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Schatten Class of Berezin Transform on Fock Spaces

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Abstract. The Schatten norm for the nuclear operator $B_\alpha^* B_\alpha$ was estimated from both sides. Here $B_\alpha : L^2_\beta \to L^2_\gamma$ is the Berezin transform regarding the Fock spaces in the plane. Also, we found the norm for the Berezin transform in case of unweighted Lebesgue spaces.

1. Introduction

Let $\mathbb{C}$ be as usual the complex plane and by $dA(z)(= dxdy)$ we denote the Lebesgue measure on the complex plane. Throughout the paper for any positive parameter α we consider the Gaussian-probability measure

$$d\mu_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} dA(z).$$

For $1 \le p < \infty$, $L^p(\mathbb{C}, d\mu_\alpha)(L^p_\alpha)$ denotes the space of all Lebesgue measurable functions f on $\mathbb{C}$ such that

$$\|f\|^p_{p,\alpha} = \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha|z|^2}{2}} dA(z) < \infty.$$

In fact, $f \in L^p_\alpha$ if and only if $f(z)e^{-\frac{\alpha|z|^2}{2}} \in L^p(\mathbb{C}, dA)$.

By F^2_α we denote the closed subspace of L^2_α which consists of all entire functions (see [4],[5],[8]). This subspace is known as the Fock space or (parameterized) Segal—Bargmann space. We refer the interested reader to [6] and [7] for analogous approach to the harmonic Fock space.

The orthogonal projection $P_\alpha : L^2_\alpha \to F^2_\alpha$ coincides with the integral operator which acting is determined as follows

$$P_\alpha f(z) = \int_{\mathbb{C}} K_\alpha(z, w) f(w) d\mu_\alpha(w),$$

where $K_\alpha(z, w)$ is reproducing kernel given by

$$K_\alpha(z, w) = e^{\alpha z \bar{w}}.$$

It is known that P_α is bounded on L^p_α for $p \ge 1$ (see [2]).

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At this point we should recall some basic notions related to the interpolation of Banach space. We will follow the notation from [8]. Namely, if X_0 and X_1 are compatible Banach spaces and $\theta \in (0, 1)$, the interpolation space X_θ between X_0 and X_1 we will also denote by $[X_0, X_1]_\theta$.

Further, if μ is a positive Borel measure on locally compact topological space X, and $L^p = L^p(X, d\mu)$, then

$$[L^{p_0}, L^{p_1}]_\theta = L^p,$$

where $1 \leq p_0 \leq p_1 \leq \infty$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

If X_0, X_1 and Y_0, Y_1 are pairs of compatible Banach spaces and if $T : X_0 + X_1 \to Y_0 + Y_1$ is bounded linear mapping in a such a manner that $T : X_0 \to Y_0$ and $T : X_1 \to Y_1$ are bounded with norms M_0 and M_1 respectively, then T maps X_θ boundedly into Y_θ with the norm at most $M_0^{1-\theta} M_1^\theta$.

For a measurable function f in $\mathbb{C}$ the Berezin transform of f is given by

$$B_\alpha f(z) = \frac{\alpha}{\pi} \int_{\mathbb{C}} f(w) e^{-\alpha|z-w|^2} dA(w).$$

Computation of the reproducing kernel and asymptotic expansion for the Berezin transform on the harmonic Fock space is given in [3].

The sufficient and necessary condition for boundedness of the Berezin transform $B_\alpha : L^p_\beta \to L^p_\gamma$ in the context of various L^p_α and different parameters can be summarized in the following theorem (see [8]).

Theorem 1.1. *Let $1 \leq p \leq \infty$. Suppose α, β and γ are positive weight parameters. Then $B_\alpha L^p_\beta \subset L^p_\gamma$ if and only if $\gamma(2\alpha - \beta) \geq 2\alpha\beta$.*

We should note that the above conditions imply $\alpha > \frac{\beta}{2}$ and $\gamma > \beta$.

The direct consequence of the above theorem is the following result, (see Proposition 3.20 in [8]).

Proposition 1.2. *Let $\alpha > 0$ and $1 \leq p \leq \infty$. Then*

$$B_\alpha : L^p(\mathbb{C}, dA) \to L^p(\mathbb{C}, dA)$$

is a contraction.

