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Minimal 3-triangulations of p-toroids

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Abstract. It is known that we can always 3-triangulate (i.e. divide into tetrahedra with the original vertices) convex polyhedra but not always non-convex ones. Polyhedra topologically equivalent to ball with p handles, shortly p-toroids, cannot be convex. So, it is interesting to investigate possibilities and properties of their 3-triangulations. Here we study the minimal number of necessary tetrahedra for the triangulation of a 3-triangulable p-toroid. For that purpose, we developed the concept of piecewise convex polyhedron and that of its connection graph.

1. Introduction

We can start with a visual elementary picture of polygon, then polyhedron and d-dimensional polytope, by induction hypothesis on d, simply by defining usual triangle disk, then d-simplex. Afterwards we continue with "ordinary" 3-polyhedron, d-polytope, as connected union of finitely many d-simplices in a simple way.

Dividing a polygon by diagonals into triangles is called triangulation. It is known that we can triangulate each polygon with n vertices by $n-3$ diagonals into $n-2$ triangles.

Generalization of this process to higher $d \geq 3$ dimensions is also called triangulation. It consists of dividing polyhedron (polytope) into tetrahedra (simplices) using only the original vertices. There are two kinds of problems with triangulation in higher dimensions. It is proved that there is no possibility to triangulate certain non-convex polyhedra [8], [9] in three-dimensional space, and it is also proved that different triangulations of the same polyhedron may have different numbers of tetrahedra [5], [10], [11], [12]. Considering the smallest and the largest number of tetrahedra in triangulation (the minimal and the maximal triangulation), the authors obtained values, which linearly, resp. squarely, depend on the number of vertices. Interesting triangulations are described in papers of Edelsbrunner, Preparata, West [5] and Sleator, Tarjan, Thurston [10].

By the term "polyhedron" we usually mean a simple polyhedron solid, topologically equivalent to a ball. Though there are classes of polyhedra topologically equivalent to torus or p-torus (ball with p handles).

By Szilassi [18], torus-like polyhedra are called toroids. Generalizing that intuitive definition, we shall use the term p-toroid ($p \in \mathbb{N}$ is a given natural number) for p-torus-like polyhedron, and term toroid as a common name for any p-toroid (the Szilassi's toroid would be called 1-toroid). Since a toroid is not convex, it is questionable whether it is possible to 3-triangulate them. The 1-toroid with the smallest number of

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vertices is the Császár polyhedron [1], [2], [4], [17], [18]. It has 7 vertices and it is triangulable with 7 tetrahedra. Also, it is a polyhedron without diagonals as it was discussed in [4], [15], [16]. Some other examples of 1-toroids are given in [17], [18], while in [13], [14] 3-triangulations of 1-toroids and 2-toroids are discussed. In [3], [6] some combinatorial properties of p-toroids are given.

In Section 2 there are described some characteristic polyhedra, while in Section 3 we give some necessary definitions and properties of 3-triangulation of 1-toroids and 2-toroids. Later on, *convex d-polytope* and especially convex 3-polyhedron will be our standard and we introduce *piecewise convex polyhedron* as our main object (Definition 3.3). So we concretize the definition of p-toroid as well, restricted for our purposes in this paper. In Section 4, we prove in Theorem 4.1 that if a p-toroid with n vertices can be triangulated, then the minimal number of necessary tetrahedra is $T_{min} \geq n + 3(p-1)$, and in Theorem 4.2 that for each $n \geq 4p + 3$ exists 3-triangulable p-toroid with $T_{min} = n + 3(p-1)$. In Section 5 we discuss whether the statement of the Theorem 4.2 holds for p-toroids with smaller number n of vertices and mention some open questions.

