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Fixed point theorems in generalized locally convex spaces and applications

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Abstract. In this paper, we present the concept of generalized locally convex spaces, by introducing the so-called family of vector-valued seminorms. Some extensions of classical fixed point theorems in Hausdorff complete generalized locally convex spaces are given. These results are formulated in terms of continuity and $\mathcal{L}$-contractions. As applications, we study the existence and uniqueness of solutions for systems of first order differential equations with impulsive.

1. Introduction and preliminaries

The fixed point in metric spaces has been of interest for many authors and various fixed points theorems have been stated [1–5, 8, 12, 25]. In 1964, Perov [16] gave the definition of a generalized metric space (or a vector metric space) by introducing the notion of metric with values in $\mathbb{R}^n$. Then, he defined a new class of mappings called Perov contractions which are $\mathcal{L}$-contractive, where $\mathcal{L}$ is a $n \times n$-matrix with non negative elements instead of a constant k. Also, he extended Banach fixed point theorem on spaces endowed with vector-valued metrics [16, 17]. Recently, Many authors studied well-known fixed point theorems of single-valued and multivalued mappings such as Schauder, Krasnoselskii, Schafer,....etc, in a generalized Banach spaces (see for example [7, 15, 18–20, 22, 24]). More recently, Nieto et al. in [14] proved some versions of Perov, Schauder, and Krasnoselskii type fixed point theorems in vector Banach algebras. As an application they studied the existence of solutions for nonlinear integral equations in Banach algebras. For more details about generalized metric or Banach spaces and fixed point theorems, the readers can refer to [9, 11].

In this work, we define generalized locally convex spaces (in short, GLCSs) by introducing the concept of generalized seminorms (seminorms with values in $\mathbb{R}^n$). Furthermore, we give some new fixed point theorems which generalize classical ones proved in locally convex spaces. Also, we apply our results to study the existence and uniqueness of some systems of differential equations with impulsive.

Firstly, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper. We denote by $\mathbb{R}_+$ the set of non-negative real numbers, by $\mathbb{R}^n$ be the set of $n \times 1$ real

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matrices. By $\preceq$ we mean coordinate-wise ordering on $\mathbb{R}^n$, that is, for $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, $x \preceq y$ if, and only if, $x_i \leq y_i$ for all $i = 1, \cdots, n$. For $x, y \in \mathbb{R}^n$, we denote by

$$\widehat{\max}(x,y) = \begin{pmatrix} \max(x_1,y_1) \\ \max(x_2,y_2) \\ \vdots \\ \max(x_n,y_n) \end{pmatrix}.$$

Let $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$, we denote by $c \succ \mathbf{0}$ if $c_i > 0$ for each $i = 1, \cdots, n$, where $\mathbf{0}$ is the zero $n \times 1$ matrix. Moreover, we denote by $\mathbb{R}^n_+$ the set of $n \times 1$ matrices with non negative elements and by $\mathcal{M}_{n\times n}(\mathbb{R}_+)$ the set of $n \times n$ matrices with nonnegative elements. We denote by I and θ the identity in $\mathcal{M}_{n\times n}(\mathbb{R}_+)$ and the zero $n \times n$ matrix respectively.

In what follows, let X be a linear space on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We introduce the definition of a vector-valued seminorm as follows:

Definition 1.1. A mapping p of X into $\mathbb{R}^n$ is called a vector-valued seminorm or generalized seminorm (GSN, in short) if it satisfies the following axioms:

(i) $p(\alpha x) = |\alpha| \ p(x)$, $x \in X$, $\alpha \in \mathbb{R}$.

(ii) $p(x + y) \preceq p(x) + p(y)$.

$$p(x) := \begin{pmatrix} p^1(x) \\ p^2(x) \\ \vdots \\ p^n(x) \end{pmatrix}.$$ ◊

Note that p is a GSN on X if, and only if, p^s are seminorms on X, for $s = 1, \cdots, n$, and from (i) , (ii) we obtain that $p(x) \succeq \mathbf{0}$, for each $x \in X$.

