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On (σ, τ)-derivations of Lie superalgebras

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Abstract. This paper is primarily devoted to studying (σ, τ)-derivations of finite-dimensional Lie superalgebras over an algebraically closed field $\mathbb{F}$. We research some properties of (σ, τ)-derivations and the relationship between the (σ, τ)-derivations and other generalized derivations. Under certain conditions, a left-multiplication structure concerned with (σ, τ)-derivations can induces a left-symmetric superalgebra structure. Let L be a Lie superalgebra, we give a subgroup G of $\mathrm{Aut}(L)$, exploiting fundamental properties, we introduce and analyze their interiors, especially focusing on the rationality of the corresponding Hilbert series when G is a cyclic group.

1. Introduction

As everyone knows, derivations and generalized derivations are very important subjects both in the study of Lie superalgebras. Many geometric and algebraic properties of Lie superalgebras can be studied based on knowledge of the corresponding super-spaces of derivations. Derivations are also applied to physical problems, such as the study of the interaction of particles (see [26]). A lot of work on derivations of Lie superalgebras was done by Kac in [22] and Scheunert in [27].

Many papers have summarized the concept of derivations of a ring in different ways to suit different purposes (see [24], [6], [17], [19], etc). The concept of generalized derivations of rings was introduced by Brešar [5]. Arzhach, Albash [1] and Argaç, Inceboz [2] introduced the notion of (σ, τ)-derivation based on Brešar. In [31], derivation superalgebras, quasiderivation superalgebras, centroids, quasi-centroids and generalized derivation superalgebras of finite-dimensional Lie superalgebras in two different characteristic domains are studied. In the earlier period, Kharchenko and Popov [21], Bergen and Grzeszczuk [9], Chuang and Lee [10] studied the skew derivations. Jung and Park. [20] study the generalized (α, β)-derivations and commutativity in prime rings. Hartwig, Larsson and Silvestrov [18] introduced the (σ, τ)-derivations of associative C-algebras and developed an approach to deformations of the Witt and Virasoro algebras based on σ-derivations. Recent years, Filippis and Wei [14] study b-generalized (α, β)-derivations and b-generalized (α, β)-biderivations of prime rings. H. Chang [11] studies a kind of new generalized derivation of Lie algebras. In order to study the theory and application of generalized derivations. In the following,

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we study the (σ,τ)-derivations of finite-dimensional Lie superalgebras over an algebraically closed field $\mathbb{F}$ of characteristic zero.

This paper is organized as follows. In section 2, we introduce the (σ,τ)-derivations and show their pertinent notions. In section 3, we research some properties of (σ,τ)-derivations and the relationship with derivations of Lie superalgebras. In particular, if a (σ,σ)-derivation is invertible, we use the properties of left-symmetric superalgebras, obtain a left-symmetric superalgebra structure induced by a left-multiplication structure concerned with (σ,τ)-derivations. In section 4, we refer to the relationship with quasiderivations, (α,β,γ)-derivations of Lie superalgebras, and give some examples. In the last section, we study the interior of G-derivations, giving related concepts and properties. In Particular, when G is a cyclic group, we calculate the corresponding Hilbert series.

2. Preliminary

Suppose that $L = L_{\bar{0}} \oplus L_{\bar{1}}$ and $\mathrm{Aut}(L)$ denotes the automorphism groups of L. If $|x|$ occurs in some expression in this paper, we regard x as a Z_2-homogeneous element and $|x|$ as the Z_2-degree of x. We denote by $h(L)$ as the set of all Z_2-homogeneous elements of L, i.e., $h(L) = h(L_i)$, $i = \bar{0}, \bar{1}$ and the notations θ, λ, μ denote the elements of Z_2-degree unless stated otherwise. We denote

$$\mathrm{pl}_\theta(L) = \{D \in \mathrm{Hom}(L,L) | D(L_\mu) \subseteq L_{\mu+\theta},\ \forall \mu, \theta \in Z_2\}.$$

We know that $\mathrm{pl}(L) = \mathrm{pl}_{\bar{0}}(L) \oplus \mathrm{pl}_{\bar{1}}(L)$ is a Lie superalgebra over $\mathbb{F}$ with the bracket product

$$[D_\theta, D_\mu] := D_\theta D_\mu - (-1)^{\theta\mu} D_\mu D_\theta,\ \forall D_\theta, D_\mu \in \mathrm{pl}_\theta(L).$$

A linear transformation $D \in \mathrm{pl}_\theta(L)$ is called a derivation of Z_2-degree θ if

$$D[x,y] = [D(x), y] + (-1)^{\theta|x|}[x, D(y)] \tag{1}$$

for any $x, y \in h(L)$.

Denote by $\mathrm{Der}(L) = \mathrm{Der}_{\bar{0}}(L) \oplus \mathrm{Der}_{\bar{1}}(L)$ as the set of all derivation superalgebras of L.

Definition 2.1. *[31] A linear transformation $D \in \mathrm{pl}_\theta(L)$ is called a quasiderivation of Z_2-degree θ if there exists an element $D' \in \mathrm{pl}_\theta(L)$ such that*

$$D'[x,y] = [D(x), y] + (-1)^{\theta|x|}[x, D(y)] \tag{2}$$

for any $x, y \in h(L)$. Let us denote the set of all quasiderivations by $\mathrm{QDer}(L)$.

Definition 2.2. *[29] A linear transformation $D \in \mathrm{pl}_\theta(L)$ is said to be an (α,β,γ)-derivation of Z_2-degree θ if there exist $\alpha,\beta,\gamma \in \mathbb{F}$ such that*

$$\alpha D([x,y]) = \beta[D(x), y] + (-1)^{\theta|x|}\gamma[x, D(y)], \tag{3}$$

for any $x, y \in h(L)$. For $\alpha,\beta,\gamma \in \mathbb{F}$, the set of all (α,β,γ)-derivations of Z_2-degree θ is denoted by $D(\alpha,\beta,\gamma)_\theta$, i.e.,

$$D(\alpha,\beta,\gamma)_\theta = \left\{D \in \mathrm{pl}_\theta(L) \,\middle|\, \alpha D([x,y]) = \beta[D(x), y] + (-1)^{\theta|x|}\gamma[x, D(y)],\ \forall x, y \in h(L)\right\}.$$

Let $D(\alpha,\beta,\gamma) = D(\alpha,\beta,\gamma)_{\bar{0}} \oplus D(\alpha,\beta,\gamma)_{\bar{1}}$ be the (α,β,γ)-derivations of L.

Definition 2.3. *Let G be a subgroup of* $\mathrm{Aut}(L)$. *A linear transformation $D \in \mathrm{pl}_\theta(L)$ is said to be a G-derivation of Z_2-degree θ if there exist two automorphisms $\sigma, \tau \in G$ such that*

$$D[x,y] = [D(x), \sigma(y)] + (-1)^{\theta|x|}[\tau(x), D(y)], \tag{4}$$

for any $x, y \in h(L)$.

In the following, we write $\mathrm{Der}_G(L)$ be the set of all G-derivations of L. When $G = 1$ is the trivial subgroup, then $\mathrm{Der}_G(L) = \mathrm{Der}(L)$. Therefore, G-derivations can be used as a kind of generalized derivations. If S and G are two subgroups of $\mathrm{Aut}(L)$, and $S \leqslant G$, we obtain $\mathrm{Der}_S(L) \subseteq \mathrm{Der}_G(L)$. In particular, when $G \leqslant \mathrm{Aut}(L)$, we have $\mathrm{Der}(L)$ is contained in $\mathrm{Der}_G(L)$. In this paper, we suppose that G is a subgroup of $\mathrm{Aut}(L)$.

