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# On interpolative fuzzy contractions with applications

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**Abstract.** In this paper, following a new interpolation approach in fixed point theory, we introduce the concepts of interpolative Hardy-Rogers-type fuzzy contraction and interpolative Reich-Rus-Ćirić type fuzzy contraction in the framework of metric spaces, and we analyze the existence of fuzzy fixed points for such contractions equipped with some suitable hypotheses. A few consequences in single-valued mappings which include the conclusion of the main result of Karapinar et al. [On interpolative Hardy-Rogers type contractions. Symmetry, 2019, 11(1), 8] are obtained. On the basis that fixed point of a single-valued mapping satisfying interpolative type contractive inequality is not necessarily unique, and thereby making the notions more appropriate for fixed point theorems of multifunctions, new multivalued analogues of the fuzzy fixed point theorems presented herein are deduced as corollaries. In addition, nontrivial examples which dwell upon the generality of our results are provided. Finally, one of our results is applied to investigate solvability conditions of a Fredholm integral inclusion.

## 1. Introduction and Preliminaries

One of the most celebrated fixed point theorems with metric space structure appeared in Banach thesis in 1922 (see [1]), where it was originally used to obtain the existence of a solution to an integral equation. The theorem is now well-known as Banach fixed point theorem (or the contraction mapping principle). Actually, Banach contraction principle is a reformulation of the successive approximation methods initially used by some earlier mathematicians, namely Cauchy, Liouville, Picard, Lipschitz and so on. Meanwhile, the main result in [1] has been modified and applied in different directions. In some generalizations of the contraction mapping principle, the inequality is weakened, see, for instance [2], and in other, the topology of the underlying space is weakened, see [3] and the references therein. Along the line, one prominent improvement of the Banach fixed point theorem was presented by Hardy-Rogers [4]. The prototype of this result (in [4]) is the following.

**Theorem 1.1.** *[4] Let $(\Lambda, \mu)$ be a complete metric space and $\Upsilon$ be a selfmapping on $\Lambda$ satisfying the conditions:*

$$\mu(\Upsilon x, \Upsilon y) \leq a\mu(x, \Upsilon x) + b\mu(y, \Upsilon y) + c\mu(x, \Upsilon y) + e\mu(y, \Upsilon x) + l\mu(x, y),$$

*where $a, b, c, e, l$ are nonnegative reals. If $a + b + c + e + l < 1$, then $\Upsilon$ has a unique fixed point in $\Lambda$.*

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Recently, Roldán et al. [16] established some new fixed point theorems for a family of contractions depending on two functions and some parameters under the name multiparametric contractions and pointed out significant number of Hardy-Roger's type contractions in the setting of both metric and $b$-metric spaces. Theorem 1.1 has also been extended by many other authors. Other important versions of the Banach contraction mapping principle were independently presented by Ciric [2], Reich [14] and Rus [17].

**Definition 1.2.** *[2, 14, 17] Let $(\Lambda, \mu)$ be a metric. A single-valued mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is called:*

*(i) Rus contraction if there exist $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that for all $x, y \in \Lambda$ ,*

$$\mu(\Upsilon x, \Upsilon y) \leq a\mu(x, y) + b\mu(y, \Upsilon y).$$

*(ii) Ciric-Reich contraction if there exist $a, b, c \in \mathbb{R}_+$ with $a + b + c < 1$ such that for all $x, y \in \Lambda$,*

$$\mu(\Upsilon x, \Upsilon y) \leq a\mu(x, y) + b\mu(x, \Upsilon x) + c\mu(y, \Upsilon y).$$

A unified form of these results, which is known as Ciric-Reich-Rus fixed point theorem is given as follows:

**Theorem 1.3.** *[2, 14, 17] Let $(\Lambda, \mu)$ be a complete metric space and the selfmapping $\Upsilon : \Lambda \longrightarrow \Lambda$ be a Ciric-Reich-Rus contraction, that is,*

$$\mu(\Upsilon x, \Upsilon y) \leq \lambda[\mu(x, y) + \mu(x, \Upsilon x) + \mu(y, \Upsilon y)]$$

*for all $x, y \in \Lambda$, where $\lambda \in \left[0, \frac{1}{3}\right)$. Then $\Upsilon$ has a unique fixed point in $\Lambda$.*

Not long ago, motivated by the interpolation theory, Karapinar et al. [6] introduced the notion of interpolative Hardy-Rogers type contraction in the following manner:

**Definition 1.4.** *[6] Let $(\Lambda, \mu)$ be a metric space. The single-valued mapping $\Upsilon : \Lambda \longrightarrow \Lambda$ is called an interpolation Hardy-Rogers type contraction if there exist $\lambda \in [0, 1)$ and $a, b, c \in (0, 1)$ with $a + b + c < 1$ such that*

$$\mu(\Upsilon x, \Upsilon y) \leq \lambda[\mu(x, y)]^b[\mu(x, \Upsilon x)]^a[\mu(y, \Upsilon y)]^c\left[\frac{1}{2}(\mu(x, \Upsilon y) + \mu(y, \Upsilon x))\right]^{1-a-b-c} \tag{1}$$

*for all $x, y \in \Lambda \setminus \mathcal{F}_{ix}(\Upsilon)$, where $\mathcal{F}_{ix}(\Upsilon)$ is the set of all fixed points of $\Upsilon$.*

For similar fixed point results availing the interpolation theory, the reader is referred to [7–9]. An inherent property of the existing fixed point results via the interpolative type contraction is that the fixed point of the concerned mapping is not necessarily unique; for example, see [7, Example 1]. This restriction is an indication that fixed point theorems using interpolative notions are more suitable for fixed point theory of point-to-set-valued maps.

On the other hand, one of the challenges in mathematical modeling of practical phenomena concerns the indeterminacy induced by our inability to categorize events with adequate precision. It has been understood that conventional mathematics in the setting of crisp sets, cannot cope effectively with imprecisions. As an attempt at reducing the aforementioned obstacles and as a generalization of the ideas of crisp set theory, the evolvement of fuzzy mathematics started with the introduction of the concepts of fuzzy sets by Zadeh [19] in 1965. Fuzzy set theory is now well-known as one of the mathematical tools for handling information with nonstatistical uncertainty. As a result, the theory of fuzzy sets has gained greater applications in diverse domains such as management sciences, engineering, environmental sciences, medical sciences and in other emerging fields. Meanwhile, the basic notions of fuzzy sets have been modified and improved in different directions; for example, see [11]. In 1981, Heilpern [5] employed the concept of fuzzy set to initiate a class of fuzzy set-valued maps and established a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of fixed point theorems due to Nadler [13] and Banach [1]. Subsequently, several authors have studied the existence of fixed points of fuzzy set-valued maps, see for example, Mohammed and Azam [12] and the references therein.