Example 1.2. (i) Every vector-valued norm is a GSN (see [9, Definition 7.20]).

(ii) Every seminorm defines a GSN. ◊

Definition 1.3. Let $\Lambda = \{p_i, \ i \in I_X\}$ be a family of GSNs on X indexed by I_X, and let M be a subset of X.

(i) M is bounded, if for every $i \in I_X$ there exists $c \in \mathbb{R}^n$, $c \succ \mathbf{0}$ such that

$$p_i(x) \preceq c, \ \text{ for all } \ x \in M.$$

(ii) M is called open if for each $x \in M$, there exist a finite set $I_{X_k} \subset I_X$ and $r \in \mathbb{R}^n$, $r \succ \mathbf{0}$, such that

$$\text{if } \ y \in X, \ \text{ and } \ \widehat{\max_{i \in I_{X_k}}} p_i(y - x) \preceq r, \ \text{ then, } \ y \in M,$$

where $\widehat{\max_{i \in I_{X_k}}} p_i(y - x) := \widehat{\max_{i \in I_{X_k}}} \{p_{i_1}(x - y), \cdots, p_{i_k}(x - y)\}$.

(iii) M is closed if $X \backslash M$ is open. ◊

Definition 1.4. Let $\Lambda = \{p_i, \ \ i \in I_X\}$ be a family of GSNs on X. Let $x \in X$, $r \in \mathbb{R}^n$, $r > \mathbf{0}$ and I_{X_k} be a finite subset of I_X. We denote by $B(x, r)$ the open ball centered at x with radius r and defined by

$$B(x, r) = \bigcap_{i \in I_{X_k}} B_i(x, r),$$

where
$$B_i(x, r) = \{y \in X, \ \ p_i(y - x) < r\}. \qquad \diamond$$

The collection of open balls centered in x generates the topology of the family Λ of GSNs. Clearly, X equipped with this topology is a topological vector space. Moreover, we have $B(x, r) = x + B(0, r)$ which is a convex subset of X. Then, each point x of X possesses for this topology a fundamental system of neighborhoods formed of convex sets. Then, the topology associated with the family of GSNs is locally convex. Now, we can introduce the following definition:

Definition 1.5. Let $\Lambda = \{p_i, \ \ i \in I_X\}$ be a family of GSNs on X. Then, the pair $(X, p_i)_{i \in I_X}$ is said to be generalized locally convex space (GLCS, in short), if each point of X possesses a fundamental system of neighborhoods formed of convex sets with respect to the topology associated with the family of Λ GSNs.

Example 1.6. Every locally convex space defines a GLCS. Consider $C(\mathbb{R}^+, \mathbb{R})$ the space of continuous functions $x : \mathbb{R}^+ \to \mathbb{R}$ endowed with the family of seminorms $|x|_T = \sup\{|x(t)|, \ \ t \in [0, T]\}$. Then, $C(\mathbb{R}^+, \mathbb{R})$ is GLCS with the family of GSNs

$$p_T : C(\mathbb{R}^+, \mathbb{R}) \to \mathbb{R}^2, \ \ p_T(x) = \begin{pmatrix} |x|_T \\ |x|_T \end{pmatrix}. \qquad \diamond$$

Remark 1.7. (*i*) The topology generated by the family of GSNs, $\Lambda = \{p_i, \ \ i \in I_X\}$ is Hausdorff if, and only if, the family Λ is separating i.e., given $p_i(x) = 0$, for all $i \in I_X$, then $x = 0$.

(*ii*) If Λ_1 and Λ_2 are two families of GSNs with $\Lambda_1 \subset \Lambda_2$, then the associated topology Λ_2 is finer than that associated with Λ_1. $\diamond$

Definition 1.8. Let $(X, p_i)_{i \in I}$ be a GLCS and M be a nonempty subset of X. M is compact if every cover of M by open sets has a finite subcover. $\diamond$

Definition 1.9. Let $(X, p_i)_{i \in I_X}$ be a Hausdorff GLCS, and $(x_n)_n$ be a sequence of X.