Definition 2.4. *Let G be a subgroup of $\mathrm{Aut}(L)$. Fix two automorphisms $\sigma, \tau \in G$, if there exists a linear transformation $D_{\sigma,\tau} \in \mathrm{pl}_\theta(L)$ of Z_2-degree θ such that*

$$D_{\sigma,\tau}[x, y] = [D_{\sigma,\tau}(x), \sigma(y)] + (-1)^{\theta|x|}[\tau(x), D_{\sigma,\tau}(y)],\ \forall x, y \in h(L), \tag{5}$$

then $D_{\sigma,\tau}$ is called a (σ,τ)-derivation.

For $\sigma, \tau \in G$, the set of all (σ, τ)-derivations of Z_2-degree θ is denoted by $\mathrm{Der}_{\sigma,\tau}(L)$, i.e.,

$$\mathrm{Der}_{\sigma,\tau}(L) = \{D : L \to L | D[x, y] = [D(x), \sigma(y)] + (-1)^{\theta|x|}[\tau(x), D(y)],\ \forall x, y \in h(L)\}.$$

It is easy to see that $\mathrm{Der}_{\sigma,\tau}(L) \subseteq \mathrm{Der}_G(L)$ is a vector space. However $\mathrm{Der}_G(L)$ is usually not a vector space, because many different (σ, τ)-derivations are contained in G-derivations. In particular, if $\sigma = \tau = 1$, then $\mathrm{Der}_{1,1}(L) = \mathrm{Der}(L)$. Hence, we can see that $\mathrm{Der}_\sigma(L) = \mathrm{Der}_{\sigma,1}(L)$.

Definition 2.5. *Let G be a subgroup of $\mathrm{Aut(L)}$. Fix two automorphisms $\sigma, \tau \in G$, if there exists a linear transformation $D_{\sigma^n,\tau^m} \in \mathrm{pl}_\theta(L)$ of Z_2-degree θ and $m, n \in \mathbb{N}^+$ such that*

$$D_{\sigma^n,\tau^m}[x, y] = [D_{\sigma^n,\tau^m}(x), \sigma^n(y)] + (-1)^{\theta|x|}[\tau^m(x), D_{\sigma^n,\tau^m}(y)],\ \forall x, y \in h(L), \tag{6}$$

then D_{σ^n,τ^m} is called a (σ^n,τ^m)-derivations.

For $\sigma, \tau \in G$, the set of all (σ^n, τ^m)-derivations of Z_2-degree θ is denoted by $\mathrm{Der}_{\sigma^n,\tau^m}(L)$, i.e.,

$$\mathrm{Der}_{\sigma^n,\tau^m}(L) = \{D : L \to L | D[x, y] = [D(x), \sigma^n(y)] + (-1)^{\theta|x|}[\tau^m(x), D(y)],\ \forall x, y \in h(L)\}.$$

When σ, τ are involutive automorphisms of G, then $\mathrm{Der}_{\sigma^n,\tau^m}(L) = \mathrm{Der}(L)$.

3. The relationship with derivations

Theorem 3.1. *Let L be a Lie superalgebra. For every $\sigma, \tau \in G$, then $\mathrm{Der}_{\sigma,\tau}(L) \cong \mathrm{Der}_{\tau^{-1}\sigma}(L)$.*

Proof. We can define a map

$$\phi_\tau : \mathrm{Der}_{\sigma,\tau}(L) \to \mathrm{Der}_{\tau^{-1}\sigma}(L)\ (i.e., \mathrm{Der}_{\tau^{-1}\sigma,1}(L))$$

by $D \mapsto \tau^{-1} \circ D$, $D \in \mathrm{Der}_{\sigma,\tau}(L)$.

First, we check that this map is well defined. For any $D \in \mathrm{Der}_{\sigma,\tau}(L)$, we have

$$\begin{aligned} \tau^{-1} \circ (D[x, y]) &= \tau^{-1}([D(x), \sigma(y)] + (-1)^{\theta|x|}[\tau(x), D(y)]) \\ &= [\tau^{-1} \circ D(x), \tau^{-1}\sigma(y)] + (-1)^{\theta|x|}[x, \tau^{-1} \circ D(y)]. \end{aligned}$$

Hence $\tau^{-1} \circ D \in \mathrm{Der}_{\tau^{-1}\sigma}(L)$. It is easy to see

$$\phi_\tau(D_1 + D_2) = \tau^{-1} \circ (D_1 + D_2) = (\tau^{-1} \circ D_1 + \tau^{-1} \circ D_2),$$

where $D, D_1, D_2 \in \mathrm{Der}_{\sigma,\tau}(L)$. Similarly, we have

$$\phi_\tau(a \cdot D) = \tau^{-1} \circ (a \cdot D) = a \cdot \tau^{-1} \circ D,$$

where $D \in \mathrm{Der}_{\sigma,\tau}(L)$, $a \in \mathbb{F}$. Which means that ϕ_τ is a linear map. In an analogous way, one can define a map

$$\psi_\tau : \mathrm{Der}_{\tau^{-1}\sigma}(L) \to \mathrm{Der}_{\sigma,\tau}(L)$$

by $D \mapsto \tau \circ D$, $D \in \mathrm{pl}_\theta(L)$. It is easy to see ψ_τ a linear map. Then $\phi_\tau^{-1} = \psi_\tau$. This implies that ϕ_τ is a linear isomorphism, thus $\mathrm{Der}_{\sigma,\tau}(L)$ is isomorphic to $\mathrm{Der}_{\tau^{-1}\sigma}(L)$. □

Remark 3.2. *The above result shows that the study of* $\mathrm{Der}_{\sigma,\tau}(L)$ *with two parameters* σ *and* τ *can actually be simplified to that of* $\mathrm{Der}_{\sigma}(L)$ *with one parameter* $\sigma \in G$. *In particular, when* $\sigma = \tau$, *we see that* $\mathrm{Der}_{\sigma,\tau}(L)$ *and* $\mathrm{Der}(L)$ *are isomorphic into vector space. The following result shows that it can be extended to such an isomorphism of Lie superalgebras.*

Lemma 3.3. *Let any* $\sigma \in G$, *then* $\mathrm{Der}_{\sigma,\sigma}(L)$ *is a Lie superalgebra with the bracket product* $[\cdot,\cdot]_\sigma$ *and* $\mathrm{Der}_{\sigma,\sigma}(L)$ *is isomorphic to* $\mathrm{Der}(L)$.

Proof. For any $\sigma \in G$, define the bilinear map

$$\phi_\sigma : \mathrm{Der}_{\sigma,\sigma}(L) \to \mathrm{Der}(L)$$

by $D \mapsto \sigma^{-1} \circ D$, $D \in \mathrm{pl}_\theta(L)$. The proof is similar to the proof in Theorem 3.1, so we can easy to see $\mathrm{Der}_{\sigma,\sigma}(L) \cong \mathrm{Der}(L)$. □

Recall the definition of the left-symmetric superalgebra in [23]. A superalgebra L is called a left-symmetric superalgebra, if the associator

$$(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z)$$

of L satisfies

$$(x, y, z) = (-1)^{\theta\lambda}(y, x, z)$$

where $x \in L_\theta$, $y \in L_\lambda$, $z \in L$, $\theta, \lambda \in Z_2$.