2. Some characteristic examples of polyhedra and their 3-triangulation

2.1 Though we can triangulate all convex polyhedra, but this is not the case with non-convex ones. Lennes [7] was the first who presented a polyhedron whose interior cannot be triangulated without new vertices. The more famous example, however, was given by Schönhardt [9] and referred to in [8]. Schönhardt's polyhedron is obtained in the following way: triangulate the lateral faces of a trigonal prism $A_1B_1C_1A_2B_2C_2$ by the diagonals A_1B_2, B_1C_2 and C_1A_2. Then "twist" the top face $A_2B_2C_2$ by a small amount in the positive direction. In such a polyhedron, none of tetrahedra with vertices in the set $\{A_1, B_1, C_1, A_2, B_2, C_2\}$ is inner, so the triangulation is not possible.

2.2 It is proved that the smallest possible number of tetrahedra in the triangulation of a polyhedron with n vertices is $n-3$. An example of polyhedron triangulable with $n-3$ tetrahedra is a pyramid with $n-1$ vertices in the basis (i.e., a total of n vertices). We can triangulate it in the following way: we have to do any 2-triangulation of the basis into $(n-1)-2 = n-3$ triangles. Each of these triangles makes with the apex one of tetrahedra in 3-triangulation.

But, it is not possible to triangulate each polyhedron into $n-3$ tetrahedra. We shall see later that every triangulation of an octahedron (6 vertices) yields 4 tetrahedra.

2.3 Let us now consider triangulations of a bipyramid with $n-2$ vertices in the basis. The first method is to divide bipyramid into two pyramids and triangulate each of them, taking care of a common 2-triangulation of the basis, then we obtain $2(n-4)$ tetrahedra. In the second method, each of $n-2$ tetrahedra has a common edge joining the apices of the bipyramid, and moreover, each of them contains a pair of the neighbouring vertices of the basis (i.e., one of the edges of the basis).

If $n = 5$, such a bipyramid has a triangular basis. Then, the first method is "better", i.e. it gives smaller number of tetrahedra. For $n = 6$ (the octahedron), both methods give 4 tetrahedra, and for $n \geq 7$, the second method is "better". In Figure 1, triangulations of a bipyramid with a pentagonal basis (i.e. $n = 7$) are given. Dividing a bipyramid into two pyramids leads to triangulation with 6 tetrahedra, and dividing it around the axis V_1V_2 gives triangulation with 5 tetrahedra.

2.4 In [18], Szilassi intuitively introduced the term toroid. Here we shall use 1-toroid for that. He speaks on *"ordinary" polyhedron*, if for each vertex the edges and faces, incident to it afterwards each other, form a cycle. Dually, it is always assumed that each face is a *simple polygon*, i.e. its incident edges and vertices form a cycle.

Definition 2.1. *(Szilassi) An ordinary polyhedron is called* 1-toroid *if it is topologically torus-like, i.e. as a solid, it can be converted to a solid torus by continuous deformation.*

A 1-toroid with the smallest number of vertices is the Császár polyhedron (Figure 2). It has 7 vertices and no diagonals, i.e. each vertex is connected to six others by edges. In [1] Bokowski and Eggert proved

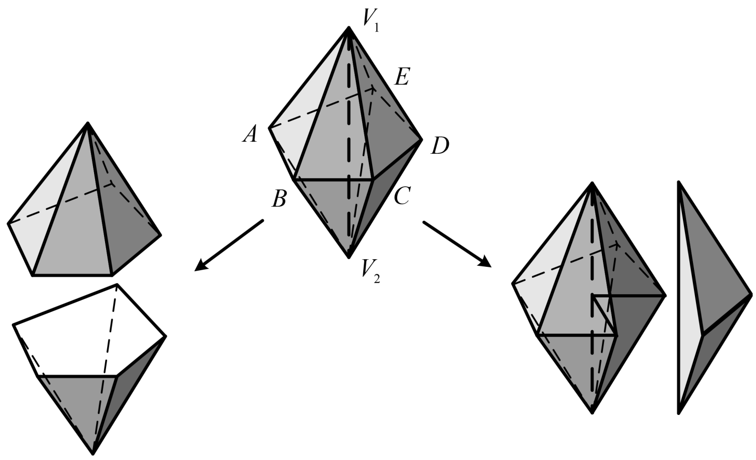

Figure 1: Triangulations of pentagonal bipyramid

that Császár polyhedron has four essentially different versions. It is to be noted that in topological terms various versions of Császár polyhedron are isomorphic – there is only one way to draw the full graph with seven vertices on the torus. In Wolfram Demonstrations Project [19] Szilassi shows that Császár polyhedron is 3-triangulable with 7 tetrahedra.