(*i*) $(x_n)_n$ converges to a point $x \in X$, denoted by $x_n \underset{n \to \infty}{\longrightarrow} x$, if for each $i \in I_X$ and $\varepsilon \in \mathbb{R}^n$, $\varepsilon > \mathbf{0}$, there exists $m(i, \varepsilon) \in \mathbb{N}$ such that for each $n \geq m(i, \varepsilon)$, $p_i(x_n - x) \leq \varepsilon$.

(*ii*) $(x_n)_n$ is called a Cauchy sequence if for each $i \in I_X$ and $\varepsilon \in \mathbb{R}^n$ $\varepsilon > \mathbf{0}$, there exists $m(i, \varepsilon) \in \mathbb{N}$ such that for each $n_1, n_2 \geq m(i, \varepsilon)$, $p_i(x_{n_1} - x_{n_2}) \leq \varepsilon$.

(*iii*) The space X is called complete if each Cauchy sequence in X converges to a limit in X. $\diamond$

Definition 1.10. Let X and Y be two Hausdorff GLCSs, $\Lambda_X = \{p_i, \ \ i \in I_X\}$ and $\Lambda_Y = \{q_j, \ \ j \in I_Y\}$ their families of GSNs. Let M be a subset of X and A be a mapping from M into Y.

(*i*) A is continuous at a point x of M if, for all $j \in I_Y$ and $\varepsilon \in \mathbb{R}^n$, $\varepsilon > \mathbf{0}$, there exist a finite set $\{i_1, \cdots, i_k\} \subset I_X$ and $\delta \in \mathbb{R}^n$, $\delta > \mathbf{0}$ such that for $y \in M$

$$\widehat{\max_{1 \leq m \leq k}} \{p_{i_m}(x - y)\} := \widehat{\max}\{p_{i_1}(x - y), \cdots, p_{i_k}(x - y)\} \leq \delta,$$

then
$$q_j(Ax - Ay) \leq \varepsilon.$$

We say that A is continuous if it is at any point of M.

(*ii*) A is sequentially continuous at a point x of X if, for every sequence $(x_n)_n \subset X$ converging to x in X, then $(Ax_n)_n$ converges to Ax in Y. We say that A is sequentially continuous if it is at any point of M. $\diamond$

Remark 1.11. In GLCS, the concepts of open, (resp., closed, compact, convex) sets convergence, Cauchy sequences and completeness are similar to those in the usual locally convex spaces. ◊

In the following, $(X, p_i)_{i\in I_X}$ denotes a Hausdorff GLCS, and $\Lambda = \{p_i, \ \ i \in I_X\}$ its family of GSNs which generates the topology of X. We need the following definition.

Definition 1.12. Let $F : \mathbb{R}^n \to \mathbb{R}$ be a function, we say that F is nondecreasing, if for all $x, y \in \mathbb{R}^n$ such that $x \le y$, then $F(x) \le F(y)$. ◊

Definition 1.13. Let $F : [0, +\infty)^n \to \mathbb{R}$ be a function, we say that F is semi-linear, if for all $x, y \in [0, +\infty)^n$ and $\alpha \ge 0$, we have $F(x + y) = F(x) + F(y)$ and $F(\alpha.x) = \alpha F(x)$. ◊

Before presenting our main results let us state the following technical lemmas.