By definition, the product $x \cdot y$ in L satisfies the following conditions

$$x \cdot (y \cdot z) - (x \cdot y) \cdot z = (-1)^{\theta\lambda} y \cdot (x \cdot z) - (-1)^{\theta\lambda}(y \cdot x) \cdot z$$

$$[x, y] = x \cdot y - (-1)^{\theta\lambda} y \cdot x$$

where $x \in L_\theta$, $y \in L_\lambda$, $z \in L$, $\theta, \lambda \in Z_2$. The left-multiplication $\mathfrak{L}$ in L by $\mathfrak{L}(x)(y) = x \cdot y$.

The two conditions are equivalent to

(1) $\mathfrak{L} : L \to \mathrm{pl}_\theta(L)$ is a Lie superalgebra homomorphism;

(2) $\mathrm{Id} : L \to L_{\mathfrak{L}}$ is a 1-cocycle in $Z^1(L, L_{\mathfrak{L}})$,

where $L_{\mathfrak{L}}$ denotes the L-module by $\mathfrak{L}$. $Z^1(L, L_{\mathfrak{L}})$ is the space of 1-cocycles in regard to $L_{\mathfrak{L}}$.

Recall a L-module V, the space of 1-cocycles is given by

$$Z^1(L, V) = \{\alpha \in \mathrm{Hom}(L, V) | \, \alpha([x, y]) = x \cdot \alpha(y) - (-1)^{\theta\lambda} y \cdot \alpha(x)\}.$$

Proposition 3.4. *A finite-dimensional Lie superalgebra L admits a left-symmetric structure if and only if there is an L-module V such that the vector space* $Z^1(L, V)$ *contains a nonsingular* 1*-cocycle.*

Proof. Suppose that $\varphi \in Z^1(L, V)$ is a nonsingular 1-cocycle with inverse transformation φ^{-1}. The module V corresponds to a linear representation $\chi : L \to \mathrm{pl}_\theta(L)$. Then $\mathfrak{L}(x) := \varphi^{-1} \circ \chi(x) \circ \varphi$ defines an L-module W such that $\varphi^{-1} \circ \varphi = \mathrm{Id} \in Z^1(L, W)$. It follows that

$$\mathfrak{L} : L \to \mathrm{pl}_\theta(L)$$

is a representation of L, and

$$\mathrm{Id}([x, y]) = \mathrm{Id}(x)y - (-1)^{|x||y|}\mathrm{Id}(y)x$$

is a bijective 1-cocycle in $Z^1(L, L_{\mathfrak{L}})$. Hence $\mathfrak{L}(x)y = x \cdot y$ defines a left-symmetric structure on L. Conversely, L admits a left-symmetric structure, then Id is a nonsingular 1-cocycle. □

Corollary 3.5. *Suppose that* $D \in \mathrm{Der}_{\sigma,\sigma}(L)$ *is invertible. Then* $\mathfrak{L}_*(x)(y) := D^{-1}([\sigma(x), D(y)])$ *induces a left-symmetric superalgebra structure on L.*

Proof. Define a bilinear map

$$\phi_\sigma : \mathrm{Der}_{\sigma,\sigma}(L) \to \mathrm{Der}(L)$$

by $D \mapsto \sigma^{-1} \circ D$, $D \in \mathrm{Der}_{\sigma,\sigma}(L)$. Let $D_1 \in \mathrm{Der}(L)$ is invertible, according to Proposition 3.4, we have the left-multiplication $\mathfrak{L}_*$ in L by $\mathfrak{L}_*(x)(y) := [D_1{}^{-1}(x), y] = D_1{}^{-1}([x, D_1(y)])$ induces a left-symmetric superalgebra on L. Let $L_{\mathfrak{L}_*}$ denotes the L-module by $\mathfrak{L}_*$. Putting $D_1 = \phi_\sigma(D)$, we have

$$\mathfrak{L}_*(x)(y) = (\phi_\sigma(D))^{-1}([x, \phi_\sigma(D)(y)]) = D^{-1} \circ \sigma([x, \sigma^{-1} \circ D(y)]) = D^{-1}([\sigma(x), D(y)]),$$

thus, $Z^1(L, L_{\mathfrak{L}_*})$ contains a nonsingular 1-cocycle. By Proposition 3.4, we obtain

$$\mathfrak{L}_*(x)(y) = D^{-1}([\sigma(x), D(y)])$$

can gives a left-symmetric superalgebra on L. □

In fact, the derivation superalgebra $\mathrm{Der}(L)$ is also a Lie superalgebra, and vector space $\mathrm{Der}_\sigma(L)$ is usually not a subalgebra of $\mathrm{pl}_\theta(L)$. From the following results, it can be concluded that under certain conditions, $\mathrm{Der}_\sigma(L)$ and $\mathrm{Der}(L)$ may be equal, and $\mathrm{Der}_\sigma(L)$ is a Lie superalgebra.

Let $y \in L$, then $Z_x(L)$ called a center, if it satisfies

$$Z_x(L) = \{x \in L | [x, y] = 0\}.$$

Lemma 3.6. *Suppose that $\sigma, \tau \in G$. If $(\sigma - \tau)(L) \in Z(L)$, then $\mathrm{Der}_\sigma(L) = \mathrm{Der}_\tau(L)$. In particular, if $(\sigma - \mathrm{Id})(L) \in Z(L)$, then $\mathrm{Der}_\sigma(L) = \mathrm{Der}(L)$.*

Proof. Since $\sigma(y) - \tau(y)$ is contained in $Z(L)$ with $y \in L$, therefore

$$[D(x), \sigma(y)] = [D(x), \tau(y)]$$

with $x \in L$ and $D \in \mathrm{pl}_\theta(L)$. In particular, let any $D \in \mathrm{Der}_\sigma(L)$ of Z_2-degree θ, we have

$$D([x, y]) = [D(x), \sigma(y)] + (-1)^{\theta|x|}[x, D(y)] = [D(x), \tau(y)] + (-1)^{\theta|x|}[x, D(y)] \subseteq \mathrm{Der}_\tau(L).$$

It follows that $D \in \mathrm{Der}_\tau(L)$ and $\mathrm{Der}_\sigma(L) \subseteq \mathrm{Der}_\tau(L)$. Similarly, the reverse is also true. Hence, we can obtain $\mathrm{Der}_\sigma(L)$ is equal to $\mathrm{Der}_\tau(L)$. □

Lemma 3.7. *Suppose that σ is a involutive element of G with $\sigma \neq \mathrm{id}$. If σ commutes with every element of $\mathrm{Der}(L)$ and $\mathrm{Der}_\sigma(L)$, then $\mathrm{Der}(L) + \mathrm{Der}_\sigma(L)$ is a Lie superalgebra.*

Proof. It is sufficient to verify that $\mathrm{Der}(L) + \mathrm{Der}_\sigma(L)$ is close under the usual bracket

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{\theta\lambda} D_2 \circ D_1 \in \mathrm{Der}(L) + \mathrm{Der}_\sigma(L).$$