In this paper, motivated by the new interpolation approach to the study of fixed point theorems initiated in [6–9], we introduce the concepts of interpolative Hardy-Rogers-type fuzzy contraction and interpolative Reich-Rus-Ciric type fuzzy contraction in the setting of metric spaces and discuss the existence of fuzzy fixed points for such contractions coupled with some suitable hypotheses. As earlier pointed out, fixed point of a single-valued mapping satisfying interpolative type contractive inequality is not necessarily unique; and thus making the notions more appropriate for fixed point theorems of multifunctions. In light of the latter observation, new multivalued analogues of the fuzzy fixed point theorems presented herein are deduced as corollaries. In addition, nontrivial examples which dwell upon the generality of our results are provided. Finally, one of our results is applied to inaugurate solvability conditions of a Fredholm-type integral inclusion.

Hereafter, we record a few preliminary concepts and notations which are essential to our main results. Denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$, the sets of natural numbers, non-negative reals and real numbers, respectively. Let $(\Lambda, \mu)$ be a metric space and $\mathcal{K}(\Lambda)$ be the family of nonempty compact subsets of $\Lambda$. Let $A, B \in \mathcal{K}(\Lambda)$ and $\epsilon > 0$ be arbitrary. Then the sets $N_\mu(\epsilon, A)$, $N_\mu(\epsilon, B)$ and $E^\mu_{(A,B)}$ and the distance function $\mu(A, B)$, are respectively defined as follows:

$$N_\mu(\epsilon, A) = \{x \in \Lambda : \mu(x, a) < \epsilon, \text{ for some } a \in A\}.$$

$$N_\mu(\epsilon, B) = \{x \in \Lambda : \mu(x, b) < \epsilon, \text{ for some } b \in B\}.$$

$$E^\mu_{(A,B)} = \{\epsilon > 0 : A \subseteq N_\mu(\epsilon, B),\ B \subseteq N_\mu(\epsilon, A)\}.$$

$$\mu(A, B) = \inf_{x\in A, y\in B} \mu(x, y).$$

Then, the Hausdorff metric $\aleph$ on $\mathcal{K}(\Lambda)$ induced by the metric $\mu$ is defined as: $\aleph(A, B) = \inf E^\mu_{(A,B)}$ (see [13, P. 3]).

Recall that an ordinary subset $A$ of $\Lambda$ is determined by its characteristic function $\chi_A$, defined by $\chi_A : A \longrightarrow \{0, 1\}$:

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A. \end{cases}$$

The value $\chi_A(x)$ specifies whether an element belongs to $A$ or not. This idea is used to define fuzzy sets by allowing an element $x \in A$ to assume any possible value in the interval $[0, 1]$. Thus, a fuzzy set in $\Lambda$ is a function with domain $\Lambda$ and values in $[0, 1] = I$. The collection of all fuzzy sets in $\Lambda$ is denoted by $I^\Lambda$. If $A$ is a fuzzy set in $\Lambda$, then the function value $A(x)$ is called the grade of membership of $x$ in $A$. The $\alpha$-level set of a fuzzy set $A$ is denoted by $[A]_\alpha$ and is defined as follows:

$$[A]_\alpha = \begin{cases} \overline{\{x \in \Lambda : A(x) > 0\}}, & \text{if } \alpha = 0 \\ \{x \in \Lambda : A(x) \geq \alpha\}, & \text{if } \alpha \in (0, 1]. \end{cases}$$

where by $\overline{M}$, we mean the closure of the crisp set $M$. A fuzzy set $A$ in $\Lambda$ is said to be convex if for all $x, y \in \Lambda$ and $t \in (0, 1)$, $A(tx + (1 - t)y) \geq \min\{A(x), A(y)\}$. A fuzzy set $A$ in a metric space $\Lambda$ is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in $\Lambda$ and $\sup_{x\in\Lambda} A(x) = 1$ (see [19]). We denote the collection of all approximate quantities in $\Lambda$ by $W(\Lambda)$. If there exists an $\alpha \in [0, 1]$ such that $[A]_\alpha, [B]_\alpha \in \mathcal{K}(\Lambda)$, then define

$$p_\alpha(A, B) = \inf_{x\in[A]_\alpha, y\in[B]_\alpha} \mu(x, y).$$

$$D_\alpha(A, B) = \aleph([A]_\alpha, [B]_\alpha).$$

$$p(A, B) = \sup_\alpha p_\alpha(A, B).$$

$$\mu_\infty(A,B)=\sup_\alpha D_\alpha(A,B).$$

Note that $p_\alpha$ is an increasing function of $\alpha$ (see [5]), $\mu_\infty$ is a metric on $\mathcal{K}(\Lambda)$ (induced by the Hausdorff metric $\aleph$) and the completeness of $(\Lambda,\mu)$ implies the completeness of the corresponding metric space $(\mathcal{K}_{\mathcal{F}}(\Lambda),\mu_\infty)$ (see [5]). Furthermore, $(\Lambda,\mu)\longmapsto(\mathcal{K}(\Lambda),\aleph)\longmapsto(\mathcal{K}_{\mathcal{F}}(\Lambda),\mu_\infty)$, are isometric embeddings via the relations $x\longrightarrow\{x\}$ (crisp set) and $M\longrightarrow\chi_M$, respectively; where

$$\mathcal{K}_{\mathcal{F}}(\Lambda)=\{A\in I^\Lambda:[A]_\alpha\in\mathcal{K}(\Lambda),\ \text{for each }\alpha\in[0,1]\}.$$

**Definition 1.5.** *[5] Let $\Lambda$ be a nonempty set. A mapping $\Upsilon:\Lambda\longrightarrow I^\Lambda$ is called fuzzy set-valued map. A fuzzy set-valued map $\Upsilon$ is a fuzzy subset of $\Lambda\times\Lambda$. The function value $\Upsilon(x)(y)$ is called the grade of membership of $y$ in the fuzzy set $\Upsilon(x)$. A point $u\in\Lambda$ is called a fuzzy fixed point of $\Upsilon$ if there exists an $\alpha(u)\in(0,1]$ such that $u\in[\Upsilon u]_{\alpha(u)}$.*

**Definition 1.6.** *[5] Let $(\Lambda,\mu)$ be a metric space. A mapping $\Upsilon:\Lambda\longrightarrow W(\Lambda)$ is called fuzzy $\lambda$-contraction if there exists $\lambda\in(0,1)$ such that for all $x,y\in\Lambda$,*

$$\mu_\infty(\Upsilon(x),\Upsilon(y))\leq\lambda\mu(x,y).$$

The following result due to Heilpern [5] is the first metric fixed point theorem for fuzzy set-valued maps.

**Theorem 1.7.** *[5, Th. 3.1] Every fuzzy $\lambda$-contraction on a complete metric space has a fuzzy fixed point.*

**Lemma 1.8.** *[13] Let $A$ and $B$ be nonempty closed and bounded subsets of a metric space $\Lambda$. If $a\in A$, then $\mu(a,B)\leq\aleph(A,B)$.*

## 2. Main Results

We start this section by introducing the notion of Hardy Rogers-type fuzzy contraction and establish the corresponding fixed point theorem.