Lemma 1.14. Let $(X, p_i)_{i\in I_X}$ be a Hausdorff GLCS. Let $F : [0, +\infty)^n \to [0, +\infty)$ be a semi-linear and nondecreasing function. Then, for $i \in I_X$,

$$\widetilde{p_i}(x) = F\begin{pmatrix} p_i^1(x) \\ \vdots \\ p_i^n(x) \end{pmatrix},$$

defines a seminorm on X. Also $\widetilde{X} := (X, \widetilde{p_i})_{i\in I_X}$ is a locally convex space, and $\widetilde{\Lambda} = \{\widetilde{p_i}, \ \ i \in I_X\}$ its family of seminorms. ◊

Proof. We can easily check the proof with definition. □

Example 1.15. Let $(X, p_i)_{i\in I_X}$ be a GLCS. Then, for $i \in I_X$,

$$\widetilde{p_i}(x) = \frac{1}{n}\sum_{s=1}^{n} p_i^s(x), \ \ x \in X$$

defines a seminorms on X. ◊

Lemma 1.16. Let $(X, p_i)_{i\in I_X}$ be a GLCS. Let $F : [0, +\infty)^n \to [0, +\infty)$ be a semi-linear nondecreasing function. If

$$F\begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} = x, \ \ \text{for } \ x \in [0, +\infty),$$

then $\widetilde{X}$ is homeomorphic to $(X, p_i)_{i\in I_X}$. ◊

Proof. Let $G : X \to \widetilde{X}$, $G(x) = x$. Obviously G is bijective. Now let us prove that G is continuous. To do this, let $x_0 \in X$ and $W_{x_0} \subset \widetilde{X}$ be an open subset of $\widetilde{X}$ containing x_0, so there exists a finite subset $I_{X_k} \subset I_X$ such that

$$\{x \in \widetilde{X}, \ \ \widetilde{p_i}(x - x_0) < \xi; \ \ i \in I_{X_k}\} \subset W_{x_0}, \ \text{for } \xi > 0.$$

Let $\Upsilon = \begin{pmatrix} \xi \\ \vdots \\ \xi \end{pmatrix}$, $\xi > 0$ and put $V_{x_0} = \{x \in X, \ p_i(x - x_0) \le \Upsilon, \ \ i \in I_{X_k}\}$. Clearly, $G(V_{x_0}) \subset W_{x_0}$. Indeed, let $x \in V_{x_0}$, then $p_i(x - x_0) \preceq \Upsilon$, for all $i \in I_{X_k}$. Thus,

$$\begin{aligned}
\widetilde{p_i}(G(x) - G(x_0)) &= \widetilde{p_i}(x - x_0) \\
&= F\begin{pmatrix} p_i^1(x - x_0) \\ \vdots \\ p_i^n(x - x_0) \end{pmatrix} \\
&\leq F\begin{pmatrix} \xi \\ \vdots \\ \xi \end{pmatrix} \\
&= \xi.
\end{aligned}$$

Hence, G is continuous.
Next, let us prove that $G^{-1} : \widetilde{X} \to X$, $G^{-1}(x) = x$ is continuous. For this purpose, let $i \in I_X$ and for $\varepsilon \in \mathbb{R}^n$, $\varepsilon \succ \mathbf{0}$. Put $I_1 = \{i\} \subset I_X$ and $\delta = \frac{\min\limits_{1\leq s\leq n} \varepsilon_s}{n}$. Then, for $x, y \in X$ such that

$$\widetilde{p_i}(x - y) \leq \delta, \tag{1}$$

and taking into account the semi-linearity of F we obtain:

$$\begin{aligned}
\widetilde{p_i}(x - y) &= F\begin{pmatrix} p_i^1(x - y) \\ \vdots \\ p_i^n(x - y) \end{pmatrix} \\
&= F\begin{pmatrix} p_i^1(x - y) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + F\begin{pmatrix} 0 \\ p_i^s(x - y) \\ \vdots \\ 0 \end{pmatrix} + \cdots + F\begin{pmatrix} 0 \\ \vdots \\ 0 \\ p_i^n(x - y) \end{pmatrix}, \quad s = 1, \cdots, n.
\end{aligned} \tag{2}$$

From Eqs. (1) and (2), we obtain

$$\begin{cases}
F\begin{pmatrix} p_i^1(x - y) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leq \delta \\
\quad\vdots \\
F\begin{pmatrix} 0 \\ \vdots \\ 0 \\ \\ p_i^n(x - y) \end{pmatrix} \leq \delta
\end{cases} \tag{3}$$