For any $D_1 \in \mathrm{Der}_{\sigma^k}(L)$ and $D_2 \in \mathrm{Der}_{\sigma^l}(L)$ with $k, l \in \{0, 1\}$ and $x, y \in L$, we have

$$\begin{aligned} D_1 \circ D_2([x, y]) &= D_1([D_2(x), \sigma^l(y)] + (-1)^{\lambda|x|}[x, D_2(y)]) \\ &= [D_1 \circ D_2(x), \sigma^{k+l}(y)] + (-1)^{\theta\lambda + \theta|x|}[D_2(x), D_1 \circ \sigma^l(y)] \\ &\quad + (-1)^{\lambda|x|}[D_1(x), \sigma^k \circ D_2(y)] + (-1)^{(\theta+\lambda)|x|}[x, D_1 \circ D_2(y)] \end{aligned}$$

and

$$\begin{aligned} D_2 \circ D_1([x, y]) &= D_2([D_1(x), \sigma^k(y)] + (-1)^{\theta|x|}[x, D_1(y)]) \\ &= [D_2 \circ D_1(x), \sigma^{l+k} \circ (y)] + (-1)^{\lambda\theta + \lambda|x|}[D_1(x), D_2 \circ \sigma^k(y)] \\ &\quad + (-1)^{\theta|x|}[D_2(x), \sigma^l \circ D_1(y)] + (-1)^{(\theta+\lambda)|x|}[x, D_2 \circ D_1(y)]. \end{aligned}$$

notice that σ commutes with every element of $\mathrm{Der}(L)$ and $\mathrm{Der}_\sigma(L)$, so we have

$$\begin{aligned}
[D_1, D_2]([x, y]) &= (D_1 \circ D_2 - (-1)^{\theta\lambda} D_2 \circ D_1)([x, y]) \\
&= [D_1 \circ D_2(x), \sigma^{k+l}(y)] + (-1)^{\theta\lambda+\theta|x|}[D_2(x), D_1 \circ \sigma^l(y)] \\
&\quad +(-1)^{\lambda|x|}[D_1(x), \sigma^k \circ D_2(y)] + (-1)^{(\theta+\lambda)|x|}[x, D_1 \circ D_2(y)] \\
&\quad -[D_2 \circ D_1(x), \sigma^{l+k} \circ (y)] - (-1)^{\lambda\theta+\lambda|x|}[D_1(x), D_2 \circ \sigma^k(y)] \\
&\quad -(-1)^{\theta|x|}[D_2(x), \sigma^l \circ D_1(y)] - (-1)^{(\theta+\lambda)|x|}[x, D_2 \circ D_1(y)] \\
&= [D_1 \circ D_2(x), \sigma^{k+l}(y)] + (-1)^{(\theta+\lambda)|x|}[x, D_1 \circ D_2(y)] \\
&\quad -[D_2 \circ D_1(x), \sigma^{l+k} \circ (y)] - (-1)^{(\theta+\lambda)|x|}[x, D_2 \circ D_1(y)] \\
&= [[D_1, D_2](x), \sigma^{k+l}(y)] + (-1)^{(\theta+\lambda)|x|}[x, [D_1, D_2](y)].
\end{aligned}$$

which implies that $[D_1, D_2] \in \mathrm{Der}_\sigma^{k+l}(L)$. Note that $k, l \in \{0, 1\}$ and σ is an involution, we have $\sigma^{k+l} = \sigma$ or $\sigma^{k+l} = \mathrm{id}$. Hence $[D_1, D_2] \in \mathrm{Der}_\sigma(L)$ or $[D_1, D_2] \in \mathrm{Der}(L)$. Therefore, $\mathrm{Der}(L)+\mathrm{Der}_\sigma(L)$ is a Lie superalgebra. □

Lemma 3.8. *Let $\sigma, \tau \in G$ be two automorphisms of L such that $(\sigma^{-1}\tau)(x)$ does not belong to $Z_x(L)$ with $x \in L$ and $x \neq 0$, then $\mathrm{Der}_\sigma(L) \cap \mathrm{Der}_\tau(L) = 0$.*

Proof. Suppose there is a non-zero $D \in \mathrm{Der}_\sigma(L) \cap \mathrm{Der}_\tau(L)$ to prove the contradiction. Since $D \in \mathrm{Der}_\sigma(L) \cap \mathrm{Der}_\tau(L)$, which means that

$$[D(x), \sigma(y)] = [D(x), \tau(y)],$$

where any $x, y \in L$. Thus,

$$[\sigma^{-1} \circ D(x), y - (\sigma^{-1}\tau)(y)] = 0. \tag{7}$$

There exists a $D \in L$, such that $y = \sigma^{-1} \circ D(x)$, substitute equation (7), we can obtain

$$[y, y - (\sigma^{-1}\tau)(y)] = 0.$$

Then $[y, (\sigma^{-1}\tau)(y)] = 0$, it follows that $(\sigma^{-1}\tau)(y) \in Z_y(L)$. But by hypothesis, $(\sigma^{-1}\tau)(x)$ does not belong to $Z_x(L)$ with $x \in L$ and $x \neq 0$, it means that $y = 0$. By the arbitrariness of x and σ^{-1} is bijective, hence $D = 0$. This contradiction shows that the conclusion is established. □

Let $D \in \mathrm{Der}_\sigma(L)$ be a derivation of a Lie superalgebra L. Then L is called periodic if there exists a positive integer $k \in \mathbb{N}^+$ such that $D^k = \mathrm{Id}$. The minimum k such that $D^k = \mathrm{Id}$ is called the order of D.

Proposition 3.9. *Suppose that L is a Lie superalgebra and $D \in \mathrm{Der}_\sigma(L)$ is a periodic derivation of order k and Z_2-degree θ. Let α, β are two eigenvectors of D such that $[\alpha, \beta] \neq 0$ and $(\sigma - \mathrm{Id}) \in Z(L)$, then k is divisible by 6.*

Proof. On account of D is of finite order, which means that D is diagonalizable. Suppose the eigenvectors α, β correspond to eigenvalues r, r' respectively, such that

$$D(\alpha) = r\alpha, \;\; D(\beta) = r'\beta.$$

Hence

$$\begin{aligned}
D([\alpha, \beta]) &= [D(\alpha), \sigma(\beta)] + (-1)^{\theta|\alpha|}[\alpha, D(\beta)] \\
&= [r'\alpha, \sigma(\beta)] + (-1)^{\theta|\alpha|}[\alpha, r\beta] \\
&= [r'\alpha, \beta] + (-1)^{\theta|\alpha|}[\alpha, r\beta] \\
&= (r' + (-1)^{\theta|\alpha|}r)[\alpha, \beta].
\end{aligned}$$

It follows that $r' + (-1)^{\theta|\alpha|}r$ is an eigenvalue of D. Since $D^k = \mathrm{Id}$, we have

$$r^k = r'^k = (r + (-1)^{\theta|\alpha|}r')^k = 1. \tag{8}$$

According to Lemma 2.2 of [3], we can see that $r' = s \cdot r$, where s denotes a primitive third root of unity. Which means that $r'^k = s^k \cdot r^k$, thus $s^k = 1$. It follows that $k = 3k'$ for some $k' \in \mathbb{N}^+$. In addition, substitute equation (8), we can get

$$1 = (r + (-1)^{\theta|\alpha|}r')^k = (r + (-1)^{\theta|\alpha|}s \cdot r)^k,$$

then $(1 + (-1)^{\theta|\alpha|}s)^k = 1$. Since k is a primitive root, it follows that $0 = s^3 \pm 1 = (s \pm 1)(s^2 \mp s + 1)$, thus $s^2 \mp s + 1 = 0$. This implies that $1 \mp s = -s^2$. It means that

$$1 = (1 \mp s)^k = (-1)^k s^{2k} = (-1)^k (s^k)^2 = (-1)^k.$$

We observe that k is an even. Consequently, 2 divides k and $k = 6k''$ for some $k'' \in \mathbb{N}^+$. □

The following result proves the commutativity of $D \in \mathrm{Der}_\sigma(L)$ and σ of perfect Lie superalgebras. Recall a Lie superalgebra L is called perfect if $[L, L] = L$.