**Definition 2.1.** *Let $(\Lambda,\mu)$ be a metric space. Then, the fuzzy set-valued map $\Upsilon:\Lambda\longrightarrow I^\Lambda$ is called an interpolative Hardy-Rogers-type fuzzy contraction if there exists a mapping $\alpha:\Lambda\longrightarrow(0,1]$ and $\lambda,a,b,c\in(0,1)$ with $a+b+c<1$ such that for all $x,y\in\Lambda\setminus\mathcal{F}_{ix}(\Upsilon)$,*

$$\begin{aligned}&\aleph([\Upsilon x]_{\alpha(x)},[\Upsilon y]_{\alpha(y)})\\ \leq\ &\lambda[\mu(x,y)]^b[\mu(x,[\Upsilon x]_{\alpha(x)})]^a[\mu(y,[\Upsilon y]_{\alpha(y)})]^c\left[\frac{1}{2}(\mu(x,[\Upsilon y]_{\alpha(y)})+\mu(y,[\Upsilon x]_{\alpha(x)}))\right]^{1-a-b-c},\end{aligned}\tag{2}$$

*where*

$$\mathcal{F}_{ix}(\Upsilon)=\{u\in\Lambda:u\in[\Upsilon u]_{\alpha(u)},\ \alpha(u)\in(0,1]\}.$$

**Theorem 2.2.** *Let $(\Lambda,\mu)$ be a complete metric space and $\Upsilon:\Lambda\to I^\Lambda$ be an interpolative Hardy-Rogers-type fuzzy contraction. Assume that $[\Upsilon x]_{\alpha(x)}$ is a nonempty compact subset of $\Lambda$ for each $x\in\Lambda$. Then $\Upsilon$ has a fuzzy fixed point in $\Lambda$.*

*Proof.* Let $x_0\in\Lambda$ be arbitrary. Then, by hypothesis, $[\Upsilon x_0]_{\alpha(x_0)}\in\mathcal{K}(\Lambda)$. Choose $x_1\in[\Upsilon x_0]_{\alpha(x_0)}$, then for this $x_1\in\Lambda$, $[\Upsilon x_1]_{\alpha(x_1)}$ is a nonempty compact subset of $\Lambda$. Hence, we can find $x_2\in[\Upsilon x_1]_{\alpha(x_1)}$ such that

$$\mu(x_1,x_2)=\mu(x_1,[\Upsilon x_1]_{\alpha(x_1)})\leq\aleph([\Upsilon x_0]_{\alpha(x_0)},[\Upsilon x_1]_{\alpha(x_1)}]).\tag{3}$$

Setting $x = x_0$ and $y = x_1$ in (2), we have

$$\begin{aligned}
&\aleph([\Upsilon x_0]_{\alpha(x_0)}, [\Upsilon x_1]_{\alpha(x_1)}) \\
&\leq \lambda[\mu(x_0, x_1)]^b[\mu(x_0, [\Upsilon x_0]_{\alpha(x_0)})]^a[\mu(x_1, [\Upsilon x_1]_{\alpha(x_1)})]^c\left[\frac{1}{2}(\mu(x_0, [\Upsilon x_1]_{\alpha(x_1)}) + \mu(x_1, [\Upsilon x_0]_{\alpha(x_0)}))\right]^{1-a-b-c} \\
&\leq \lambda[\mu(x_0, x_1)]^b[\mu(x_0, x_1)]^a\mu(x_1, x_2)]^c\left[\frac{1}{2}(\mu(x_0, x_2) + \mu(x_1, x_1))\right]^{1-a-b-c} \\
&\leq \lambda[\mu(x_0, x_1)]^b[\mu(x_0, x_1)]^a\mu(x_1, x_2)]^c\left[\frac{1}{2}(\mu(x_0, x_1) + \mu(x_1, x_2))\right]^{1-a-b-c}.
\end{aligned} \tag{4}$$

Suppose that $\mu(x_0, x_1) \leq \mu(x_1, x_2)$, then (4) becomes

$$\begin{aligned}
\aleph([\Upsilon x_0]_{\alpha(x_0)}, [\Upsilon x_1]_{\alpha(x_1)}) &\leq \lambda[\mu(x_1, x_2)]^{a+b+c}[\mu(x_1, x_2)]^{1-a-b-c} \\
&\leq \lambda(\mu(x_1, x_2)) \\
&< \mu(x_1, x_2).
\end{aligned} \tag{5}$$

Notice that the combination of (3) and (5) gives a contraction . Hence $\mu(x_1, x_2) < \mu(x_0, x_1)$. Consequently, for $\zeta = \sqrt{\lambda}$ and $\omega = \zeta\mu(x_0, x_1)$, (4) yields

$$\begin{aligned}
\aleph([\Upsilon x_0]_{\alpha(x_0)}, [\Upsilon x_1]_{\alpha(x_1)}) \leq \quad &\leq \lambda[\mu(x_0, x_1)]^{a+b+c}[\mu(x_0, x_1)]^{1-a-b-c} \\
&\leq \lambda\mu(x_0, x_1) \\
&\leq \omega.
\end{aligned}$$

It follows that $\mu(x_1, x_2) < \omega$ for some $x_2 \in [\Upsilon x_1]_{\alpha(x_1)}$. Thus, $\omega \in E^{\mu}_{([\Upsilon x_0]_{\alpha(x_0)}, [\Upsilon x_1]_{\alpha(x_1)})}$. This implies that $[\Upsilon x_0]_\alpha \subseteq N_\mu(\omega, [\Upsilon x_0]_{\alpha(x_0)})$ and $x_1 \in N_\mu(\omega, [\Upsilon x_1]_{\alpha(x_1)})$. On similar steps, there exists $x_2 \in N_\mu(\zeta d(x_0, x_1), [\Upsilon x_2]_{\alpha(x_2)})$ and $x_3 \in [\Upsilon x_2]_{\alpha(x_2)}$ such that for $\omega^2 = \zeta^2\mu(x_0, x_1)$, we have

$$\begin{aligned}
\mu(x_2, x_3) &\leq \zeta\mu(x_1, x_2) \\
&\leq \omega^2.
\end{aligned}$$

Hence, $\omega^2 \in E^{\mu}_{([\Upsilon x_1]_{\alpha(x_1)}, [\Upsilon x_2]_{\alpha(x_2)})}$. Recursively, we generate a sequence $\{x_n\}_{n\geq 1}$ in $\Lambda$ such that $x_{n+1} \in [\Upsilon x_n]_{\alpha(x_n)}$ and

$$\mu(x_n, x_{n+1}) \leq \zeta^n\mu(x_0, x_1) \text{ for all } n \geq 1.$$

Next, by standard arguments, we show that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\Lambda$. By triangular inequality , for all $k \geq 1$,

$$\begin{aligned}
\mu(x_n, x_{n+k}) &\leq \mu(x_n, x_{n+1}) + \mu(x_{n+1}, x_{n+2}) + \cdots + \mu(x_{n+k-1}, x_{n+k}) \\
&\vdots \\
&\leq \frac{\zeta^n}{1-\zeta}\mu(x_0, x_1).
\end{aligned} \tag{6}$$

