and a weakly efficient solutions of (P) under ∂^*-invexity assumptions.

Theorem 6.2. *Let $(\bar{x},\ \bar{\lambda},\ \bar{\mu})$ satisfy the Theorem 2.8. Suppose that the objective function f and the constraint g are ∂^*-invex with respect to η at $\bar{x} \in Q$ over Q. Then $(\bar{x},\ \bar{\mu})$ is a weak vector saddle-point of (P).*

Proof. Since $(\bar{x},\ \bar{\lambda},\ \bar{\mu})$ satisfies the condition (i) and (ii), then

$$0 \in \sum_{i=1}^{m} \bar{\lambda}_i co\partial^* f_i(\bar{x}) + \sum_{j=1}^{n} \bar{\mu}_j co\partial^* g_j(\bar{x}) + N(Q,\ \bar{x}), \tag{15}$$

$$\bar{\mu}_j g_j(\bar{x}) = 0,\ \forall j = 1,\ \dots,\ n, \tag{16}$$

which implies that, there exists $\xi_i \in co\partial^* f_i(\bar{x}),\ i \in I,\ \zeta_j \in co\partial^* g_j(\bar{x}),\ j \in J$ and $v \in N(Q,\ \bar{x})$ such that

$$0 = \sum_{i=1}^{m} \bar{\lambda}_i \xi_i + \sum_{j=1}^{n} \bar{\mu}_j \zeta_j + v. \tag{17}$$

Since f and g are ∂^*-invex with respect to η at $\bar{x}$ over Q, by Definition 2.6, the inequalities

$$f_i(x) - f_i(\bar{x}) \geq \langle \xi_i,\ \eta(x,\ \bar{x})\rangle,\ \forall \xi_i \in \partial^* f_i(\bar{x}),\ \forall i \in I, \tag{18}$$

$$g_j(x) - g_j(\bar{x}) \geq \langle \zeta_j,\ \eta(x,\ \bar{x})\rangle,\ \forall \zeta_j \in \partial^* g_j(\bar{x}),\ \forall j \in J \tag{19}$$

hold. Since $\bar{\lambda}_i \geq 0,\ i = 1,\ \dots,\ m,\ \sum_{i=1}^m \bar{\lambda}_i = 1,\ \bar{\mu}_j \geqq 0,\ j = 1,\ \dots,\ n$, then multiplying $\bar{\mu}_j$ in (19) and adding all of these inequalities, we get

$$\begin{aligned} f_i(x) - f_i(\bar{x}) &+ \sum_{j=1}^{n} \bar{\mu}_j g_j(x) - \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x}) \\ &\geq \left\langle \xi_i + \sum_{j=1}^{n} \bar{\mu}_j \zeta_j,\ \eta(x,\ \bar{x}) \right\rangle,\ \forall \zeta_j \in \partial^* g_j(\bar{x}),\ j \in J,\ \forall \xi_i \in \partial^* f_i(\bar{x}),\ \forall i \in I. \end{aligned} \tag{20}$$

Now, multiplying $\bar{\lambda}_i$ in (20) and adding all of the inequalities. Therefore, for every $\xi_i \in co\partial^* f_i(\bar{x}),\ i \in I$ and $\zeta_j \in co\partial^* g_j(\bar{x}),\ j \in J$, one has

$$\begin{aligned}\sum_{i=1}^{m}\left(\bar{\lambda}_i\Big(f_i(x)-f_i(\bar{x})\Big)\right)+\sum_{i=1}^{m}\bar{\lambda}_i\left(\sum_{j=1}^{n}\bar{\mu}_j g_j(x)-\sum_{j=1}^{n}\bar{\mu}_j g_j(\bar{x})\right)\\ &\geq \left\langle \sum_{i=1}^{m}\bar{\lambda}_i\xi_i+\sum_{j=1}^{n}\bar{\mu}_j\zeta_j,\ \eta(x,\ \bar{x})\right\rangle\\ &\geq \left\langle \sum_{i=1}^{m}\bar{\lambda}_i\xi_i+\sum_{j=1}^{n}\bar{\mu}_j\zeta_j+v,\ \eta(x,\ \bar{x})\right\rangle\\ &= 0 \quad \text{(by using equation (17))},\end{aligned}$$