Proposition 3.10. *Suppose that L is a nonabelian Lie superalgebra. Let $D \in \mathrm{Der}_\sigma(L)$ of Z_2-degree θ such that $[D, \sigma](L) \subseteq Z(L)$, then $[L, L] \subseteq \ker([D, \sigma])$. In particular, if L is perfect, then D commutes with σ.*

Proof. Let any $x, y \in L$, in fact that

$$D \circ \sigma([x, y]) = [D \circ \sigma(x), \sigma^2(y)] + (-1)^{\theta|x|}[\sigma^2(x), D \circ \sigma(y)]$$

and

$$\sigma \circ D([x, y]) = [\sigma \circ D(x), \sigma^2(y)] + (-1)^{\theta|x|}[\sigma^2(x), \sigma \circ D(y)].$$

Since the assumption $[D, \sigma](L) \subseteq Z(L)$, which means that

$$\begin{aligned} [D, \sigma]([x, y]) &= (D \circ \sigma - (-1)^{\theta|\sigma|}\sigma \circ D)[x, y] \\ &= [[D, \sigma](x), \sigma^2(y)] + (-1)^{\theta|x|}[\sigma^2(x), [D, \sigma](y)] \\ &= 0. \end{aligned}$$

Hence, $[L, L] \subseteq \ker([D, \sigma])$. In addition, when L is perfect, we have

$$L = [L, L] \subseteq \ker([D, \sigma]) \subseteq L.$$

It is easy to see that $\ker([D, \sigma]) = L$ and $[D, \sigma] = 0$, as desired. □

Recall the definition of centroid. Let L be a Lie superalgebra, and the centroid of L of Z_2-degree θ is a space of linear transformations on L given by

$$D([x, y]) = [D(x), y] = (-1)^{\theta|x|}[x, D(y)],$$

where $x, y \in L$. The centroid $C(L)$ is a subalgebra of $\mathrm{pl}_\theta(L)$. Recall the mapping $\mathrm{ad} : L \to \mathrm{pl}_\theta(L)$ denotes the adjoint map by $x \mapsto \mathrm{ad}x$.

Definition 3.11. *Let G be a subgroup of* Aut(L), *$\sigma \in G$ be an automorphism, we call D be a σ-centroid of Lie superalgebra if*

$$D([x, y]) = [D(x), \sigma(y)] = (-1)^{\theta|x|}[x, D(y)], \tag{9}$$

for all $x, y \in L$, $D \in \mathrm{Der}_\sigma(L)$ of Z_2-degree θ.

Lemma 3.12. *Suppose that any element $\sigma \in G$. If $D \in C(L) \cap \mathrm{Der}_\sigma(L)$, then $\mathrm{ad} \circ D = 0$. In particular, if $Z(L) = 0$, then $C(L) \cap \mathrm{Der}_\sigma(L) = 0$.*

Proof. By the assumption that $D \in C(L) \cap \mathrm{Der}_\sigma(L)$, we have

$$[D(x), \sigma(y)] = 0$$

where any $x, y \in L$. In fact that $D(x) \in Z(L) = \ker(\mathrm{ad})$, since σ is bijective. Thus $\mathrm{ad} \circ D = 0$. In particular, if $Z(L) = 0$, the map ad is injective. We know that ad is an adjoint map, thus it has a right inverse. Thus $D = 0$, as desired. □

Lemma 3.13. *Suppose that arbitrary element $\sigma \in G$, $D \in \mathrm{Der}_\sigma(L)$ of Z_2-degree θ. Then*

$$[D, \mathrm{ad}x] = \sigma \circ \mathrm{ad}_{\sigma^{-1} \circ D(x)},$$

for any $x \in L$.

Proof. In fact,

$$\begin{aligned} [D, \mathrm{ad}x](y) &= (D \circ \mathrm{ad}x - (-1)^{\theta|x|}\mathrm{ad}x \circ D)(y) \\ &= D([x, y]) - (-1)^{\theta|x|}[x, D(y)] \\ &= [D(x), \sigma(y)] + (-1)^{\theta|x|}[x, D(y)] - (-1)^{\theta|x|}[x, D(y)] \\ &= [D(x), \sigma(y)] \\ &= \sigma([\sigma^{-1} \circ D(x), y]) \\ &= \sigma \circ \mathrm{ad}_{\sigma^{-1} \circ D(x)}(y), \end{aligned}$$

for all $x, y \in L$. Consequently,$[D, \mathrm{ad}x] = \sigma \circ \mathrm{ad}_{\sigma^{-1} \circ D(x)}$. □

Lemma 3.14. *Let any element $x \in L$ and $\sigma \in G$, we define a map $\phi_x^\sigma : \mathrm{Der}_\sigma(L) \to \mathrm{ad}(L)$, by $D \mapsto \mathrm{ad}_{\sigma^{-1} \circ D(x)}$. Then ϕ_x^σ is linear.*

Proof. Set any $D_1, D_2 \in \mathrm{Der}_\sigma(L)$ and $y \in L$, then

$$\begin{aligned} \phi_x^\sigma(D_1 + D_2)(y) &= \mathrm{ad}_{\sigma^{-1} \circ (D_1 + D_2)(x)}(y) = [\sigma^{-1} \circ (D_1 + D_2)(x), y] \\ &= \sigma^{-1}([D_1(x), \sigma(y)] + [D_2(x), \sigma(y)]) \\ &= [\sigma^{-1} \circ D_1(x), y] + [\sigma^{-1} \circ D_2(x), y] \\ &= (\phi_x^\sigma(D_1) + \phi_x^\sigma(D_2))(y), \end{aligned}$$

thus,

$$\phi_x^\sigma(D_1 + D_2)(y) = (\phi_x^\sigma(D_1) + \phi_x^\sigma(D_2))(y).$$

And

$$\begin{aligned} \phi_x^\sigma(a \cdot D)(y) &= \mathrm{ad}_{\sigma^{-1} \circ (a \cdot D)(x)}(y) \\ &= [\sigma^{-1} \circ (a \cdot D)(x), y] \\ &= [a \cdot (\sigma^{-1} \circ D)(x), y] \\ &= a \cdot \phi_x^\sigma(D)(y), \end{aligned}$$

where $a \in \mathbb{F}$. It follows that $\phi_x^\sigma(a \cdot D) = a \cdot \phi_x^\sigma(D)$. Which means that ϕ_x^σ is linear. □

Proposition 3.15. *Suppose that arbitrary element $x \in L$ and $\sigma \in G$. Then we have $\ker(\phi_x^\sigma) = \{D \in \mathrm{Der}_\sigma(L) | D(x) \in Z(L)\}$. In addition, $\ker(\phi_x^\sigma)$ is a subalgebra of $\mathrm{pl}_\theta(L)$.*

Proof. According to the Lemma 3.13 and Lemma 3.14, we have

$$\begin{aligned}\ker(\phi_x^\sigma) &= \{D \in \mathrm{Der}_\sigma(L) | \mathrm{ad}_{\sigma^{-1}\circ D(x)}(y) = 0,\ \forall y \in L\}\\ &= \{D \in \mathrm{Der}_\sigma(L) | \sigma \circ \mathrm{ad}_{\sigma^{-1}\circ D(x)}(y) = 0,\ \forall y \in L\}\\ &= \{D \in \mathrm{Der}_\sigma(L) | [D, \mathrm{ad}x](y) = 0,\ \forall y \in L\}\\ &= \{D \in \mathrm{Der}_\sigma(L) | [D(x), \sigma(y)] = 0,\ \forall y \in L\}.\end{aligned}$$