In the following theorem estimates from Proposition 1.2 are revisited and the norm of the Berezin transform is precisely determined.

Theorem 1.3. *Let $\alpha > 0$, and $1 \leq p \leq \infty$. Then*

$$\|B_\alpha\|_{L^p \to L^p} = 1.$$

Proof. At the beginning we shall give a brief observation for the limit cases when $p = \infty$ and $p = 1$.

It is easy to see that $\|B_\alpha f\|_\infty \leq \|f\|_\infty, f \in L^\infty$. Taking the function $f \equiv 1$ which is identically equal to 1, the last inequality becomes equality, i.e. $\|B_\alpha\|_{L^\infty \to L^\infty} = 1$.

On the other hand, using Fubini's theorem it is not hard to obtain that

$$\|B_\alpha f\|_{L^1} \leq \|f\|_{L^1}, f \in L^1(\mathbb{C}, dA),$$

and specially for $f(w) = \frac{1}{|B(0,R)|} \chi_{B(0,R)}(w)$, where by $|B(0, R)|$ we denote the measure of the ball $B(0, R)$ and $\chi_{B(0,R)}$ is the characteristic function of the ball $B(0, R)$, we get that $\|B_\alpha f\|_{L^1} = 1$.

We include the observation for the case $p = 2$.

Using Plancherel theorem for $f \in L^2(\mathbb{C}, dA)$ one gets

$$\|B_\alpha f\|^2_{L^2} = \|\mathcal{F}(B_\alpha f)\|^2_{L^2} = \|\hat{\psi} \hat{f}\|^2_{L^2},$$

where $\psi(x) = \frac{\alpha}{\pi} e^{-\alpha|x|^2}$ and as usual

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^2} f(x) e^{-2\pi i x\cdot\xi} dx.$$

Note that $\hat{\psi}(\xi) = e^{-\frac{\pi^2|\xi|^2}{\alpha}}$.
Then,

$$\|B_\alpha\|^2_{L^2\to L^2} = \sup_{\|\hat{f}\|^2_{L^2}\le 1} \int_{\mathbb{C}} |\hat{\psi}|^2 |\hat{f}(\xi)|^2 dA(\xi) = \sup_{\xi\in\mathbb{C}} |\hat{\psi}(\xi)|^2 = 1.$$

Using the Interpolation of spaces $L^p(\mathbb{C}, dA)$ for $1 < p < \infty$, we derive

$$\|B_\alpha\|_{L^p\to L^p} \le 1. \tag{1}$$

Further, let $f_{n,\gamma}(w) = w^n e^{-\gamma|w|^2}$, where $\gamma > 0$ and n is nonnegative integer, then $\|f_{n,\gamma}\|_p = \frac{\pi^{1/p}\Gamma^{1/p}(\frac{np}{2}+1)}{(p\gamma)^{\frac{n}{2}+\frac{1}{p}}}$.
On the other hand,

$$\begin{aligned} &B_\alpha f_{n,\gamma}(z) \\ &= \frac{\alpha}{\pi} \int_{\mathbb{C}} e^{-\alpha|z-w|^2} f_{n,\gamma}(w) dA(w) \\ &= \frac{\alpha e^{-\alpha|z|^2}}{\pi} \sum_{l,k\ge 0} \frac{\alpha^{k+l} z^l \bar{z}^k}{k! l!} \int_{\mathbb{C}} f_{n,\gamma}(w) \bar{w}^l w^k e^{-\alpha|w|^2} dA(w) \\ &= \frac{\alpha e^{-\alpha|z|^2} z^n}{\pi} \sum_{k=0}^{\infty} \frac{\alpha^{2k+n}|z|^{2k}}{k!(k+n)!} \int_{\mathbb{C}} |w|^{2(k+n)} e^{-(\alpha+\gamma)|w|^2} dA(w) \\ &= \left(\frac{\alpha}{\alpha+\gamma}\right)^{n+1} z^n e^{-\frac{\alpha\gamma}{\alpha+\gamma}|z|^2}. \end{aligned} \tag{2}$$

Therefore,

$$\|B_\alpha f_{n,\gamma}\|_{L^p} = \left(\frac{\alpha}{\alpha+\gamma}\right)^{\frac{n}{2}+\frac{1}{q}} \frac{(\pi\Gamma(\frac{np}{2}+1))^{1/p}}{(p\gamma)^{\frac{n}{2}+\frac{1}{p}}},$$

and

$$\frac{\|B_\alpha f_{n,\gamma}\|_{L^p}}{\|f\|_{L^p}} = \left(\frac{\alpha}{\alpha+\gamma}\right)^{\frac{n}{2}+\frac{1}{q}},$$

where q is conjugate exponent to p, $\frac{1}{p} + \frac{1}{q} = 1$.