the above inequality imply that,

$$\sum_{i=1}^{m}\left\{\bar{\lambda}_i\left(\Big(f_i(x)-f_i(\bar{x})\Big)+\Big(\sum_{j=1}^{n}\bar{\mu}_j g_j(x)-\sum_{j=1}^{n}\bar{\mu}_j g_j(\bar{x})\Big)\right)\right\}\geq 0. \tag{21}$$

Since, $\bar{\lambda}_i \geq 0$, not all zero, it follows that

$$f(x)+\bar{\mu}^T g(x)e \nless f(\bar{x})+\bar{\mu}^T g(\bar{x})e \quad \text{for any } x\in Q, \tag{22}$$

i.e., $L(x,\ \bar{\mu}) \nless L(\bar{x},\ \bar{\mu})$ for any $x \in Q$.

Since, $(\bar{x},\ \bar{\lambda},\ \bar{\mu})$ satisfies the Theorem 2.8, then $\bar{x} \in S$ *i.e.*, $g_j(\bar{x}) \leq 0$, for all $j \in J$. Then $\mu^T g(\bar{x})e \leq 0$ for any $\mu \in \mathbb{R}^n_+$. Now, by (16) $\bar{\mu}^T g(\bar{x})e = 0$. We get

$$\bar{\mu}^T g(\bar{x})e - \mu^T g(\bar{x})e \geq 0 \text{ for any } \mu \in \mathbb{R}^n_+.$$

Thus

$$f(\bar{x}) + \bar{\mu}^T g(\bar{x})e - \{f(\bar{x}) + \mu^T g(\bar{x})e\} \in \mathbb{R}^m_+$$

hence $L(\bar{x},\ \bar{\mu}) \nless L(\bar{x},\ \mu)$ for any $\mu \in \mathbb{R}^n_+$ and the proof is complete. □

Theorem 6.3. *If $(\bar{x},\ \bar{\mu})$ is a weak vector saddle-point of (P), then $\bar{x}$ is a weakly efficient solution of (P).*

Proof. Since $(\bar{x},\ \bar{\mu})$ is a weak vector saddle-point of (P), by Definition 6.1, the inequality

$$L(x,\ \bar{\mu}) \nless L(\bar{x},\ \bar{\mu}),\ \forall x \in Q, \tag{23}$$

$$L(\bar{x},\ \bar{\mu}) \nless L(\bar{x},\ \mu),\ \forall \mu \in \mathbb{R}^n_+ \tag{24}$$

satisfied. Then, by the inequality (24)

$$L_i(\bar{x},\ \bar{\mu}) \geq L_i(\bar{x},\ \mu),\ \forall \mu \in \mathbb{R}^n_+$$

for at least one $i = 1,\ 2,\ \ldots,\ m$.

Then,

$$f_i(\bar{x}) + \bar{\mu}^T g(\bar{x})e \geq f_i(\bar{x}) + \mu^T g(\bar{x})e,\ \forall \mu \in \mathbb{R}^n_+$$

for at least one i. Hence,

$$(\mu - \bar{\mu}^T)g(\bar{x}) \leq 0,\ \forall \mu \in \mathbb{R}^n_+ \tag{25}$$

for any $j = 1,\ 2,\ \ldots,\ n$ set $\mu_t = \bar{\mu}_t$, for $t = 1,\ 2,\ \ldots,\ j-1,\ j+1,\ \ldots,\ n$ and $\mu_j = \bar{\mu}_j + 1$. From which, we get