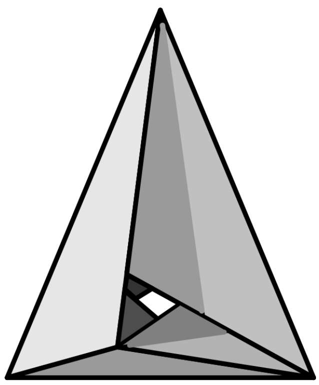

Figure 2: Császár polyhedron

3. Preliminaries

According to definition 2.1, we intuitively introduce the term p-toroid ($p \in \mathbb{N}$)

Definition 3.1. *A polyhedron as a solid is called* p-toroid, *$p \in \mathbb{N}$, if it is topologically equivalent to a ball with p handles (p-torus).*

There is an obvious formulation for polyhedral surfaces, but it needs additional assumptions for dimensions $d > 2$, where the naive concept of connected sum is not clear yet.

Let *toroid* be a common name for all p-toroids.

Remark 3.2. *Here we want to remind that in the surface theory* p-torus *is a cyclic polygon with paired sides. Any side s and its pair S are oppositely directed (then glued together), related to the fixed orientation of the polygon. Then - by a standard combinatorial procedure - the polygon can be divided and glued to a cyclic normal form*

$a_1b_1A_1B_1a_2b_2A_2B_2...a_pb_pA_pB_p$, as a p-torus. This combinatorial procedure is independent of the future spatial placement of the surface. So from any spatial knot (as a topological circle in the space) we can form a torus. Its surface can be triangulated, of course, to be a polyhedron. But this indicates that "knot polyhedra" would be our topic, implicitly, that we want to exclude later, see Definition 3.7.

In our consideration we shall also use the following

Definition 3.3. *A polyhedron is* piecewise convex *if it can be divided into finitely many convex polyhedra P_i, $i = 1, \dots, m$, with disjoint interiors. A pair of above polyhedra P_i, P_j is said to be* neighbouring *if they have a common face called* contact face.

If the above polyhedra P_i and P_j are not neighbouring, they may have a common edge e or a common vertex v. That is possible iff there is a sequence of neighbouring polyhedra $P_i, P_{i+1}, \dots, P_{i+k} \equiv P_j$ such that the edge e, or the vertex v belongs to each contact face f_l common to P_l and P_{l+1}, $l \in \{i, \dots, i+k-1\}$. Otherwise, polyhedra P_i and P_j do not have common points.

Remark 3.4. *Since a convex polyhedron can be 3-triangulated, the same holds for piecewise convex one, especially for a piecewise convex toroid.*

Remark 3.5. *Each 3-triangulable polyhedron is a collection of connected tetrahedra, so that is piecewise convex.*

In Figure 3, we give an example of 1-toroid P_9^1 with $n = 9$ vertices. It is composed of three pieces of convex polyhedra A which are topologically equivalent to triangular prisms. Polyhedron P_9^1 is an example of cyclically piecewise convex 1-toroid defined below.

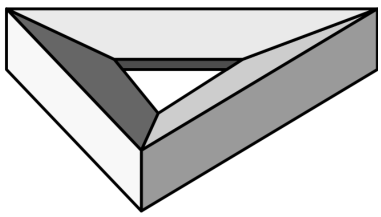

Figure 3: Cyclically piecewise convex polyhedron (1-toroid) P_9^1

Definition 3.6. *An 1-toroid is* cyclically piecewise convex *if it is possible to divide it into a cycle of convex polyhedra P_i, $i = 1, \dots, n$, such that P_i and P_{i+1}, $i = 1, \dots, n-1$ and P_n and P_1 are neighbours.*

If a polyhedron P is piecewise convex, let us form its *graph of connection* (or its *connection graph*), in such a way that *nodes* represent convex polyhedra P_i, $i = 1, \dots, m$, the pieces of P, while *edges* represent contact faces between them.