Then we can get $\{D \in \mathrm{Der}_\sigma(L) | [D(x), y] = 0,\ \forall\, y \in L\}$, hence $\{D \in \mathrm{Der}_\sigma(L) | D(x) \in Z(L)\}$. Which means that $\ker(\phi_x^\sigma)$ is a vector space. In the following, we will show that $\ker(\phi_x^\sigma)$ is a Lie superalgebra. Let any $y \in L$ and $D_1, D_2 \in \ker(\phi_x^\sigma)$ of Z_2-degree θ, λ respectively. It follows that

$$\begin{aligned}[[D_1, D_2](x), \sigma(y)] &= [D_1 \circ D_2(x), \sigma(y)] - (-1)^{\theta+\lambda}[D_2 \circ D_1(x), \sigma(y)]\\ &= D_1([D_2(x), y]) - (-1)^{\theta|x|}[D_2(x), D_1(y)] - D_2([D_1(x), y])\\ &\quad +(-1)^{\lambda|x|+\theta+\lambda}[D_1(x), D_2(y)]\\ &= 0,\end{aligned}$$

since $D_2(x), D_1(x) \in Z(L)$. We know that σ is a bijective, thus $[D_1, D_2](x) \in Z(L)$. Which means that $[D_1, D_2] \in \ker(\phi_x^\sigma)$, hence $\ker(\phi_x^\sigma)$ is a Lie superalgebra. Consequently, $\ker(\phi_x^\sigma)$ is a subalgebra of $\mathrm{pl}_\theta(L)$. □

Corollary 3.16. *Suppose that L is a centerless Lie superalgebra. If there exists an element $x \in L$ such that*

$$D(x) \neq 0,\ \forall\, D \in \mathrm{Der}_\sigma(L),$$

then $\dim(\mathrm{Der}_\sigma(L)) \leq \dim(L)$.

Proof. In fact that $\mathrm{ad} : L \to \mathrm{ad}(L)$ is an isomorphism, since $Z(L) = 0$. According to Proposition 3.15, we can obtain ϕ_x^σ is injective. Thus, we have $\phi_x^\sigma(D)$ is a subspace of L, it follows that $\phi_x^\sigma(D)$ can be embedded into L. □

The results of the Proposition 3.15 and Corollary 3.16 describe the kernel of ϕ_σ^x.

4. Relationship with (Generalized) Derivations

In this section, we research the relationship between (σ, τ)-derivations and certain (generalized) derivations of Lie superalgebras. Including the relationships with (α, β, γ)-derivations and quasiderivations of Lie superalgebras.

4.1. The relation with (α, β, γ)-derivations

Theorem 4.1. *Let any element $\sigma \in G$, there exists an element $a \in \mathbb{F}(a \neq 1)$ such that*

$$(\sigma - a \cdot \mathrm{Id})(L) \subseteq Z(L),$$

then $\mathrm{Der}_\sigma(L) = D_{(1/(a+1),1,0)}(L)$.

Proof. Let any element $D \in \mathrm{Der}_\sigma(L)$ of Z_2-degree θ. Since $(\sigma - a \cdot \mathrm{Id})(L) \subseteq Z(L)$, which means that

$$[D(x), \sigma(y)] = [D(x), ay]$$

where any $x, y \in L$. Therefore

$$D([x, y]) = [D(x), \sigma(y)] + (-1)^{\theta|x|}[x, D(y)]$$

where any $x, y \in L$. It follows that

$$D([x, y]) = a \cdot [D(x), y] + (-1)^{\theta|x|}[x, D(y)]$$

where any $x, y \in L$. Thus $D \in D_{(1,a,1)}(L)$, i.e., $\mathrm{Der}_\sigma(L) = D_{(1,a,1)}(L)$. We know $a - 1 \neq 0$, according to the proof of the Theorem 2, part 2(a) in [29], we can obtain $D_{(1,a,1)}(L) = D_{(1/(a+1),1,0)}(L)$. The proof is complete. □

Example 4.2. *Let L_2 be a nonabelian two dimensional Lie superalgebra with a basis $\{e_1, e_2\}$ that only nonzero commutation relation is $[e_1, e_2] = e_2$, where $|e_1| = \bar{0}$.*

(1)When $(\sigma - \mathrm{Id})(L_2) \in Z(L_2)$, by the Example 5 in [29], we can get

$$\mathrm{Der}_\sigma(L_2) = \mathrm{span}_C \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong L_2.$$

(2)When $(\sigma - a \cdot \mathrm{Id})(L_2) \in Z(L_2)$ where $a \in \mathbb{F}(a \neq 0, 1)$, we have $\mathrm{Der}_\sigma(L_2) = \{0\}$ by the Example 5 in [29].

Example 4.3. *Let $\mathfrak{sl}(1,1)$ be a non-simple three dimensional Lie superalgebra with a basis $\{e_1, e_2, e_3\}$, where $e_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ that the only nonzero commutation relation of the basis is $[e_1, e_2] = e_3$. And $|e_1| = |e_2| = \bar{1}, |e_3| = \bar{0}$. When $(\sigma - a \cdot I)(\mathfrak{sl}(1,1)) \in Z(\mathfrak{sl}(1,1))$, where $a \in \mathbb{F}(a \neq 1)$, by the Example 6 in [29], we can get*

$$\mathrm{Der}_\sigma(\mathfrak{sl}(1,1)) = \mathrm{span}_C \left\{ \begin{pmatrix} 1/(a+1) & 0 & 0 \\ 0 & 1/(a+1) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}.$$

4.2. The relation with quasiderivations

We denote $\mathrm{Der}_{\sigma,H}(L)$ as follow

$$\mathrm{Der}_{\sigma,H}(L) = \{D \in \mathrm{Der}_\sigma(L) | D(H) \subseteq H\}.$$

the set of all σ-derivations of L that stabilizes H. Clearly, $\mathrm{Der}_{\sigma,H}(L)$ is a subspace of $\mathrm{Der}_\sigma(L)$.

Lemma 4.4. *Let H be a subalgebra of Lie superalgebra L. If H is a perfect ideal of L, σ is a cyclic automorphism of G such that σ commutes with every element of $\mathrm{Der}_\sigma(L)$, then*

$$\mathrm{Der}_{\sigma,H}(L) = \mathrm{Der}_\sigma(L).$$

Proof. Let arbitrary $D \in \mathrm{Der}_\sigma(L)$ of Z_2-degree θ and $x \in H$. Since $[H, H] = H$, we have $x = [y, z]$ with certain $y, z \in H$. Further, as H is stabilized by σ and Lemma 3.7, thus

$$D(x) = D([y, z]) = [D(y), \sigma(z)] + (-1)^{\theta|y|}[y, D(z)] \in H.$$

It implies that $D(H) \subseteq H$. Thus

$$\mathrm{Der}_{\sigma,H}(L) = \mathrm{Der}_\sigma(L),$$

as desired. □

Since H is stabilized by σ, it follows that $\sigma|_H$ of H as an automorphism by σ restricts to H, i.e., $\sigma(H) \subseteq H$. The restriction map induces a natural linear map as follows

$$\delta : \mathrm{Der}_{\sigma,H}(L) \to \mathrm{Der}_{\sigma|_H}(L)$$

by $D \to D|_H$. Set $D \in \mathrm{Der}_{\sigma,H}(L)$, defining a map

$$\mathfrak{D} : [L, L] \to L$$

by

$$[x, y] \mapsto 2D([x, y]) + (-1)^{|x||y|}[D(y), \sigma(x)] - [D(x), \sigma(y)]$$

where $x, y \in L$.