Clearly, for any positive $\epsilon \in (0,1)$ there is some $\gamma > 0$ such that

$$\left(\frac{\alpha}{\alpha+\gamma}\right)^{\frac{n}{2}+\frac{1}{q}} > 1 - \epsilon,$$

i.e., there is $f_{n,\gamma} \in L^p(\mathbb{C}, dA)$ such that

$$\|B_\alpha f_{n,\gamma}\|_{L^p} > (1-\epsilon)\|f_{n,\gamma}\|_{L^p}.$$

From the last inequality and relation (1) we conclude that

$$\|B_\alpha\|_{L^p\to L^p} = 1.$$

□

2. Schatten class of Berezin transform

In general, by H we denote a Hilbert space. Recall that all compact linear operators $T : H \to H$ satisfying

$$\|T\|_{S_p} = \left(\sum_{n=1}^{\infty} s_n^p(T) \right)^{1/p}, 0 < p < \infty$$

constitute the Schatten classes S_p.

For $1 \leq p \leq \infty$, S_p is a separable symmetrically-normed ideal with the norm

$$\|T\|_{S_p} = \|T\|_p = \left(\sum_{n=1}^{\infty} s_n^p(T) \right)^{1/p}.$$

The quantity $\| \cdot \|_{S_p}$ is called the Schatten(-–von Neumann) norm. In this paper we discuss such a type of norm for the Berezin transform and its product with the adjoint operator.

The duality pairing for the particular type spaces $L^p_\alpha, 1 \leq p < \infty$ $(L^p_\alpha)^* = L^q_\beta$ is given by

$$\langle f, g \rangle_\gamma = \frac{\gamma}{\pi} \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-\gamma|z|^2} dA(z), \tag{3}$$

where $\gamma = \frac{\alpha+\beta}{2}$.

According to the introduced duality (3), using Fubbini's theorem the adjoint operator $B^*_\alpha : L^2_\gamma \to L^2_\beta$ can be determined as follows

$$B^*_\alpha f(z) = \frac{\alpha\gamma}{\pi\beta} e^{\beta|z|^2} \int_{\mathbb{C}} e^{-\alpha|z-w|^2 - \gamma|w|^2} f(w) dA(w).$$

Here, we should noticed that

$$B^*_\alpha B_\alpha : L^2_\beta \to L^2_\beta. \tag{4}$$

The operator $B^*_\alpha B_\alpha$ is the integral operator given by the sequent formula

$$B^*_\alpha B_\alpha f(z) = \frac{\alpha^2\gamma}{\beta^2} \int_{\mathbb{C}} H(z,t) f(t) d\mu_\beta(t),$$

where

$$H(z,t) = e^{\beta(|z|^2+|t|^2)} \int_{\mathbb{C}} e^{-\alpha|z-w|^2 - \alpha|w-t|^2 - \gamma|w|^2} dA(w),$$

or

$$H(z,t) = e^{\beta(|z|^2+|t|^2)} e^{-\frac{\alpha}{2}|z-t|^2} \int_{\mathbb{C}} e^{-\frac{\alpha}{2}|2w-(z+t)|^2 - \gamma|w|^2} dA(w).$$

Lemma 2.1. *The kernel $H(z,t)$ of the operator $B^*_\alpha B_\alpha$ defined in* (4) *is given by the sequel formula*

$$H(z,t) = \frac{\pi}{2\alpha+\gamma} e^{(\beta-\alpha)(|z|^2+|t|^2)} e^{\frac{\alpha^2}{2\alpha+\gamma}|z+t|^2}.$$