It is obvious that if a 1-toroid is cyclically piecewise convex, then its graph of connection is a single cycle. Other piecewise convex 1-toroids have graphs with a cycle and additional branches. It may also happen that the graph of connection for some piecewise convex 1-toroid have more cycles in situations when the convex pieces are cyclically connected to each other, but "glued" in such a way that there is no handle between them. In such cases, we can assume that the cycle, we shall call it "false", corresponds to a degenerate handle.

Similarly, a piecewise convex p-toroid has a graph of connection with p basic cycles, and eventually additional branches, but it can also has false cycles without solid handle. The 2-toroid P_{14}^2 given in Figure 4 have two cycles in both of its graphs in Figure 5. This 2-toroid has $n = 14$ vertices and it is composed of six pieces of A or of two 1-toroids P_9^1. It has two graphs of connection because the union of two A parts in the middle form convex polyhedron marked with $2 \cdot A$ in the second of graphs.

This example shows us that division of polyhedron to convex pieces is **not** necessarily unique.

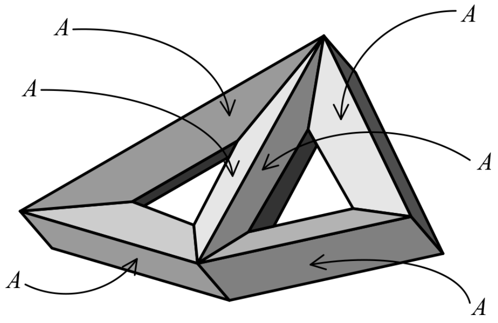

Figure 4: Piecewise convex 2-toroid P^2_{14}

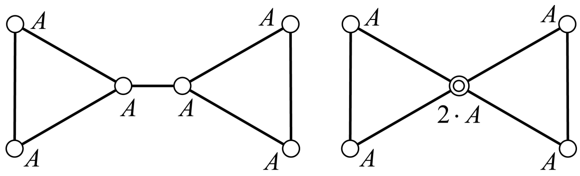

Figure 5: Two graphs of connection for the 2-toroid P^2_{14}

Two different cycles in a graph of 2-toroid can have a common node, or to be connected by an edge or by a branch. The first two cases we observe on the graphs of connection of P^2_{14} in Figure 5. As in this example, if two cycles of graph for toroid P have a common node, then corresponding cyclically piecewise convex pieces of P share common convex piece, and if they are connected by an edge, they have a contact face. In the third case, two cyclically piecewise convex pieces of P are connected by contact faces with a simple piecewise convex polyhedron inducing branch in the graph of connection. An example of such graph is given in Figure 7 describing 2-toroid P^2_{20} (Figure 6). P^2_{20} has $n = 20$ vertices and it is composed of two 1-toroids with $n = 10$ vertices connected by polyhedron A. In both of the figures convex polyhedron with $n = 7$ vertices is marked with B.

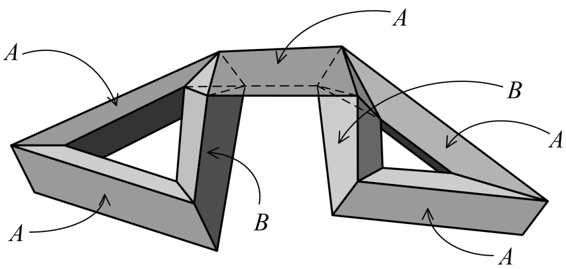

Figure 6: Piecewise convex 2-toroid P^2_{20}

In this paper we consider the minimal number of necessary simplices in 3-triangulation of a toroid P. So, it would be useful to deal with divisions and graphs of toroids in which the minimal 3-triangulation of P is in accordance with the minimal 3-triangulation of their pieces, namely, not to take care about this accordance. **It must not happen** that the sum of tetrahedra in minimal 3-triangulations of pieces would be greater than the number of tetrahedra in 3-triangulation of the whole toroid. Really, if contact face of two pieces is with $t \geq 5$ vertices, it may happen that we have around it the bipyramid R with t vertices in the basis. Then, each of two considered pieces would contain one of pyramids as a piece of the bipyramid R. As we have observed in 2.3, minimal 3-triangulation of R gives smaller number of tetrahedra then the sum of separate 3-triangulations of pyramids belonging to the pieces. So, let us define:

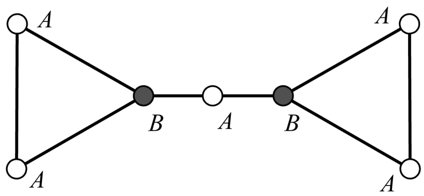

Figure 7: Graph of connection for the 2-toroid P^2_{20}

Definition 3.7. *An* m-division *of a polyhedron is a division in which the tetrahedra participating in the minimal 3-triangulations of the pieces are at the same time participating in the minimal 3-triangulation of the whole polyhedron. A graph of connection of a given polyhedron is* m-graph *if it represents m-division of that polyhedron.*

Remark 3.8. *We see that m-division and thus m-graph of a polyhedron is not unique. Note that convex pieces of division (m-division) can be either separated tetrahedra or their different collections. Besides that, more possibilities can occur for minimal 3-triangulation of the same polyhedron.*

On the other hand, it is obvious that there exists at least one m-division of a given 3-triangulable polyhedron. That is its partition into tetrahedra participating in the minimal 3-triangulation.

In [13] we have proved the next theorems for 1-toroids:

Theorem 3.9. *If a 1-toroid with $n \geq 7$ vertices can be 3-triangulated, then the minimal number of tetrahedra in that triangulation is $T_{min} \geq n$.*

Theorem 3.10. *For each $n \geq 7$, there is a 1-toroid, with n vertices, that can be 3-triangulated.*

Corresponding theorems for 2-toroids have been given in [14].

Theorem 3.11. *If it is possible to 3-triangulate 2-toroid with $n \geq 10$ vertices, then the minimal number of tetrahedra for that triangulation is $T_{min} \geq n + 3$.*

Theorem 3.12. *For $10 \leq n \in \mathbb{N}$, there is a 2-toroid, with n vertices, which can be 3-triangulated.*

4. 3-triangulations of *p*-toroids

In this section we shall discuss what is the minimal number of tetrahedra necessary for 3-triangulation of p-toroid with n vertices. First, we shall prove the next statement.

Theorem 4.1. *If a p-toroid with n vertices can be 3-triangulated, then the minimal number of tetrahedra necessary for its 3-triangulation is $T_{min} \geq n + 3(p - 1)$.*

There are more possibilities to connect pieces in m-graph of p-toroid, but in the proof of this theorem it is not necessary to consider them. Here, we shall prove Theorem 4.1 using the mathematical induction by the number of handles p.

Since false cycles can interfere us in proving the theorem, we shall introduce *optimized graph* of connection. Consider a toroid P and its graph of connection G that have one or more false cycles. For each of the false cycles, notice all the nodes that belong to it and the corresponding convex pieces of P. The union of such convex pieces for each false cycle builds a new node of optimized graph $\hat{G}$. The other nodes of the graph G remain in $\hat{G}$ and we shall call them the old ones. The edges between the old nodes remain in $\hat{G}$. The edges of G between some old node and some node belonging to a false cycle are converted to the edge of $\hat{G}$ between that old node and the new one.

The optimized graph $\hat{G}$ has the same number of basic cycles as it is the number of handles of the starting toroid P. Note that it is not necessary for the new nodes of the optimized graph to correspond to convex polyhedra, they only correspond to simple piecewise convex polyhedra. Also, if the graph G is an m-graph, the same property holds for graph $\hat{G}$.