$$g_j(\bar{x}) \leq 0.$$

This process are repeating for $j = 1, 2, \ldots, n$, we have

$$g(\bar{x}) \leq 0.$$

Which implies that $\bar{x} \in S$ and $\bar{\mu} \in \mathbb{R}^n_+$, we have

$$\bar{\mu}^T g(\bar{x}) \leq 0.$$

Set $\mu = 0$ in (25), we get

$$\bar{\mu}^T g(\bar{x}) \geq 0.$$

Thus $\bar{\mu}^T g(\bar{x}) \leq 0$ and $\bar{\mu}^T g(\bar{x}) \geq 0$ yields

$$\bar{\mu}^T g(\bar{x}) = 0. \tag{26}$$

By the inequality (23)

$$f(x) + \bar{\mu}^T g(x)e \nless f(\bar{x}) + \bar{\mu}^T g(\bar{x})e, \ \forall x \in Q.$$

Then, by the equation (26), we get the inequalities

$$f(x) + \sum_{j=1}^{n} \bar{\mu}_j^T g_j(x)e \nless f(\bar{x}), \ \forall x \in Q, \tag{27}$$

if $x \in S$, then $\sum_{j=1}^{n} \bar{\mu}_j^T g_j(x) \leq 0$. Then, the inequality (27) imply that $f(x) \nless f(\bar{x})$. Hence, $\bar{x}$ is a weakly efficient solution of (P). □

6.2. *Saddle-point analysis under generalized ∂^*-invexity*

In this section, we derive weak vector saddle-point theorems of (P) under generalized ∂^*-invexity assumptions.

Theorem 6.4. *Let $(\bar{x}, \bar{\lambda}, \bar{\mu})$ satisfy the Theorem 2.8 and $0 \in co\partial^*(\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) + \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x})) + N(Q, \bar{x})$. Suppose that the function $\sum_{i=1}^{m} \bar{\lambda}_i f_i(\cdot) + \sum_{j=1}^{n} \bar{\mu}_j g_j(\cdot)$ is ∂^*-pseudoinvex with respect to η at $\bar{x} \in Q$ over Q. Then $(\bar{x}, \bar{\mu})$ is a weak vector saddle-point of (P).*

Proof. Since $(\bar{x}, \bar{\lambda}, \bar{\mu})$ satisfies the condition (i), (ii) and $0 \in co\partial^*(\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) + \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x})) + N(Q, \bar{x})$, then there exist $\xi \in co\partial^*(\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) + \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x}))$ and $v \in N(Q, \bar{x})$ such that

$$0 = \xi + v. \tag{28}$$

Since $\sum_{i=1}^{m} \bar{\lambda}_i f_i(\cdot) + \sum_{j=1}^{n} \bar{\mu}_j g_j(\cdot)$ is ∂^*-pseudoinvex with respect to η at $\bar{x}$ over Q, by Definition 2.6, the inequalities

$$\begin{aligned} &\sum_{i=1}^{m} \bar{\lambda}_i f_i(x) + \sum_{j=1}^{n} \bar{\mu}_j g_j(x) < \sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) + \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x}) \\ &\qquad \Rightarrow \langle \xi, \eta(x, \bar{x}) \rangle < 0, \ \forall \xi \in \partial^*\Big(\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) + \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x})\Big). \end{aligned} \tag{29}$$

Suppose that

$$\sum_{i=1}^{m} \Big(\bar{\lambda}_i\big(f_i(x) - f_i(\bar{x})\big)\Big) + \sum_{i=1}^{m} \bar{\lambda}_i \left(\sum_{j=1}^{n} \bar{\mu}_j g_j(x) - \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x})\right) < 0, \tag{30}$$

since $\bar{\lambda}_i \geq 0,\ i = 1,\ \ldots,\ m,\ \sum_{i=1}^{m} \bar{\lambda}_i = 1$, then, by (30) we get

$$\sum_{i=1}^{m} \left(\bar{\lambda}_i\Big(f_i(x) - f_i(\bar{x})\Big)\right) + \left(\sum_{j=1}^{n} \bar{\mu}_j g_j(x) - \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x})\right) < 0. \tag{31}$$