Proof. Let us denote by $\omega_0 = \frac{z+t}{2}$, then using the polar-coordinates $w = re^{is}, \omega_0 = |\omega_0|e^{i\theta}$ we get

$$\begin{aligned}
&\int_{\mathbb{C}} e^{-2\alpha|w-\omega_0|^2-\gamma|w|^2}dA(w) \\
&\quad = e^{-2\alpha|\omega_0|^2}\int_{\mathbb{C}} e^{4\alpha\Re w\bar{\omega_0}-(2\alpha+\gamma)|w|^2}dA(w) \\
&\quad = e^{-2\alpha|\omega_0|^2}\int_0^\infty e^{-(2\alpha+\gamma)r^2}r\int_0^{2\pi} e^{4\alpha r|\omega_0|\cos(s-\theta)}dsdr \\
&\quad = e^{-2\alpha|\omega_0|^2}\int_0^\infty e^{-(2\alpha+\gamma)r^2}r\int_0^{2\pi} e^{4\alpha r|\omega_0|\cos s}dsdr \\
&\quad = e^{-2\alpha|\omega_0|^2}\int_0^\infty e^{-(2\alpha+\gamma)r^2}r\left(\int_{|\xi|=1}\frac{e^{2\alpha r|\omega_0|(\xi+\frac{1}{\xi})}}{i\xi}d\xi\right)dr \\
&\quad = 2\pi e^{-2\alpha|\omega_0|^2}\int_0^\infty e^{-(2\alpha+\gamma)r^2}r\left(Res_{\xi=0}\frac{e^{2\alpha r|\omega_0|(\xi+\frac{1}{\xi})}}{\xi}\right)dr \\
&\quad = 2\pi e^{-2\alpha|\omega_0|^2}\int_0^\infty e^{-(2\alpha+\gamma)r^2}rJ(0,4r\alpha|\omega_0|)dr,
\end{aligned} \tag{5}$$

where $J(0,z)$ is a modified Bessel function of the first kind given by the formula $J(0,z) = \sum_{k=0}^{\infty}\frac{(\frac{z}{2})^{2k}}{(k!)^2}$. By direct calculation one obtains

$$\int_0^\infty e^{-(2\alpha+\gamma)r^2}rJ(0,4r\alpha|\omega_0|)dr = \frac{e^{\frac{4\alpha^2|\frac{z+t}{2}|^2}{2\alpha+\gamma}}}{4\alpha+2\gamma}.$$

□

Using the similar type of calculations as it was done in the Lema 2.1 it is not hard to check that the operator $B_\alpha^*B_\alpha$ is a Hilbert-Schmidt.

Lemma 2.2. *The operator $B_\alpha^*B_\alpha$ is a Hilbert-Schmidt on L^2_β, and*

$$||B_\alpha^*B_\alpha||_2 = \frac{\pi^2\alpha}{\beta}\left(\frac{\gamma - \frac{2\alpha^2\gamma}{2\alpha\beta-2\alpha\gamma+\beta\gamma}}{(2\alpha+\gamma)(2\alpha-\beta)(2\alpha^2-\beta\gamma+2\alpha(\gamma-\beta))}\right)^{1/2}.$$

Proof. By direct calculation one obtains

$$\begin{aligned}
&\int_{\mathbb{C}}\int_{\mathbb{C}} |H(z,t)|^2 d\mu_\beta(z)d\mu_\beta(t) \\
&= C\times\int_{\mathbb{C}} e^{(\beta-2\alpha+\frac{2\alpha^2}{2\alpha+\gamma})|z|^2}\int_{\mathbb{C}} e^{(\beta-2\alpha+\frac{2\alpha^2}{2\alpha+\gamma})|t|^2+\frac{4\alpha^2}{2\alpha+\gamma}\Re t\bar{z}}dA(t)dA(z) \\
&= C\times\int_{\mathbb{C}} e^{(\beta-2\alpha+\frac{2\alpha^2}{2\alpha+\gamma})|z|^2} e^{\frac{4\alpha^4|z|^2}{(2\alpha+\gamma)(2\alpha^2-2\alpha\beta+2\alpha\gamma-\beta\gamma)}}dA(z) \\
&= C\times\int_{\mathbb{C}} e^{\frac{4\alpha^2\beta-2\alpha\beta^2-4\alpha^2\gamma+4\alpha\beta\gamma-\beta^2\gamma}{2\alpha^2-\beta\gamma+2\alpha(-\beta+\gamma)}|z|^2}dA(z).
\end{aligned} \tag{6}$$

Since

$$4\alpha^2\beta - 2\alpha\beta^2 - 4\alpha^2\gamma + 4\alpha\beta\gamma - \beta^2\gamma = (2\alpha\gamma - 2\alpha\beta - \beta\gamma)(\beta - 2\alpha) < 0$$

the last integral is finite the claim of the lemma follows.