Figure 8 shows a 6-toroid P^6_{32} with $n = 32$ vertices, which was created by gluing six 1-toroids P^1_9. It has six real handles and one degenerate. The two possible graphs of connection for P^6_{32} are given in Figure 9. On the left is given a graph G_1 whose nodes represent convex pieces A while a graph G_2 given on the right has six nodes representing $2 \cdot A$, and six representing A. Both graphs G_1 and G_2 have the same optimized graph $\hat{G}$, given in Figure 10. As in this example, we can see that in optimized graph it can appear cycle with only two nodes and two edges. That can happen if one of the nodes is old and the other is new, i.e. one piece of P is convex and the other is piecewise convex, so they can have two different common contact faces.

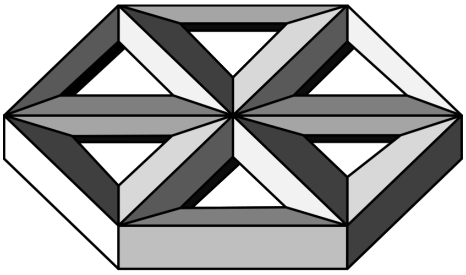

Figure 8: 6-toroid P^6_{32}

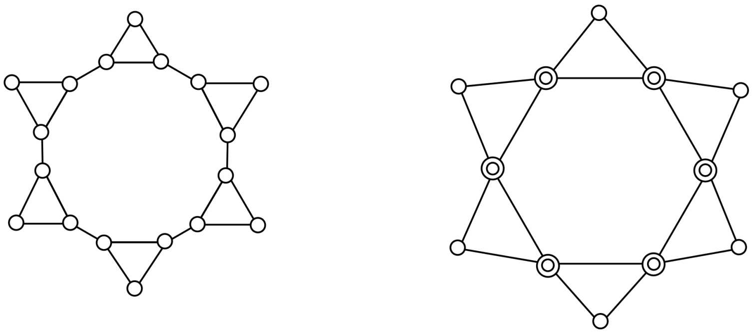

Figure 9: Two graphs of connection for P^6_{32}

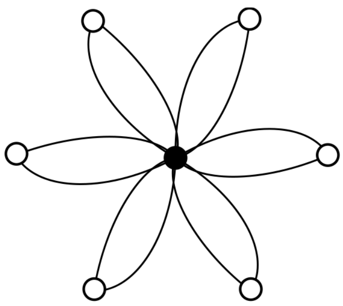

Figure 10: Optimized graph of connection for P^6_{32}

The proof of the Theorem 4.1 follows.

Proof. Note that Theorems 3.9 and 3.11 guarantee that statement is true for $p = 1, 2$. Let us suppose that statement is true for $p = k$, ($k \in \mathbb{N}$) i.e.

If a k-toroid ($k \in \mathbb{N}$) with n vertices can be 3-triangulated, then the minimal number of tetrahedra necessary for its 3-triangulation is $T_{min} \geq n + 3(k-1)$.

Observe in m-graph G of $(k+1)$-toroid P, optimized if it is necessary, an edge e which belongs to one or more cycles. Let us form a new graph $\bar{G}$ by excluding e from G. From Definition 3.7 of m-graph, it holds that graph $\bar{G}$ is also m-graph.

Denote by $\bar{P}$ the corresponding polyhedron of $\bar{G}$ and by $\bar{n}$ its number of vertices. $\bar{P}$ is obtained from P by "separating" convex pieces and by "duplicating" contact face with t vertices ($t \geq 3$) appropriate to the edge e in graph G. Thinking about definition of $(k+1)$-torus, excluding edge e from G is equivalent to cutting it by $a_{k+1}b_{k+1}A_{k+1}B_{k+1}$ and after that deforming it little bit.