Taking limit in (6) as $n \longrightarrow \infty$, we have $\lim_{n\to\infty} \mu(x_n, x_{n+k}) = 0$. Hence, $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\Lambda$. By completeness of $\Lambda$, there exist $u \in \Lambda$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Now, to prove that $u$ is a fuzzy fixed point of $\Upsilon$, assume that $u \notin [\Upsilon u]_{\alpha(u)}$. Replacing $x$ with $x_n$ and $y$ with $u$ in (2), we get

$$\begin{aligned}
&\mu(x_{n+1}, [\Upsilon u]_{\alpha(u)}) \\
&\leq \aleph([\Upsilon x_n]_{\alpha(x_n)}, [\Upsilon u]_{\alpha(u)}) \\
&\leq \lambda[\mu(x_n, u)]^b[\mu(x_n, [\Upsilon x_n]_{\alpha(x_n)})]^a[\mu(u, [\Upsilon u]_{\alpha(u)})]^c\left[\frac{1}{2}(\mu(x_n, [\Upsilon u]_{\alpha(u)}) + \mu(u, [\Upsilon x_n]_\alpha)\right]^{1-a-b-c} \\
&\leq \lambda[\mu(x_n, u)]^b[\mu(x_n, x_{n+1})]^a[\mu(u, [\Upsilon u]_{\alpha(u)})]^c\left[\frac{1}{2}(\mu(x_n, [\Upsilon u]_{\alpha(u)}) + \mu(u, x_{n+1})\right]^{1-a-b-c}.
\end{aligned} \tag{7}$$

Taking limit in (7) as $n \longrightarrow \infty$ and using the continuity of $\mu$, we have $\mu(u, [\Upsilon u]_{\alpha(u)}) = 0$. This proves that $u \in [\Upsilon u]_{\alpha(u)}$. □

**Example 2.3.** *Let $\Lambda = \{1, 2, 3, 4\}$ be endowed with the usual metric, then $(\Lambda, \mu)$ is a complete metric space. Let $\alpha \in (0, 1]$ and $\Upsilon : \Lambda \longrightarrow I^{\Lambda}$ be a fuzzy set-valued map such that for each $x \in \Lambda$ , $\Upsilon(x) : \Lambda \longrightarrow [0, 1]$ is defined as:*

$$\Upsilon(x)(t) = \begin{cases} \alpha, & if\, t = 1 \\ \frac{\alpha}{3}, & if\, t = 2 \\ \frac{\alpha}{7}, & if\, t = 3 \\ \frac{\alpha}{9}, & if\, t = 4. \end{cases}$$

*Take $\alpha(x) = \frac{\alpha}{4} \in (0, 1]$ for all $x \in X$. Then,*

$$\begin{aligned} [\Upsilon x]_{\frac{\alpha}{4}} &= \{t : \Upsilon(x)(t) \geq \alpha(x)\} \\ &= \{1, 2\}. \end{aligned}$$

*To show that $\Upsilon$ is a Hardy-Rogers-type fuzzy contraction, let $x, y \in \Lambda \setminus \mathcal{F}_{ix}(\Upsilon)$. Clearly, $x, y \in \{3, 4\}$. Therefore,*

$$\begin{aligned} \aleph([\Upsilon 3]_{\alpha(3)}, [\Upsilon 4]_{\alpha(4)}) &= \aleph([\Upsilon 4]_{\alpha(4)}, [\Upsilon 3]_{\alpha(3)}) \\ &= \aleph(\{1, 2\}, \{1, 2\}) = 0. \end{aligned}$$

*Consequently, all the hypotheses of Theorem 2.2 are satisfied. In this case, the set of all fuzzy fixed points of $\Upsilon$ is given by $\mathcal{F}_{ix}(\Upsilon) = \{1, 2\}$.*

Next, motivated by Theorem 1.3 and the result of Karapinar et al. [9, Th. 4], we introduce the concept of interpolative Reich-Rus-Ciric type fuzzy contraction and investigate the existence of fuzzy fixed point for such contraction.

**Definition 2.4.** *Let $(\Lambda, \mu)$ be a metric space. A fuzzy set-valued map $\Upsilon : \Lambda \longrightarrow I^{\Lambda}$ is called interpolative Reich-Rus-Ciric fuzzy contraction if there exists a mapping $\alpha : \Lambda \longrightarrow (0, 1]$ and constants $\eta \in [0, 1)$, $a, b \in (0, 1)$ with $a + b < 1$ such that*

$$\aleph([\Upsilon x]_{\alpha(x)}, [\Upsilon y]_{\alpha(y)}) \leq \eta\, [\mu(x, y)]^{a}\, [\mu(x, \Upsilon x)]^{b}\, [\mu(y, \Upsilon y)]^{1-a-b}, \tag{8}$$

*for all $x, y \in \Lambda \setminus \mathcal{F}_{ix}(\Upsilon)$.*

**Theorem 2.5.** *Let $(\Lambda, \mu)$ be a complete metric space and $\Upsilon : \Lambda \longrightarrow I^{\Lambda}$ be an interpolative Reich-Rus-Ciric fuzzy contraction. Assume further that $[\Upsilon x]_{\alpha(x)}$ is a nonempty compact subset of $\Lambda$ for each $x \in X$. Then $\Upsilon$ has a fuzzy fixed point in $\Lambda$.*

*Proof.* Let $x_0 \in X$ be arbitrary. Then, by hypothesis, there exists $\alpha(x_0) \in (0, 1]$ such that $[\Upsilon x_0]_{\alpha(x_0)} \in \mathcal{K}(\Lambda)$. By compactness of $[\Upsilon x_0]_{\alpha(x_0)}$, we can find $x_1 \in [\Upsilon x_0]_{\alpha(x_0)}$ with $\mu(x_0, x_1) > 0$ such that $\mu(x_0, x_1) = \mu(x_0, [\Upsilon x_0]_{\alpha(x_0)})$. Note that if there exists no such $x_1$, then $x_0$ is already a fuzzy fixed point of $\Upsilon$. Similarly, by assumption, there exists $\alpha(x_1) \in (0, 1]$ such that $[\Upsilon x_1]_{\alpha(x_1)}$ is a nonempty compact subset of $\Lambda$. Thus, there exists $x_2 \in [\Upsilon x_1]_{\alpha(x_1)}$ with $\mu(x_1, x_2) > 0$ such that $\mu(x_1, x_2) = \mu(x_1, [\Upsilon x_1]_{\alpha(x_1)})$. Inductively, we generate a sequence $\{x_n\}_{n\geq 1}$ of points of $\Lambda$ with $x_{n+1} \in [\Upsilon x_n]_{\alpha(x_n)}$, $\mu(x_n, x_{n+1}) > 0$ such that $\mu(x_n, x_{n+1}) = \mu(x_n, [\Upsilon x_n]_{\alpha(x_n)})$. By Lemma 1.8, we have

$$\mu(x_n, x_{n+1}) \leq \aleph([\Upsilon x_{n-1}]_{\alpha(x_{n-1})}, [\Upsilon x_{n+1}]_{\alpha(x_{n+1})}). \tag{9}$$

Now, we show that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\Lambda$. Setting $x = x_n$ and $y = x_{n-1}$ in (8), we get

$$\begin{aligned} \mu(x_n, x_{n+1}) &\leq \aleph([\Upsilon x_n]_{\alpha(x_n)}, [\Upsilon x_{n-1}]_{\alpha(x_{n-1})}) \\ &\leq \eta[\mu(x_n, x_{n-1})]^{a}[\mu(x_n, [\Upsilon x_n]_{\alpha(x_n)})]^{b}[\mu(x_{n-1}, [\Upsilon x_{n-1}]_{\alpha(x_{n-1})})]^{1-a-b} \\ &\leq \eta[\mu(x_n, x_{n-1})]^{a}[\mu(x_n, x_{n+1})]^{b}[\mu(x_{n-1}, x_n)]^{1-a-b} \\ &= \eta[\mu(x_n, x_{n-1})]^{1-b}[\mu(x_n, x_{n+1})]^{b}. \end{aligned} \tag{10}$$