Therefore, for every $\xi \in co\partial^*\Big(\sum_{i=1}^{m} \bar{\lambda}_i f_i(\bar{x}) + \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x})\Big)$, (29) and (31), gives the inequality

$$\langle \xi,\ \eta(x,\ \bar{x})\rangle < 0,\ \forall x \in Q,\ i.e.,\ \langle \xi + v,\ \eta(x,\ \bar{x})\rangle < 0,\ \forall x \in Q.$$

Then, by (28) we get the contradiction. Hence, our supposition is wrong, i.e.,

$$\sum_{i=1}^{m} \left(\bar{\lambda}_i\Big(f_i(x) - f_i(\bar{x})\Big)\right) + \sum_{i=1}^{m} \bar{\lambda}_i\left(\sum_{j=1}^{n} \bar{\mu}_j g_j(x) - \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x})\right) \geq 0, \tag{32}$$

the above inequality imply that,

$$\sum_{i=1}^{m} \left\{\bar{\lambda}_i\left(\Big(f_i(x) - f_i(\bar{x})\Big) + \Big(\sum_{j=1}^{n} \bar{\mu}_j g_j(x) - \sum_{j=1}^{n} \bar{\mu}_j g_j(\bar{x})\Big)\right)\right\} \geq 0. \tag{33}$$

Since, $\bar{\lambda}_i \geq 0$, not all zero, it follows that

$$f(x) + \bar{\mu}^T g(x)e \nless f(\bar{x}) + \bar{\mu}^T g(\bar{x})e \quad \text{for any } x \in Q, \tag{34}$$

i.e., $L(x,\ \bar{\mu}) \nless L(\bar{x},\ \bar{\mu})$ for any $x \in Q$.

Since, $(\bar{x},\ \bar{\lambda},\ \bar{\mu})$ satisfies the Theorem 2.8, then $\bar{x} \in S$ *i.e.*, $g_j(\bar{x}) \leq 0$, for all $j \in J$. Then $\mu^T g(\bar{x})e \leq 0$ for any $\mu \in \mathbb{R}^n_+$. Now, by (16) $\bar{\mu}^T g(\bar{x})e = 0$. We get

$$\bar{\mu}^T g(\bar{x})e - \mu^T g(\bar{x})e \geq 0 \text{ for any } \mu \in \mathbb{R}^n_+.$$

Thus

$$f(\bar{x}) + \bar{\mu}^T g(\bar{x})e - \{f(\bar{x}) + \mu^T g(\bar{x})e\} \in \mathbb{R}^m_+,$$

hence $L(\bar{x},\ \bar{\mu}) \nless L(\bar{x},\ \mu)$ for any $\mu \in \mathbb{R}^n_+$ and the proof is complete. □

Remark 6.5. *For generalized ∂^*-invexity, if $(\bar{x},\ \bar{\mu})$ is a weak vector saddle-point of (P), then $\bar{x}$ is a weakly efficient solution of (P).*

7. Conclusions

In this paper, we have formulated Wolfe and Mond-Weir type duals to the primal multiobjective programming problem using convexificators and established weak, strong, converse, restricted converse and strict converse duality results under the assumptions of ∂^*-invex, strict ∂^*-invex, ∂^*-pseudoinvex, strict ∂^*-pseudoinvex and ∂^*-quasiinvex functions. We have established the saddle-point under ∂^*-invexity and generalized ∂^*-invexity assumptions. Also, the mixed type dual to the primal multiobjective programming problem may be formulated in terms of convexificators and various duality results may be derived. The results of this paper may be extended by using various generalized convexity assumptions (see, e.g., [19–24]).