If the edge e belongs to only one cycle of G, then that cycle in $\bar{G}$ would be missing. If e belong to more basic cycles of G, then in $\bar{G}$ earlier cycles would be merged and form one basic cycle less. So $\bar{P}$ is k-toroid, graph $\bar{G}$ have k basic cycles and for the number of the vertices of $\bar{P}$ is true

$$\bar{n} = n + t.$$

Since $\bar{P}$ has k handles, for its minimal triangulation by induction hypothesis holds

$$T_{min}(\bar{P}) \geq \bar{n} + 3(k-1) = n + t + 3(k-1).$$

Observe that $T_{min}(P) = T_{min}(\bar{P})$. That means

$$T_{min}(P) \geq n + t + 3(k-1) \geq n + 3\left((k+1) - 1\right),$$

thus the statement is true for $p = k + 1$. □

Considering the smallest number n of vertices in a 3-triangulable p-toroid, it will be necessary to take care about possible connecting the pieces of its m-graph. Note that some polyhedron might be topologically (combinatorially) realizable but not also geometrically. That is the reason to create more examples of topologically realizable p-toroids. Some checks whether their geometric realizations exist, will be left for a future paper. For the series of p-toroids, described in the proof of the following theorem, the realization is obvious.

Theorem 4.2. *For each $n \geq 4p + 3$ there is 3-triangulable p-toroid with $T_{min} = n + 3(p-1)$.*

Proof. First we shall form the main series of p-toroids $\bar{P}^{p}_{4p+3}$ by gluing p Császár's toroids into chain. Each pair of neighbour 1-toroids have a common contact face. The m-graphs of these p-toroids are formed of p heptagons connected by $p-1$ edges, as it is shown in the Figure 11.

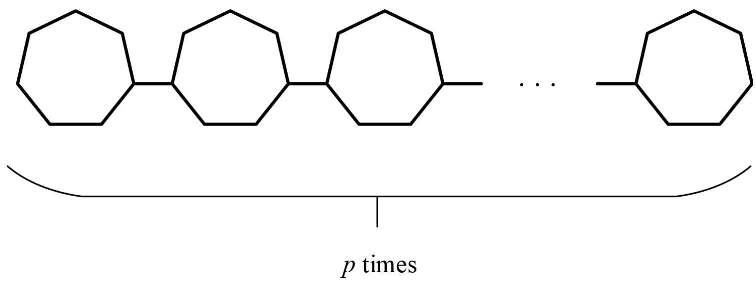

Figure 11: Graph of connection for p-toroid $\bar{P}^{p}_{4p+3}$

In p-toroids $\bar{P}^p_{4p+3}$ neighbour 1-toroids have 3 common vertices, so the total number of vertices of $\bar{P}^p_{4p+3}$ is $n = 4p + 3$. On the other hand, the number of tetrahedra in the 3-triangulation of $\bar{P}^p_{4p+3}$ is equal to $7p$, i.e. $T_{min}(\bar{P}^p_{4p+3}) = 7p$. Since for $n = 4p + 3$ holds

$$n + 3(p - 1) = (4p + 3) + 3(p - 1) = 7p = T_{min}(\bar{P}^p_{4p+3}),$$

our claim is true whenever $n = 4p + 3$.

If $n > 4p + 3$ we can take any simple polyhedron S_k with $k = n - (4p + 3) + 3 = n - 4p$ vertices, which has triangular faces and $T_{min}(S) = k - 3$, e.g. a pyramid with space $(k - 1)$-gon in the basis. Then p-toroid $\bar{P}^p_n$ can be formed by gluing p-toroid $\bar{P}^p_{4p+3}$ and S_k so that they have a common triangular face. Then, the number of vertices of $\bar{P}^p_n$ is

$$(4p + 3) + (n - 4p) - 3 = n,$$

and

$$\begin{aligned} T_{min}(\bar{P}^p_n) &= T_{min}(\bar{P}^p_{4p+3}) + T_{min}(S_k) \\ &= 7p + k - 3 \\ &= 7p + n - 4p - 3 \\ &= n + 3p - 3 \\ &= n + 3(p - 1). \end{aligned}$$