From (10), we have

$$\mu(x_n, x_{n+1}) \leq \eta^{\frac{1}{1-b}} \mu(x_{n-1}, x_n) \text{ for all } n \in \mathbb{N}. \tag{11}$$

We infer from(11) that for all $n \in \mathbb{N}$,

$$\mu(x_n, x_{n+1}) \leq \eta\mu(x_{n-1}, x_n) \leq \eta^n \mu(x_0, x_1). \tag{12}$$

From (12), following the proof of Theorem 2.2, we deduce that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in $\Lambda$. The completeness of this space implies that there exists $u \in \Lambda$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$. Now, we show that $u$ is a fuzzy fixed point of $\Lambda$. Assume that $u \notin [\Upsilon u]_{\alpha(u)}$ and $\mu(u, [\Upsilon u]_{\alpha(u)}) > 0$. Then, replacing $x$ and $y$ with $x_n$ and $u$, respectively in (8), and using Lemma 1.8, gives

$$\begin{aligned} \mu(u, [\Upsilon u]_{\alpha(u)}) \leq\ & \mu(u, x_{n+1}) + \mu(x_{n+1}, [\Upsilon u]_{\alpha(u)}) \\ \leq\ & \mu(u, x_{n+1}) + \aleph([\Upsilon x_n]_{\alpha(x_n)}, [\Upsilon u]_{\alpha(u)}) \\ \leq\ & \mu(u, x_{n+1}) + \eta[\mu(x_n, u)]^a [\mu(x_n, x_{n+1})]^b [\mu(u, [\Upsilon u]_{\alpha(u)})]^{1-a-b}. \end{aligned} \tag{13}$$

Letting $n \longrightarrow \infty$ in (13) and using the continuity of the metric $\mu$, yields $\mu(u, [\Upsilon u]_{\alpha(u)}) = 0$, a contradiction. Hence, $u \in [\Upsilon u]_{\alpha(u)}$. □

As an improvement of Theorem 1.7 due to Heilpern [5], in what follows, we study fixed point results of Hardy-Roger's type fuzzy contraction and Reich-Rus-Ciric contraction, using the interpolative approach in connection with $\mu_\infty$-metric for fuzzy sets. It is noteworthy that fuzzy fixed point results in the setting of $\mu_\infty$-metric are very useful in evaluating Hausdorff dimensions. These dimensions help us to understand the notion of $\varepsilon^\infty$-space which is of tremendous importance in higher energy physics.

**Theorem 2.6.** *Let $(\Lambda, \mu)$ be a complete metric space and $\Upsilon : \Lambda \longrightarrow \mathcal{K}_{\mathcal{F}}(\Lambda)$ be fuzzy set valued map. Assume that the following conditions are satisfied: there exist $\lambda, a, b, c \in (0, 1)$ with $a + b + c < 1$ such that for all $x, y \in \Lambda \setminus \mathcal{F}_{ix}(\Upsilon)$, we have*

$$\mu_\infty(\Upsilon x, \Upsilon y) \leq \lambda[\mu(x, y)]^b [p(x, \Upsilon(x)]^a [p(y, \Upsilon y)]^c \left[\frac{1}{2}(p(x, \Upsilon y) + p(y, \Upsilon x))\right]^{1-a-b-c}. \tag{14}$$

*Then $\Upsilon$ has a fuzzy fixed point in $\Lambda$.*

*Proof.* Let $x \in \Lambda$ be arbitrary, and define the mapping $\alpha : \Lambda \longrightarrow (0, 1]$ by $\alpha(x) = 1$. Then, by hypothesis, $[\Upsilon x]_1 \in \mathcal{K}(\Lambda)$. Now, for every $x, y \in \Lambda \setminus \mathcal{F}_{ix}(\Upsilon)$,

$$\begin{aligned} & D_1(\Upsilon x, \Upsilon y) \\ \leq\ & \mu_\infty(\Upsilon x, \Upsilon y) \\ \leq\ & \lambda[\mu(x, y)]^b [p(x, \Upsilon x)]^a [p(y, \Upsilon y)]^c \left[\frac{1}{2}(p(x, \Upsilon y) + p(y, \Upsilon x))\right]^{1-a-b-c}. \end{aligned}$$

Since $[\Upsilon x]_1 \subseteq [\Upsilon x]_{\alpha(x)} \in \mathcal{K}(\Lambda)$, therefore, $\mu(x, [\Upsilon x]_{\alpha(x)}) \leq \mu(x, [\Upsilon x]_1)$ for each $\alpha(x) \in (0, 1]$. It follows that $p(x, \Upsilon x) \leq \mu(x, [\Upsilon x]_1)$. Consequently,

$$\begin{aligned} & \aleph([\Upsilon x]_1, [\Upsilon y]_1) \\ \leq\ & \lambda[\mu(x, y)]^b [\mu(x, [\Upsilon x]_1)]^a [\mu(y, [\Upsilon y]_1)]^c \left[\frac{1}{2}(\mu(x, [\Upsilon y]_1), \mu(y, [\Upsilon x]_1))\right]^{1-a-b-c}. \end{aligned} \tag{15}$$

Hence, Theorem 2.2 can be applied to find $u \in \Lambda$ such that $u \in [\Upsilon u]_1$. □

On the same steps as in the proof of Theorem 2.6, we can establish the following result.

**Theorem 2.7.** *Let $(\Lambda,\mu)$ be a complete metric space and $\Upsilon:\Lambda\longrightarrow\mathcal{K}_{\mathcal{F}}(\Lambda)$ be fuzzy set valued map. Assume that the following conditions are satisfied: there exist $\eta\in[0,1)$ and $a,b\in(0,1)$ with $a+b<1$ such that for all $x,y\in\Lambda\setminus\mathcal{F}_{ix}(\Upsilon)$, we have*

$$\mu_\infty(\Upsilon x,\Upsilon y)\leq\eta[\mu(x,y)]^a[p(x,\Upsilon(x)]^b[p(y,\Upsilon y)]^{1-a-b}. \tag{16}$$

*Then $\Upsilon$ has a fuzzy fixed point in $\Lambda$.*

**Example 2.8.** *Let $\Lambda=\left\{\sigma_n=\frac{n(n+1)}{2}:n=1,2,\cdots\right\}\cup\{0\}$ and $\mu(x,y)=|x-y|$ for all $x,y\in\Lambda$. Then, $(\Lambda,\mu)$ is a complete metric space. Define a fuzzy set-valued map $\Upsilon:\Lambda\longrightarrow\mathcal{K}_{\mathcal{F}}(\Lambda)$ as follows:*
*For $x=0$,*

$$\Upsilon(0)(t)=\begin{cases}\frac{1}{6}, & \text{if } t=0\\ \frac{1}{2}, & \text{if } t=\sigma_1\\ \frac{3}{10}, & \text{if } t=\sigma_2\\ \frac{5}{19}, & \text{if } t=\sigma_n,\ n\geq3,\end{cases}$$