Furthermore,

$$p_i^s(x-y) = F\begin{pmatrix} p_i^s(x-y) \\ \vdots \\ p_i^s(x-y) \end{pmatrix}$$

$$= F\begin{pmatrix} p_i^s(x-y) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + F\begin{pmatrix} 0 \\ p_i^s(x-y) \\ \vdots \\ 0 \end{pmatrix} + \cdots + F\begin{pmatrix} 0 \\ \vdots \\ 0 \\ p_i^s(x-y) \end{pmatrix}, \ s = 1, \cdots, n. \tag{4}$$

Combining Eqs. (3) and (4), we get

$$p_i^s(x-y) \le n\delta, \ \ s = 1, \cdots, n.$$

and,

$$p_i(x-y) = \begin{pmatrix} p_i^1(x-y) \\ \vdots \\ p_i^n(x-y) \end{pmatrix} \le \begin{pmatrix} \min\limits_{1\le s\le n} \varepsilon_s \\ \vdots \\ \min\limits_{1\le s\le n} \varepsilon_s. \end{pmatrix}$$

Then,

$$\begin{aligned} p_i(G^{-1}(x) - G^{-1}(y)) &= p_i(x-y) \\ &\le \varepsilon \end{aligned}$$

So, G^{-1} is continuous. This completes the proof. □

Remark 1.17. (*i*) It is clear that if A continuous, then it is sequentially continuous but the converse is not true. For example, consider $l(\mathbb{N})$ the space of real sequences $(x_n)_n$ such that $\sum_n |x_n| < +\infty$. Recall that $l^1(\mathbb{N})$ has the Schur property i.e, if $(x_n)_n$ is converging weakly to x ($x_n \rightharpoonup x$), implies that $(x_n)_n$ converges to x with respect to the strong topology. Let $(l^1(\mathbb{N}), p_i)_{i\in I}$ be a Hausdorff locally convex space with family of seminorms $\{p_i, \ \ i \in I\}$ which defines the weakest topology on $l^1(\mathbb{N})$. Now, if we define the generalized seminorm on $l^1(\mathbb{N})$, for $i \in I$ by

$$\begin{aligned} \mu_i : \ & l^1(\mathbb{N}) \to \mathbb{R}^2 \\ & x \to \mu_i(x) = \begin{pmatrix} p_i(x) \\ p_i(x) \end{pmatrix}. \end{aligned}$$

Then, $(l^1(\mathbb{N}), \mu_i)_{i\in I}$ is GLCS, and if we consider the operator

$$\begin{aligned} A : \ & (l^1(\mathbb{N}), \mu_i)_{i\in I} \to \mathbb{R} \\ & x \to A(x) = \|x\| = \textstyle\sum_n |x_n|, \end{aligned}$$

then A is sequentially continuous. Indeed, let $(x_n)_n$ be a sequence in $l^1(\mathbb{N})$ that converges to x in $l^1(\mathbb{N})$. Thus, for all $i \in I$ $\mu_i(x_n - x) \underset{n\to\infty}{\longrightarrow} \mathbf{0}$, which imply that, for all $i \in I$ $p_i(x_n - x) \underset{n\to\infty}{\longrightarrow} 0$. Since $l^1(\mathbb{N})$ is a Schur space then $x_n \underset{n\to\infty}{\longrightarrow} x$. As $\|\cdot\|$ is continuous hence,

$$Ax_n \underset{n\to\infty}{\longrightarrow} Ax.$$

Now, we prove that A is continuous with respect to the topology of the family of GSN $\{\mu_i, \ \ i \in I\}$. We prove that the set

$$S := A^{-1}\{1\} = \{x \in l^1(\mathbb{N}), \ \ \|x\| = 1\},$$

is not closed with respect to the topology generated by the family of GSNs $\{\mu_i, \ \ i \in I\}$. Suppose that $\overline{S}^{\{\mu_i, \ i\in I\}} = S$, then by Lemma 1.16, we get

$$S = \overline{S}^{\{\mu_i, \ i\in I\}} = G(\overline{S}^{\{p_i, \ i\in I\}}) = \overline{S}^{\{p_i, \ i\in I\}}. \tag{5}$$

Knowing that $\overline{S}^{\{p_i, \ i\in I\}} \subset \overline{S}^{\{\mu_i, \ i\in I\}}$ and by Eq. (5), we obtain $\overline{S}^{\{p_i, \ i\in I\}} = S$ which is absurd.