□

5. Closing remarks

A smaller number of vertices in p-toroid appears if in the main series of p Császár's toroids, neighbour ones have a common tetrahedron instead of a common face (Figure 12). Then number n of vertices in such $\hat{P}^p_{3p+4}$ is $n = 3p + 4$, while $T_{min}(\hat{P}^p_{3p+4}) = 6p + 1$. Since $6p + 1 = (3p + 4) + (3p - 3) = n + 3(p - 1)$, holds

$$T_{min}(\hat{P}^p_{3p+4}) = n + 3(p - 1).$$

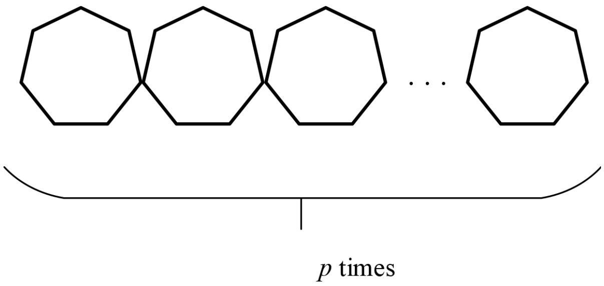

Figure 12: Graph of connection for p-toroid $\hat{P}^p_{3p+4}$

But, for $\hat{P}^p_{3p+4}$ it is questionable if it has geometric realization. In [14] double-Császár 2-toroid, which is $\hat{P}^2_{10}$ from this series was introduced. It was proved that it is 3-triangulable 2-toroid with the smallest number of vertices, $n = 10$. It seems likely that this 2-toroid has geometric realization. Next toroid in this series is $\hat{P}^3_{13}$ is composed of three Császár's 1-toroids. Since all three Császár's 1-toroids which build $\hat{P}^3_{13}$ have at least one common vertex, geometric realization of $\hat{P}^3_{13}$ is not so obvious.

On the other hand, if this chain is geometrically realizable, we can think if it is possible for some enough great p to close new circle to form $\tilde{P}^{p+1}_{3p}$ (Figure 13). Then this 'cycle of cycles' would be $(p+1)^{\text{th}}$ cycle, the number of vertices would be $n = 3p$, and again

$$\begin{aligned} T_{min}(\tilde{P}^{p+1}_{3p}) &= T_{min}(\hat{P}^{p}_{3p+4}) - 1 = \\ &= 6p \\ &= n + 3\left((p+1) - 1\right). \end{aligned}$$

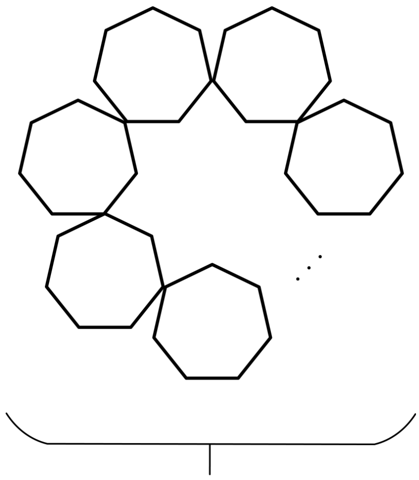

Figure 13: Graph of connection for $(p+1)$-toroid $\tilde{P}^{p+1}_{3p}$

Of course, it is also possible to think of closing cycle of Császár's 1-toroids in the chain of type $\tilde{P}^{p}_{4p+3}$. Such $(p+1)$-toroid would have $n = 4p$ vertices and T_{min} would be equal to $7p$. Again it would be $7p = n + 3\left((p+1) - 1\right)$.

Note that the smallest possible number of vertices for p-toroid is considered in [6], in a combinatorial way. It is proved for example, that the minimal number of vertices for 2-toroids and 3-toroids is $n = 10$. Geometric realization is not considered in that paper. Also, it is not known if those toroids are 3-triangulable. In [3] is proved that 3-toroid with $n = 10$ vertices have geometric realization, but its 3-triangulability remains open.

In this paper, the graph of connection is introduced for an arbitrary p-toroid. Consideration in the opposite direction, from the graph to the toroid, promises to explore new properties of 3-triangulations of the toroids and to open new problems.

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