□

The direct consequence of the last result is that

$$s_n(B_\alpha^* B_\alpha) = o(n^{-\frac{1}{2}}).$$

However, the stronger result is valid.

Theorem 2.3. *If $\gamma(2\alpha - \beta) > 2\alpha\beta$, then the operator $B_\alpha^* B_\alpha$ is nuclear and*

$$\frac{\pi\gamma}{\gamma - \beta} \leq \|B_\alpha^* B_\alpha\|_1 \leq \frac{\pi\gamma\alpha^2}{\beta(2\alpha\gamma - 2\beta\alpha - \beta\gamma)}.$$

Proof. Relying on Theorem 5.1 from [1] (pp.85) we will prove that the operator $B_\alpha^* B_\alpha$ is a weak limit of certain sequence of nuclear operators whose Schatten norms are uniformly bounded.

Let us consider the sequence of operators $\{T_n\}_{n\geq 1}$,

$$T_n : L^2_\beta \to L^2_\beta,$$

defined by

$$\begin{aligned} T_n f(z) =& C_{\alpha,\beta,\gamma} \\ &\times \sum_{k+m\leq n} \frac{(\frac{2\alpha^2}{2\alpha+\gamma})^{k+m}}{k!m!} \int_{\mathbb{C}} e^{(\beta-\alpha+\frac{\alpha^2}{2\alpha+\gamma})(|z|^2+|t|^2)} (z\bar{t})^k (t\bar{z})^m f(t) d\mu_\beta(t), \end{aligned}$$

where $C_{\alpha,\beta,\gamma} = \frac{\pi\gamma\alpha^2}{\beta^2(2\alpha+\gamma)}$, and m and k are nonnegative integers. It is not hard to check that the operators $\{T_n\}_{n\geq 1}$, belong to the class S_2. Moreover, the operators $\{T_n\}_{n\geq 1}$, are nonnegative induced with a continuous Hermitian nonnegative kernel.

Namely, if we denote by $K_n(z,t)$ the kernel of the operator T_n, then for any continuous function ϕ in $\mathbb{C}$, we have

$$\begin{aligned} &\int_{\mathbb{C}}\int_{\mathbb{C}} K_n(z,t)\phi(z)\overline{\phi(t)} d\mu_\beta(z) d\mu_\beta(t) \\ &= C_{\alpha,\beta,\gamma} \times \sum_{k+m\leq n} \frac{(\frac{\alpha^2}{2\alpha+\gamma})^{k+m}}{k!m!} \left(\int_{\mathbb{C}} e^{(\beta-\alpha+\frac{\alpha^2}{2\alpha+\gamma})|z|^2} \phi(z) z^k \bar{z}^m d\mu_\beta(z) \right)^2 \geq 0. \end{aligned} \tag{7}$$

According to the Theorem 10.1 from [1], T_n is a nuclear operator, and

$$\begin{aligned} &sp(T_n) = \|T_n\|_1 = \int_{\mathbb{C}} K_n(z,z) d\mu_\beta(z) \\ &= C'_{\alpha,\beta,\gamma} \times \\ &\sum_{k+m\leq n} \left(\frac{\alpha^2}{2\alpha^2 + 2\alpha\gamma - 2\alpha\beta - \beta\gamma} \right)^{k+m} \frac{\Gamma(1+k+m)}{\Gamma(1+k)\Gamma(1+m)} \\ &= C'_{\alpha,\beta,\gamma} \sum_{s=0}^{n} \left(\frac{2\alpha^2}{2\alpha^2 + 2\alpha\gamma - 2\alpha\beta - \beta\gamma} \right)^s, \end{aligned} \tag{8}$$

where $C'_{\alpha,\beta,\gamma} = \frac{\pi\gamma\alpha^2}{\beta(2\alpha^2+2\alpha\gamma-2\alpha\beta-\beta\gamma)}$.