(*ii*) If A is defined on a subset of metrizable space, then continuity coincides with sequential continuity(see [21, Theorem 9.1]). ◊

Definition 1.18. Let $n \in \mathbb{N}\backslash\{0\}$. A square matrix $\mathcal{L} \in \mathcal{M}_{n\times n}(\mathbb{R})$ of real numbers is said to be convergent to zero if $\mathcal{L}^k \underset{k\to\infty}{\longrightarrow} \theta$. ◊

Lemma 1.19. [9] Let $\mathcal{L}$ be a square matrix with nonnegative elements. The following assertions are equivalent:

(*i*) $\mathcal{L}$ is convergent to zero.

(*ii*) The matrix $(I - \mathcal{L})$ is non singular and

$$(I - \mathcal{L})^{-1} = I + \mathcal{L} + \mathcal{L}^2 + \cdots + \mathcal{L}^k + \cdots$$

(*iii*) $|\lambda| < 1$ for every $\lambda \in \mathbb{C}$ with $\det(\mathcal{L} - \lambda I) = 0$.

(*iv*) $(I - \mathcal{L})$ is non-singular and $(I - \mathcal{L})^{-1}$ has nonnegative elements. ◊

Example 1.20. In the literature we can find some examples of matrices that converge to zero. Let a, b and $c \in \mathbb{R}_+$.

(*i*) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, with $\max(a, \ b) < 1$.

(*ii*) $A = \begin{pmatrix} a & -c \\ 0 & b \end{pmatrix}$, with $a + b < 1$, $c < 1$.

(*iii*) $A = \begin{pmatrix} a & -a \\ b & -b \end{pmatrix}$, with $|a - b| < 1$, $a > 1$, $b > 0$.

(*iv*) $A = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$ or $A = \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, with $a + b < 1$.

◊

Definition 1.21. Let $(X, p_i)_{i\in I_X}$ be a GLCS. The operator $A : X \to X$ is said to be $\mathcal{L}$-contraction if for every $i \in I_X$ there exists a square matrix $\mathcal{L}_i$ of nonnegative numbers convergent to zero such that

$$p_i(Ax - Ay) \leq \ \mathcal{L}_i \ p_i(x - y), \ \text{ for each } \ x, \ y \in X.$$ ◊

Example 1.22. Let $X = C(I, \mathbb{R})$ the GLCS with the family of GSNs p_T (see Example 1.6). Let $0 < k < 1$, we define the operator $A : X \to X$, $A(x) = kx$, where A is $\mathcal{L}$-contraction. Indeed,

$$p_T(A(x) - A(y)) \leq \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} p_T(x - y)$$ ◊

2. Fixed point results

We state our first fixed point theorem which is a generalization of Schauder-Tychonoff fixed point theorem [23, Theorem 4.5.1].

Theorem 2.1. Let X be a Hausdorff GLCS, $M \subset X$ be a nonempty convex subset of X, and let K be a compact subset of X with $K \subset M$. If $A : M \to X$ is a continuous operator with $A(M) \subset K \subset M$, then A has at least one fixed point in M. ◊

Proof. Let $G : X \to \widetilde{X}$, $G(x) = x$. We define the operator $\widetilde{A}(x) = G \circ A \circ G^{-1}(x)$, for $x \in \widetilde{M} := G(M) = M$. It is clear that $\widetilde{M}$ is a convex subset of $\widetilde{X}$, and $\widetilde{A}(\widetilde{M}) \subset G(K) \subset \widetilde{M}$. Since K is a compact subset of X and from Lemma 1.16, G is continuous, then $G(K)$ is a compact subset of $\widetilde{X}$. By Schauder-Tychonoff fixed point theorem, there exists $\widetilde{x} \in \widetilde{M}$ such that

$$\widetilde{x} = \widetilde{A}(\widetilde{x}).$$

Consequently, there exits $x \in M$ such that $x = Ax$. This completes the proof. □

It is known that the closed convex hull of a compact set is compact (Krein-Šmulian theorem) in Banach spaces, so we can draw the following result in generalized Banach spaces. However, it is not true in GLCS.