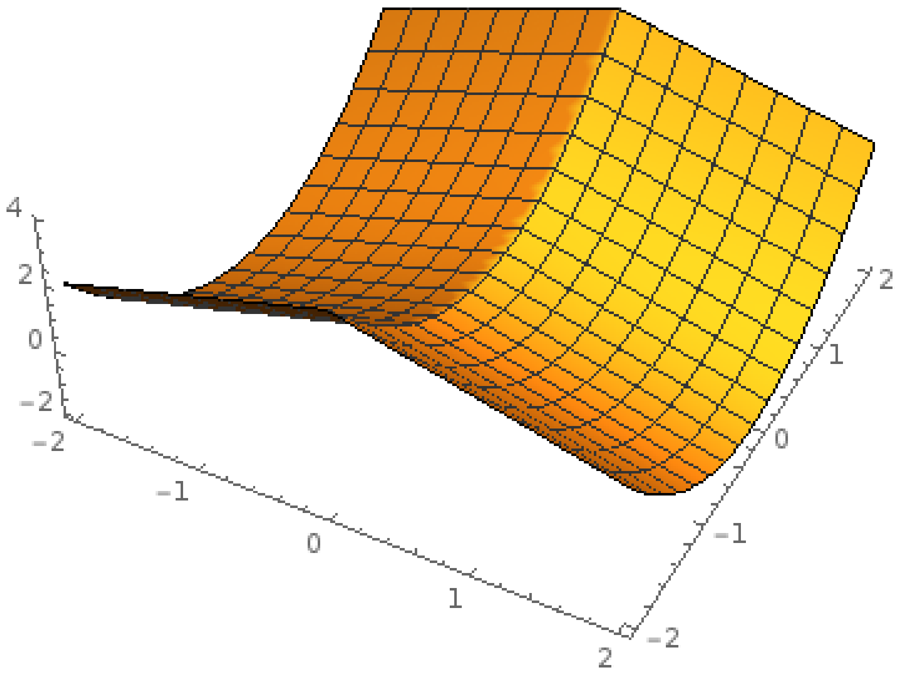

Figure 1: The function $\eta_1 = -|x_1| + x_2^2$

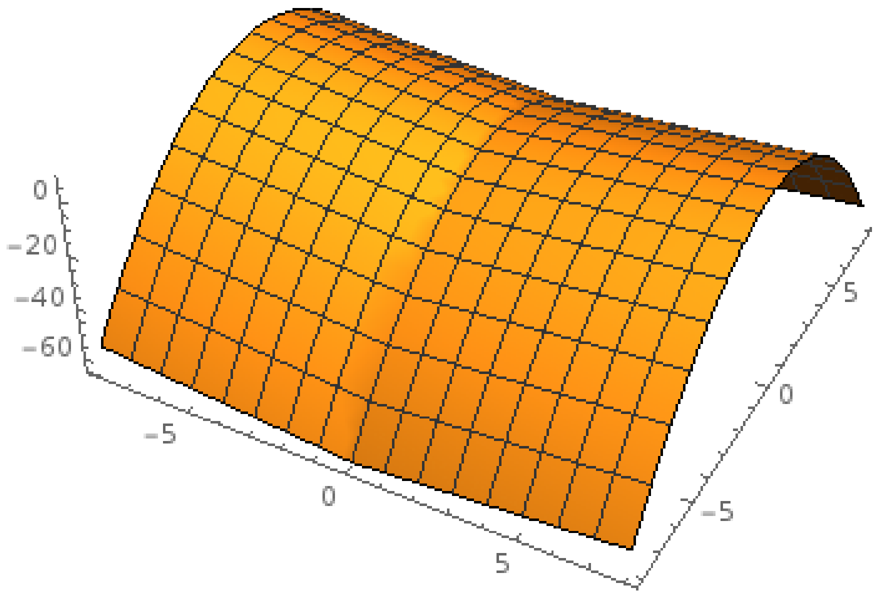

Figure 2: The function $\eta_1 = |x_1| - x_2^2$

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