Since $\frac{2\alpha^2}{2\alpha^2+2\alpha\gamma-2\alpha\beta-\beta\gamma}<1$, we have

$$\sup_{n\geq 1}\|T_n\|_1=\frac{\pi\gamma\alpha^2}{\beta(2\alpha\gamma-2\beta\alpha-\beta\gamma)}.$$

Further, let us consider $f,g\in C_c(\mathbb{C})$ (continuous functions with a compact support), then

$$\lim_{n\to+\infty}\langle T_nf,g\rangle=\langle B_\alpha^*B_\alpha f,g\rangle,$$

since the series

$$\sum_{k+m\leq n}\frac{(\frac{\alpha^2}{2\alpha+\gamma})^{k+m}}{k!m!}(z\bar{t})^k(t\bar{z})^m$$

converges uniformly on $supp(f)\times supp(g)$ to the function $e^{\frac{2\alpha^2}{2\alpha+\gamma}\Re z\bar{t}}$.

Due to the fact that for any functions $f,g\in L^2_\beta$ we may take the sequences $f_m,g_m\in C_c(\mathbb{C})$ such that f_m converges to f (g_m converges to g) in L^2_β, the difference

$$\begin{aligned}&|\langle T_nf,g\rangle_\beta-\langle Tf,g\rangle_\beta|\\&\leq|\langle T_nf,g\rangle_\beta-\langle T_nf,g_m\rangle_\beta|+|\langle T_nf,g_m\rangle_\beta-\langle T_nf_m,g_m\rangle_\beta|\\&+|\langle T_nf_m,g_m\rangle_\beta-\langle Tf,g\rangle_\beta|\end{aligned}\tag{9}$$

can be made arbitrary small for m (n) big enough. In other words, the sequence $\{T_n\}_{n\geq 1}$ converges weakly to the operator $B_\alpha^*B_\alpha$ in L^2_β.

Thus,

$$\|B_\alpha^*B_\alpha\|_1\leq\frac{\pi\gamma\alpha^2}{\beta(2\alpha\gamma-2\beta\alpha-\beta\gamma)}.$$

In order to obtain the estimate from below we consider the operator

$$P_\beta B_\alpha^*B_\alpha P_\beta:L^2_\beta\to L^2_\beta.$$

Clearly, the operator $P_\beta B_\alpha^*B_\alpha P_\beta$ acts as a restriction of the operator $B_\beta^*B_\alpha$ on the Fock space F^2_β.

Therefore,

$$\begin{aligned}\|B_\alpha^*B_\alpha\|_1&\geq\|P_\beta B_\alpha^*B_\alpha P_\beta\|_1\\&\geq\sum_{n=0}^{\infty}\left\langle P_\beta B_\alpha^*B_\alpha P_\beta\phi_n,\phi_n\right\rangle_\beta.\end{aligned}\tag{10}$$

In the last inequality of (10), the matrix trace of the operator $P_\beta B_\alpha^*B_\alpha P_\beta$ is defined with respect to the arbitrary orthonormal basis $\{\phi_n\}_{n\geq 0}$ in L^2_β.

In this particular case, for $\{\phi_n\}$ we will consider the standard orthonormal base in the Fock space F^2_β given by $\phi_n(z)=\sqrt{\frac{\beta^n}{n!}}z^n, n\geq 0$.

By direct calculation one obtains

$$B_\alpha^*B_\alpha P_\beta\phi_n=C^{\alpha,\beta,\gamma}z^n\left(\frac{\alpha}{\alpha+\gamma}\right)^n e^{(\beta-\frac{\alpha\gamma}{\alpha+\gamma})|z|^2},$$

where

$$C^{\alpha,\beta,\gamma}=\frac{\alpha\gamma\pi}{\beta(\alpha+\gamma)}\sqrt{\frac{\beta^n}{n!}}.$$

Therefore,

$$\sum_{n=0}^{\infty}\left\langle P_\beta B_\alpha^* B_\alpha P_\beta \phi_n, \phi_n\right\rangle_\beta = \sum_{n=0}^{\infty}\left\langle B_\alpha^* B_\alpha P_\beta \phi_n, \phi_n\right\rangle_\beta$$
$$= \pi \sum_{n=0}^{\infty}\left(\frac{\beta}{\gamma}\right)^n$$
$$= \frac{\pi\gamma}{\gamma-\beta}.$$

□

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