Proposition 4.5. *Assume that there exists an element $z \in L$ such that $\mathrm{ad}_z \in \mathrm{Aut}(H)$, If $\mathfrak{D}$ restricts to H there is a linear map $\mathfrak{D}|_H : H \to H$. Then $\mathfrak{D}$ is linear.*

Proof. On account of
$$\mathrm{ad}_z(H) = [z, H] \subseteq [L, L],$$
for any $x \in H$, there exists a $y \in H$ such that
$$x = [z, y],$$
as well as,
$$\begin{aligned}\mathfrak{D}(x) = \mathfrak{D}([z, y]) &= 2D([x, y]) + (-1)^{|x||y|}[D(y), \sigma(x)] - [D(x), \sigma(y)] \\ &\in D(H) + [D(H), H] - [D(H), H] \\ &\subseteq H.\end{aligned}$$
Thus $\mathfrak{D}(H) \subseteq H$. By the restriction map $\mathfrak{D}|_H$ is linear, we also assume any $x_0 \in H$. Then there is a $y_0 \in H$ such that $x_0 = [z, y_0]$, we have
$$\begin{aligned}\mathfrak{D}(ax + a_0x) &= \mathfrak{D}([z, ay]) + \mathfrak{D}([z, a_0y_0]) \\ &= 2D([z, ay]) + (-1)^{|z||y|}[D(ay), \sigma(z)] - [D(z), \sigma(ay)] \\ &\quad +2D([z, a_0y_0]) + (-1)^{|z||y_0|}[D(a_0y_0), \sigma(z)] - [D(z), \sigma(a_0y_0)] \\ &= a\mathfrak{D}(x) + a_0\mathfrak{D}(x_0)\end{aligned}$$
where $a, a_0 \in \mathbb{F}$. It is similar that
$$\begin{aligned}\mathfrak{D}(bx) &= 2D([z, by]) + (-1)^{|x||y|}[D(by), \sigma(z)] - [D(z), \sigma(by)] \\ &= b\mathfrak{D}[z, y] \\ &= b\mathfrak{D}(x)\end{aligned}$$
where $b \in \mathbb{F}$. Consequently, $\mathfrak{D}$ is linear. □

Theorem 4.6. *If there exists an $x \in L$ such that $\mathrm{ad}x \in \mathrm{Aut}(H)$ and $D(x) \in Z(H)$ where $D \in \mathrm{Der}_{\sigma,H}(L)$ of Z_2-degree θ. Then*
$$\delta(\mathrm{Der}_{\sigma,H}(L)) \subseteq \mathrm{QDer}(H).$$

Proof. Since $D(x) \in Z(H)$, which means that
$$D \circ \mathrm{ad}_x(y) = D([x, y]) = (-1)^{\theta|x|}[x, D(y)] = (-1)^{\theta|x|}\mathrm{ad}_x \circ D(y),$$
where $\forall x, y \in H$ and $D \in \mathrm{Der}_{\sigma,H}(L)$. Thus
$$D \circ \mathrm{ad}x = (-1)^{\theta|x|}\mathrm{ad}x \circ D.$$
Furthermore, we have
$$\begin{aligned}\mathfrak{D}|_H \circ \mathrm{ad}_x([y, z]) &= \mathfrak{D}|_H([\mathrm{ad}_x(y), \mathrm{ad}_x(z)]) \\ &= 2D([\mathrm{ad}_x(y), \mathrm{ad}_x(z)]) + (-1)^{|y||z|}[D \circ \mathrm{ad}_x(z), \sigma \circ \mathrm{ad}_x(y)] - [D \circ \mathrm{ad}_x(y), \sigma \circ \mathrm{ad}_x(z)] \\ &= D([\mathrm{ad}_x(y), \mathrm{ad}_x(z)]) - [D \circ \mathrm{ad}_x(y), \sigma \circ \mathrm{ad}_x(z)] \\ &\quad -(-1)^{|y||z|}D([\mathrm{ad}_x(z), \mathrm{ad}_x(y)]) + (-1)^{|y||z|}[D \circ \mathrm{ad}_x(z), \sigma \circ \mathrm{ad}_x(y)] \\ &= [D \circ \mathrm{ad}_x(y), \mathrm{ad}_x(z)] + (-1)^{\theta|y|}[\mathrm{ad}_x(y), D \circ \mathrm{ad}_x(z)] \\ &= \mathrm{ad}_x([D(y), z] + (-1)^{\theta|y|}[y, D(z)]),\end{aligned}$$
where $x, y, z \in H$. It follows that
$$[D(y), z] + (-1)^{\theta|y|}[y, D(z)] = \mathrm{ad}_x^{-1} \circ \mathfrak{D}|_H \circ \mathrm{ad}_x([y, z]).$$

It is easy to see $\mathfrak{D}|_H \in \mathrm{QDer}(H)$. Hence

$$\delta(\mathrm{Der}_{\sigma,H}(L)) \subseteq \mathrm{QDer}(H).$$

We complete the proof of this theorem. □

5. The interior of the G-derivations

We have observed from the proof of Proposition 3.7 that the commutability with σ is important to make $\mathrm{Der}_\sigma(L)$ a nice algebraic structure. In order to understand the structure of $\mathrm{Der}_\sigma(L)$ and $\mathrm{Der}_G(L)$, we study several special subspaces of $\mathrm{Der}_\sigma(L)$ and we also observe that the Hilbert series is a subspace when G is a cycle group.

For the set $\mathrm{Der}_\sigma(L)$, when it is composed of elements that are commute with σ, it is recorded as $\mathrm{Der}^+_\sigma(L)$, and when the element is commuting with every $\sigma \in G$, it is recorded as $\mathrm{Der}^-_\sigma(L)$. It is easy to see that

$$\mathrm{Der}^-_\sigma(L) \subseteq \mathrm{Der}^+_\sigma(L) \subseteq \mathrm{Der}_\sigma(L).$$

We find that $\mathrm{Der}_G(L)$ is usually not a vector space. In addition, for any $\sigma \in G$, $\mathrm{Der}^+_\sigma(L)$, $\mathrm{Der}^-_\sigma(L)$ and $\mathrm{Der}_\sigma(L)$ are subsets of $\mathrm{Der}_G(L)$, and they are vector spaces. Therefore, we consider all $\mathrm{Der}^\pm_\sigma(L)$ or some "sum"s of $\mathrm{Der}_\sigma(L)$, then we can see how close they are from these "sum"s to $\mathrm{Der}_G(L)$. In the following, we define

$$\mathrm{Der}^+_G(L) := \bigoplus_{\sigma\in G} \mathrm{Der}^+_\sigma(L)$$

is called the big interior, and

$$\mathrm{Der}^-_G(L) := \bigoplus_{\sigma\in G} \mathrm{Der}^-_\sigma(L),$$

is called the small interior of $\mathrm{Der}_G(L)$. Then, we may define

$$\mathrm{Der}^\star_G(L) := \bigoplus_{\sigma\in G} \mathrm{Der}_\sigma(L)$$

is called the interior of $\mathrm{Der}_G(L)$. Then we have

$$\mathrm{Der}^-_G(L) \subseteq \mathrm{Der}^+_G(L) \subseteq \mathrm{Der}^\star_G(L).$$

The following two examples are both valid in Lie superalgebra and Lie algebra.