*and for $x\in\Lambda\setminus\{0\}$,*

$$\Upsilon(x)(t)=\begin{cases}\frac{2}{11}, & \text{if } t=\sigma_1\\ \frac{1}{5}, & \text{if } t=\sigma_2\\ \frac{3}{7}, & \text{if } t\in\{\sigma_3,\sigma_4,\cdots,\sigma_{n-1}\},\ n\geq3.\end{cases}$$

*Also, define the mapping $\alpha:\Lambda\longrightarrow(0,1]$ by $\alpha(x)=0.4$ for all $x\in\Lambda$. Then,*

$$[\Upsilon x]_{\alpha(x)}=\begin{cases}\{\sigma_1\}, & \text{if } x=0\\ \{\sigma_3,\sigma_4,\cdots,\sigma_{n-1}\}, & \text{if } x\neq0,\ n\geq3.\end{cases}$$

*Now, to see that the contractive condition (16) holds, let $x,y\in\Lambda\setminus\mathcal{F}_{ix}(\Upsilon)$. Obviously, $x,y\in\{\sigma_1\}$. Therefore,*

$$\mu_\infty(\Upsilon(x),\Upsilon(y)) = 0\leq\eta[\mu(x,y)]^a[p(x,\Upsilon(x))]^b[p(y,\Upsilon(y))]^{1-a-b},$$

*for all $\eta\in(0,1)$. This shows that (16) holds for all $x,y\in\Lambda$. Hence, all the conditions of Theorem 2.7 are satisfied. We can see that $\Upsilon$ has many fuzzy fixed points in $\Lambda$.*

*On the other hand, $\Upsilon$ is not a fuzzy $\lambda$-contraction (see Definition 1.6), since for $x=0$ and $y=\sigma_{n-1}$, $n\geq3$, we have*

$$\begin{aligned}\sup_{n\geq3}\frac{\aleph([\Upsilon0]_{0.4},[\Upsilon\sigma_{n-1}]_{0.4})}{\mu(0,\sigma_{n-1})} &= \sup_{n\geq3}\frac{\sigma_{n-1}-1}{\sigma_{n-1}}\\ &= \sup_{n\geq3}\frac{\frac{n(n-1)}{2}-1}{\frac{n(n-1)}{2}}\\ &= \sup_{n\geq3}\left[1-\frac{2}{n(n-1)}\right]=1.\end{aligned}$$

*Hence, Theorem 1.7 due to Heilpern [5] cannot be employed to find any fuzzy fixed point of $\Upsilon$.*

## 3. Consequences in multivalued and singlevalued mappings

Let $(\Lambda,\mu)$ be a metric space, $CB(\Lambda)$ and $\mathcal{N}(\Lambda)$ be the family of nonempty closed and bounded and nonempty subsets of $\Lambda$, respectively. A setvalued mapping $F:\Lambda\longrightarrow\mathcal{N}(\Lambda)$ is called a multivalued contraction (see [13]) if there exists a constant $\lambda\in(0,1)$ such that $\aleph(Fx,Fy)\leq\lambda\mu(x,y)$ for all $x,y\in\Lambda$.

A point $u \in \Lambda$ is said to be a fixed point of $F$ if $u \in Fu$. Nadler [13, Th. 5] established that every multivalued contraction on a complete metric space has a fixed point. Among well-known generalizations of multivalued contractions due to Nadler related to our focus here are the ones presented by Reich [15] and Rus [18].

**Theorem 3.1.** *(See Rus [18]) Let $(\Lambda, \mu)$ be a complete metric space and $F : \Lambda \longrightarrow CB(\Lambda)$ be a multivalued mapping. Assume that there exist $a, b \in \mathbb{R}_+$ with $a + b < 1$ such that for all $x, y \in \Lambda$,*

$$\aleph(Fx, Fy) \leq a\mu(x, y) + b\mu(y, Fy).$$

*Then there exists $u \in \Lambda$ such that $u \in Fu$.*

**Theorem 3.2.** *(See Reich [15]) Let $(\Lambda, \mu)$ be a complete metric space and $F : X \longrightarrow CB(X)$ be a multivalued mapping. Assume that there exist $a, b \in \mathbb{R}_+$ with $a + b + c < 1$ such that for all $x, y \in \Lambda$,*

$$\aleph(Fx, Fy) \leq a\mu(x, y) + b\mu(x, Fx) + c\mu(y, Fy).$$

*Then there exists $u \in \Lambda$ such that $u \in Fu$.*

In this section, we deduce some consequences and comparative results of our main theorems in the framework of both singlevalued and multivalued mappings. First, we present multivalued analogues of Theorems 2.2 and 2.5. They are also multivalued generalizations of the recently established fixed point theorems due to Karapinar et al. [6, Th. 4] and Karapinar et al. [7, Cor. 1], respectively.

**Corollary 3.3.** *Let $(\Lambda, \mu)$ be a complete metric space and $F : \Lambda \longrightarrow \mathcal{K}(\Lambda)$ be a multi-valued mapping. Assume that there exist $\lambda, a, b, c, \in (0, 1]$ with $a + b + c < 1$ such that for all $x, y \in \Lambda \setminus \mathcal{F}_{ix}(F)$,*

$$\aleph(Fx, Fy) \leq \lambda[\mu(x, y)]^b[\mu(x, Fx)]^a[\mu(y, Fy)]^c \left[\frac{1}{2}(\mu(x, Fy) + \mu(y, Fx))\right]^{1-a-b-c}. \tag{17}$$

*Then there exists $u \in \Lambda$ such that $u \in Fu$.*

*Proof.* Consider a mapping $\vartheta : \Lambda \to (0, 1]$ and fuzzy set valued map $\Upsilon : \Lambda \to I^\Lambda$ defined by

$$\Upsilon(x)(t) = \begin{cases} \vartheta x, & \text{if } t \in Fx \\ 0, & \text{if } t \notin Fx. \end{cases}$$

Taking $\alpha(x) = \vartheta(x) \in (0, 1]$ for all $x \in \Lambda$, we have

$$[\Upsilon x]_{\alpha(x)} = \{t \in \Lambda : \Upsilon(x)(t) \geq \alpha(x)\} = Fx.$$

Consequently , Theorem 2.2 can be applied to find $u \in \Lambda$ such that $u \in Fu = [\Upsilon u]_\alpha$. □

As a consequence of Theorem 2.5 and in line with the proof of Corollary 3.3, we can also establish the next result.