Example 5.1. *Let L be a Lie superalgebra. Considering $G = 1$ is a trivial group. Then* $\mathrm{Der}^-_G(L) = \mathrm{Der}_G(L) = \mathrm{Der}(L)$, *we have*

$$\mathrm{Der}^-_G(L) = \mathrm{Der}^+_G(L) = \mathrm{Der}^\star_G(L) = \mathrm{Der}_G(L) = \mathrm{Der}(L).$$

Example 5.2. *Let L be a Lie superalgebra. Considering G is a cyclic group generated by σ. Therefore* $\mathrm{Der}^+_\sigma(L) = \mathrm{Der}^-_\sigma(L)$. *After that, let $* \in \{+, -, \star\}$, then* $\mathrm{Der}^*_G(L) = \mathrm{Der}^*_{\langle\sigma\rangle}(L) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Der}^*_{\sigma^k}(L)$ *with $\sigma^0 = 1$, $\sigma^1 = \sigma$, $\sigma^k = \sigma^{k-1} \circ \sigma$. Following, we formulate* $\mathrm{Der}^*_{\sigma^k}(L) := \mathrm{Der}_{\sigma^k}(L)$. *Under this circumstance,* $\mathrm{Der}^*_{\langle\sigma\rangle}(L)$ *is a Z-graded vector space. Recall the Hilbert series of* $\mathrm{Der}^*_{\langle\sigma\rangle}(L)$ *is defined by*

$$\mathfrak{H}(\mathrm{Der}^*_{\langle\sigma\rangle}(L), t) := \sum_{k\in\mathbb{Z}} \dim_{\mathbb{C}}(\mathrm{Der}^*_{\sigma^k}(L)) \cdot t^k.$$

When the order of σ is finite, $\mathfrak{H}(\mathrm{Der}^*_{\langle\sigma\rangle}(L), t)$ is a polynomial function in $\mathbb{Z}[t]$.

Proposition 5.3. *If G is an abelian group, then* $\mathrm{Der}^-_G(L)$ *is a Lie superalgebra with the usual bracket product.*

Proof. Since $\mathrm{Der}_G^-(L)$ is a vector space, which means that

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{\theta\lambda} D_2 \circ D_1 \in \mathrm{Der}_G^-(L)$$

where any $D_1, D_2 \in \mathrm{Der}_G^-(L)$ and D_1, D_2 of Z_2-degree θ, λ respectively. Set $D_1 \in \mathrm{Der}_{\sigma_1}^-(L)$ and $D_2 \in \mathrm{Der}_{\sigma_2}^-(L)$, where $\sigma_1, \sigma_2 \in G$ are automorphisms. Then we have

$$\begin{aligned} D_1 \circ D_2([x, y]) &= [D_1 \circ D_2(x), \sigma_1\sigma_2(y)] + (-1)^{\theta(\lambda+|x|)}[D_2(x), D_1 \circ \sigma_2(y)] \\ &\quad +(-1)^{\lambda|x|}[D_1(x), \sigma_1 \circ D_2(y)] + (-1)^{(\theta+\lambda)|x|}[x, D_1 \circ D_2(y)] \end{aligned}$$

and

$$\begin{aligned} D_2 \circ D_1([x, y]) &= [D_2 \circ D_1(x), \sigma_2\sigma_1(y)] + (-1)^{\lambda(\theta+|x|)}[D_1(x), D_2 \circ \sigma_1(y)] \\ &\quad +(-1)^{\theta|x|}[D_2(x), \sigma_2 \circ D_1(y)] + (-1)^{(\theta+\lambda)|x|}[x, D_2 \circ D_1(y)], \end{aligned}$$

where $x, y \in L$. We know that G is abelian, it is suffices to show that

$$\begin{aligned} [D_1, D_2]([x, y]) &= (D_1 \circ D_2 - (-1)^{\theta\lambda} D_2 \circ D_1)([x, y]) \\ &= [[D_1, D_2](x), \sigma_1\sigma_2(y)] + (-1)^{\theta\lambda}[x, [D_1, D_2](y)], \end{aligned}$$

which means that $[D_1, D_2] \in \mathrm{Der}_{\sigma_1\sigma_2}^-(L) \subseteq \mathrm{Der}_G^-(L)$. □

Corollary 5.4. *Suppose that G is a cyclic group. Then $\mathrm{Der}_G^+(L) = \mathrm{Der}_G^-(L)$ are both Lie superalgebras with the usual bracket product.*

Through the above results, we can see that the set $\mathrm{Der}_G(L)$ can be very large and complex. Therefore, we now focus on the interior of $\mathrm{Der}_G(L)$ and the case where G is an infinite cyclic group. Particularly, we already know that the Hilbert series of Z-graded vector space is an important invariant, which encodes the dimension of the subspace as infinite series.

Proposition 5.5. *Assume that G is an infinite cyclic group generated by σ. Let $k_0 \in \mathbb{N}^+$ and $D \in \mathrm{Der}_{\sigma^{k_0}}(L)$ such that $\mathrm{ad}D$ restricts to an invertible map on $D \in \mathrm{Der}_{\sigma^i}(L)$ for arbitrary $i \in \mathbb{Z} \setminus \{k_0\}$. Then $\mathfrak{H}(\mathrm{Der}_G^+(L), t)$ is a rational function.*

Proof. Let any $k \in \mathbb{Z} \setminus \{k_0\}$, we have

$$\mathrm{ad}D : \mathrm{Der}_{\sigma^k}(L) \to \mathrm{Der}_{\sigma^{k+k_0}}(L)$$

is a linear isomorphism, since G is an infinite cyclic group generated by σ. It follows that

$$\dim(\mathrm{Der}_{\sigma^k}(L)) = \dim(\mathrm{Der}_{\sigma^{k+k_0}}(L)) = \dim(\mathrm{Der}_{\sigma^{k-k_0}}(L))$$

where $k \in \mathbb{N} \setminus \{k_0\}$. In addition,

$$\dim(\mathrm{Der}(L)) = \dim(\mathrm{Der}_{\sigma^{k_0}}(L)) = \dim(\mathrm{Der}_{\sigma^{-k_0}}(L)) = \cdots$$

and

$$\dim(\mathrm{Der}_{\sigma^k}(L)) = \dim(\mathrm{Der}_{\sigma^{2k}}(L)) = \dim(\mathrm{Der}_{\sigma^{3k}}(L)) = \cdots.$$

Which means that

$$\mathfrak{H}(\mathrm{Der}_G^+(L), t) = \sum_{k=k_0+1}^{\infty} \dim(\mathrm{Der}_{\sigma^k}(L)) \cdot t^k + \sum_{k=k_0}^{-\infty} \dim(\mathrm{Der}_{\sigma^k}(L)) \cdot t^k \in \mathbb{Z}((t)),$$

where $\mathbb{Z}((T))$ denotes the ring of formal Laurent series over $\mathbb{Z}$, i.e., the quotient ring of formal power series ring $\mathbb{Z}[[T]]$ with respect to the powers of the indeterminate T. Consequently, $\mathfrak{H}(\mathrm{Der}_G^+(L), t)$ is a rational function, we complete the proof of this result. □

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