**Corollary 3.4.** *Let $(\Lambda, \mu)$ be a complete metric space and $F : \Lambda \longrightarrow \mathcal{K}(\Lambda)$ be a multi-valued mapping. Assume that there exist $\lambda, a, b \in (0, 1]$ with $a + b < 1$ such that for all $x, y \in \Lambda \setminus \mathcal{F}_{ix}(\Upsilon)$,*

$$\aleph(Fx, Fy) \leq \lambda[\mu(x, y)]^a[\mu(x, Fx)]^b[\mu(y, Fy)]^{1-a-b}. \tag{18}$$

*Then there exists $u \in \Lambda$ such that $u \in Fu$.*

**Example 3.5.** *Let $\Lambda = [1,5]$ and $\mu(x,y) = |x-y|$ for all $x,y \in \Lambda$. Then, $(\Lambda,\mu)$ is a complete metric space. Define $F : \Lambda \longrightarrow \mathcal{K}(\Lambda)$ by*

$$Fx = \begin{cases} [1,2], & \text{if } 1 \leq x < 2 \\ [3,5], & \text{if } 2 \leq x \leq 5. \end{cases}$$

*Let $x,y \in \Lambda \setminus \mathcal{F}_{ix}(F)$. Clearly, $x,y \in (1,2)$ and*

$$\begin{aligned} \aleph(Fx,Fy) &= \aleph([1,2],[1,2]) = 0 \\ &\leq \lambda[\mu(x,y)]^b[\mu(x,Fx)]^a[\mu(y,Fy)]^c\left[\frac{1}{2}(\mu(x,Fy)+\mu(y,Fx))\right]^{1-a-b-c}. \end{aligned}$$

*Consequently, all the hypotheses of Corollary 3.3 are satisfied. We see that $F$ has many fixed points in $\Lambda$.*

*On the other hand, $F$ is not a multivalued contraction, since for $x = 1$ and $y = 2$, we have*

$$\begin{aligned} \aleph(F1,F2) &= \aleph([1,2],[3,5]) \\ &= 3 > \lambda(1) = \lambda\mu(1,2), \end{aligned}$$

*for all $\lambda \in (0,1)$. Therefore, the result of Nadler [13, Th. 5] cannot be used in this example to obtain a fixed point of $F$.*

*Similarly, since $F1 = [1,2]$ and $F2 = [3,5]$, we have*

$$\mu(1,F1) = \inf_{\omega \in [1,2]} \mu(1,\omega) = 0,$$

$$\mu(2,F2) = \inf_{\xi \in [3,5]} \mu(2,\xi) = 1.$$

*Hence,*

$$\begin{aligned} \aleph(F1,F2) &= \aleph([1,2],[3,5]) \\ &= 3 > a+b = a(1)+b(1) \\ &= a\mu(1,2)+b\mu(2,F2), \end{aligned}$$

*for all $a,b \in \mathbb{R}_+$ satisfying $a+b<1$. This means that Theorem 3.1 due to Rus [18] is not applicable to this example to find a fixed point of $F$.*

*In like manner,*

$$\begin{aligned} \aleph(F1,F2) &= \aleph([1,2],[3,5]) \\ &= 3 > a+c = a(1)+b(0)+c(1) \\ &= a\mu(1,2)+b\mu(1,F1)+c\mu(2,F2), \end{aligned}$$

*for all $a,b,c \in \mathbb{R}_+$ with $a+b+c<1$. Hence, Theorem 3.2 due to Reich [15] cannot be applied in this case to locate any fixed point of $F$.*

**Corollary 3.6.** *(See Karapinar et al. [6, Th. 4]) Let $(\Lambda,\mu)$ be a complete metric space and $f : \Lambda \to \Lambda$ be a single-valued mapping. Assume that there exist $\lambda,a,b,c \in (0,1)$ with $a+b+c<1$ such that for all $x,y \in \Lambda \setminus \mathcal{F}_{ix}(f)$, we have*

$$\mu(fx,fy) \leq \lambda[\mu(x,y)]^b[\mu(x,fx)]^a[\mu(y,fy)]^c\left[\frac{1}{2}(\mu(x,fy)+\mu(y,fx))\right]^{1-a-b-c}. \tag{19}$$

*Then there exists $u \in \Lambda$ such that $fu = u$*

*Proof.* Let $\alpha : \Lambda \longrightarrow (0,1]$ be a mapping, and define a fuzzy set-valued map $\Upsilon : \Lambda \longrightarrow I^\Lambda$ as follows:

$$\Upsilon(x)(t) = \begin{cases} \alpha(x), & \text{if } t = fx \\ 0, & \text{if } t \neq fx. \end{cases}$$

Then,

$$[\Upsilon x]_{\alpha(x)} = \{t \in \Lambda : \Upsilon(x)(t) \geq \alpha(x)\} = \{fx\}.$$

Clearly, $\{fx\} \in \mathcal{K}(\Lambda)$ for all $x \in \Lambda$. Note that in this case, $\aleph([\Upsilon x]_{\alpha(x)}, [\Upsilon x]_{\alpha(y)}) = \mu(fx, fy)$ for all $x, y \in \Lambda$. Consequently, Theorem 2.2 can be applied to find $u \in \Lambda$ such that $u \in [\Upsilon u]_{\alpha(u)} = \{fu\}$; which further implies that $u = fu$. □

By adopting the method of deducing Corollary 3.6, we can also derive the next result.

**Corollary 3.7.** *(See Karapinar et al. [7, Cor. 1]) Let $(\Lambda, \mu)$ be a complete metric space and $f : \Lambda \to \Lambda$ be a single-valued mapping. Assume that there exist $\lambda, a, b \in (0, 1)$ with $a + b < 1$ such that for all $x, y \in \Lambda \setminus \mathcal{F}_{ix}(f)$, we have*

$$\mu(fx, fy) \leq \lambda[\mu(x, y)]^a [\mu(x, fx)]^b [\mu(y, fy)]^{1-a-b}. \tag{20}$$

*Then there exists $u \in \Lambda$ such that $fu = u$*

## 4. Applications to Fredholm Integral Inclusions

In this section, we apply Theorem 2.2 to study some sufficient conditions for the existence of solutions of Fredholm-type Integral inclusions.

Consider the following integral inclusion of Fredholm type:

$$x(t) \in \left[ g(t) + \int_a^b L(t, s, x(s))\mu s,\ t \in [a, b] \right] \tag{21}$$

where $x \in C([a, b], \mathbb{R})$ is an unknown real-valued continuous functions defined on $[a, b]$, $g \in C([a, b], \mathbb{R})$ is a given real-valued continuous function and $L : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow F_{cv}(\mathbb{R})$ is a given set-valued map, where we denote the family of nonempty compact and convex subsets of $\mathbb{R}$ by $F_{cv}(\mathbb{R})$.

Now, we study the existence of solutions of 21 under the following conditions:

**Theorem 4.1.** *Let $\Lambda = C([a, b], \mathbb{R})$ and assume that:*

- $(C_1)$ *the set-valued map $L : [a, b] \times [a, b] \times \mathbb{R} \longrightarrow F_{cv}(\mathbb{R})$ is such that for every $x \in \Lambda$, the map $L_x(t, s) := L(t, s, x(s))$ is lower semicontinuous;*
- $(C_2)$ *$g \in C([a, b], \mathbb{R})$;*
- $(C_3)$ *there exists a function $\xi : (0, \infty) \longrightarrow \mathbb{R}$ such that for all $x, y \in \Lambda$,*

$$\aleph\left(L_x(t, s), L_y(t, s)\right) \leq \pi(t, s)\xi(t)\left(|x(s) - y(s)|\right)^r$$

*for each $t, s \in [a, b]$, where $\sup_s \left(\int_a^b \pi(t, s)\mu s\right) \leq 1$, $\pi(t, .) \in L^1[a, b]$ and $r \in (0, 1)$.*

*Then, the integral inclusion* (21) *has at least one solution in $\Lambda$.*

*Proof.* Define $\mu : \Lambda \times \Lambda \longrightarrow \mathbb{R}$ by

$$\mu(x, y) = \max_{a \leq t \leq b} |x(x) - y(t)|, \text{ for all } x, y \in \Lambda,$$

then $(\Lambda, \mu)$ is a complete metric space. Let $\Upsilon : \Lambda \longrightarrow I^\Lambda$ be a fuzzy set-valued map. Consider the $\alpha$-level set of $\Upsilon$ defined as:

$$[\Upsilon x]_{\alpha(x)} = \left\{ y \in \Lambda : y(t) \in g(t) + \int_a^b L(t, s, x(s))\mu s,\ t \in [a, b] \right\}.$$

Clearly, the set of solutions of (21) coincides with the set of fuzzy fixed points of $\Upsilon$. Therefore, we have to show that under the given hypotheses, $\Upsilon$ has at least one fuzzy fixed point in $\Lambda$. For this, we shall verify that all the hypotheses of Theorem 2.2 are satisfied.

Let $x \in \Lambda$ be arbitrary. Since the set-valued map $L_x : [a,b] \times [a,b] \longrightarrow F_{cv}(\mathbb{R})$ is lower semicontinuous, it follows from Michael's selection theorem ([10, Theorem 1]) that there exists a continuous map $\rho_x : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ such that $\rho_x(t,s) \in L_x(t,s)$, for each $(t,s) \in [a,b] \times [a,b]$. Therefore, $g(t) + \int_a^b \rho_x(t,s)\mu s \in [\Upsilon x]_{\alpha(x)}$. So $[\Upsilon x]_{\alpha(x)}$ is nonempty. One can easily see that $[\Upsilon x]_{\alpha(x)}$ is a compact subset of $\Lambda$. Further, given that $g \in C([a,b])$ and $L_x(t,s)$ is continuous on $[a,b] \times [a,b]$, their range sets are compact. Hence, $[\Upsilon x]_{\alpha(x)}$ is also compact; that is, $[\Upsilon x]_{\alpha(x)} \in \mathcal{K}(\Lambda)$ for each $x \in \Lambda$.

Take $x_1, x_2 \in \Lambda$; then there exists $\alpha(x_1), \alpha(x_2) \in (0,1]$ such that $[\Upsilon x_1]_{\alpha(x_1)}$ and $[\Upsilon x_2]_{\alpha(x_2)}$ are nonempty compact subsets of $\Lambda$. Let $y_1 \in [\Upsilon x_1]_{\alpha(x_1)}$ be arbitrary such that

$$y_1(t) \in g(t) + \int_a^b L(t,s,x_1(s))\mu s, \; t \in [a,b].$$

This means for each $(t,s) \in [a,b] \times [a,b]$, there exists $\rho_{x_1} \in L_{x_1}(t,s)$ such that

$$y_1(t) = g(t) + \int_a^b \rho_{x_1}(t,s)\mu s, \; t \in [a,b].$$

Since, from $(C_2)$,

$$\aleph\left(L(t,s,x_1(s)), L(t,s,x_2(s))\right) \le \pi(t,s)\xi(t)\left(|x_1(s) - x_2(s)|\right)^r$$

for each $t, s \in [a,b]$ and $r \in (0,1)$, so there exists $\rho_{x_2} \in L_{x_2}(t,s)$ such that

$$|\rho_{x_1}(t,s) - \rho_{x_2}(t,s)| \le \pi(t,s)\xi(t)\left(|x_1(s) - x_2(s)|\right)^r$$

for all $(t,s) \in [a,b] \times [a,b]$.

Now, consider the set-valued map $\mathfrak{M}$ defined by

$$\mathfrak{M}(t,s) = L_{x_2}(t,s) \cap \left\{\omega \in \mathbb{R} : |\rho_{x_1}(t,s) - \omega| \le \pi(t,s)\xi(t)\left(|x_1(s) - x_2(s)|\right)^r\right\}.$$

Taking into account the fact that from $(C_1)$, $\mathfrak{M}$ is lower semicontinuous, therefore, there exists a continuous map $\rho_{x_2} : [a,b] \times [a,b] \longrightarrow \mathbb{R}$ such that $\rho_{x_2}(t,s) \in \mathfrak{M}(t,s)$, for all $(t,s) \in [a,b] \times [a,b]$. Then,

$$\begin{aligned} y_2(t) &= g(t) + \int_a^b \rho_{x_1}(t,s)\mu s \\ &\in g(t) + \int_a^b L(t,s,x_2(s))\mu s, \; t \in [a,b]. \end{aligned}$$

Thus, $y_2 \in [\Lambda x_2]_{\alpha(x_2)}$, and

$$\begin{aligned} |y_1(t) - y_2(t)| &\le \left(\int_a^b |\rho_{x_1}(t,s) - \rho_{x_2}(t,s)|\mu s\right) \\ &\le \sup\left(\int_a^b \pi(t,s)\mu s\right)\xi(t)\left(|x_1(s) - x_2(s)|\right)^r \\ &\le \xi(t)(|x_1(s) - x_2(s)|)^r. \end{aligned}$$

Therefore,

$$\aleph\left([\Upsilon x_1]_{\alpha(x_1)}, [\Upsilon x_2]_{\alpha(x_2)}\right) \le \xi(t)\left(\mu(x_1,x_2)\right)^r. \tag{22}$$

Hence, setting $x = x_1$ and $y = x_2$ in (22), gives

$$\aleph\left([\Upsilon x]_{\alpha(x)}, [\Upsilon y]_{\alpha(y)}\right) \le \xi(t)\left(\mu(x,y)\right)^r.$$

Hence, all the hypotheses of Theorem 2.2 are satisfied with $\xi(t) = \lambda t$, for all $t > 0$ and $\lambda \in (0,1)$. So, the conclusion of Theorem 4.1 holds consequently. □

## 5. Conclusion

The results of this paper broadened the scope of fuzzy fixed point theory and fixed point theory of multivalued mappings by incorporating the interpolative approaches. To this end, interpolative Hardy-Rogers type fuzzy contraction and interpolative Reich-Rus-Ciric type fuzzy contraction are initiated and the corresponding fixed point theorems are proved, with examples illustrating the hypotheses of the main results. To show the usability of the new ideas presented herein, some sufficient conditions for the existence of solutions of Fredholm-type integral inclusions is established. The ideas in this work, being discussed in the setting of metric spaces, are completely fundamental. Hence, they can be improved upon when presented in the framework of generalized metric spaces such as $b$-metric spaces, $G$-metric spaces, $F$-metric spaces and some other pseudo-metric or quasi metric spaces. Also, the fuzzy set-valued map's component can be extended to $L$-fuzzy mappings, intuitionistic fuzzy mappings, soft set-valued maps, and so on.

## Competing Interests

The authors declare that they have no competing interests.


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