Corollary 2.2. Let X be a generalized Banach space and M be a closed convex subset of X. Suppose that $A : M \to M$ is a continuous, compact map. Then, A has at least one fixed point in M. ◊

Proof. Since $A(M)$ is compact, then $K = \overline{conv}(A(M))$ is also a compact and convex subset in X. It is clear that $A(M) \subset K \subset M$. Hence, by Theorem 2.1, the operator A has at least one fixed point in M. □

Theorem 2.3. Let $(X, p_i)_{i\in I_X}$ be a Hausdorff complete GLCS. Suppose that the mapping $A : X \to X$ is $\mathcal{L}$-contraction . Then, A has a unique fixed point x^*, and $A^k x \to x^*$ for every $x \in X$. ◊

Proof. Let $x \in X$ and $i \in I_X$, then there exists $\mathcal{L}_i$ a square matrix of nonnegative numbers converging to θ such that

$$p_i(Ax - Ay) \leq \mathcal{L}_i \ p_i(x - y), \ \text{ for all } \ x, y \in X.$$

We define the sequence $x_k = A^k x$, then

$$p_i(x_{k+1} - x_k) \ \leq \ \mathcal{L}_i^k p_i(Ax - x). \tag{6}$$

As a consequence of Eq.(6) and Lemma 1.19 (*ii*), we get

$$p_i(x_k - x_{k+m}) \ \leq \ \mathcal{L}_i^k (I - \mathcal{L}_i)^{-1} p_i(Ax - x).$$

Since $\mathcal{L}_i$ converge to θ , then $(x_k)_k$ is a Cauchy sequence in X, and so $(x_k)_k$ converge to a point $x^* \in X$. Using the continuity of A, we get

$$x^* = \lim_{k\to\infty} A^k x = A(\lim_{k\to\infty} A^{k-1} x) = Ax^*.$$

Since X is a Hausdorff space, we obtain the uniqueness of the fixed point. □

Corollary 2.4. Let $(X, p_i)_{i\in I_X}$ be a Hausdorff complete GLCS, and $A : X \to X$ is $\mathcal{L}$-contraction. Then, $(I - A)$ is a homeomorphism. ◊

Proof. Clearly $(I - A)$ is continuous. Now, we prove that $(I - A)$ is bijective. Let $y \in X$ such that

$$y = (I - A)(x), \ \ x \in X.$$

We define the operator $\widetilde{A} : X \to X$, $\widetilde{A}(x) = A(x) + y$. Since A is $\mathcal{L}$-contraction, then $\widetilde{A}$ is $\mathcal{L}$-contraction too. By Theorem 2.3, the operator $\widetilde{A}$ has a unique fixed point in X. Hence, $(I - A)$ is bijective. Now, we show that $(I-A)^{-1} : (I-A)(X) \to X$ is continuous. Let $x, y \in (I-A)(X)$. So, there exists $a, b \in X$ such that $a = (I-A)^{-1}x$, and $b = (I-A)^{-1}y$. So, for $i \in I_X$, we get

$$p_i(a-b) = p_i((I-A)^{-1}x - (I-A)^{-1}y),$$

and

$$\begin{aligned} p_i(x-y) &= p_i((I-A)a - (I-A)b) \\ &\geq p_i(a-b) - p_i(A(a) - A(b)) \\ &\geq (I - \mathcal{L}_i)p_i(a-b). \end{aligned}$$

Using hypothesis and Lemma 1.19, we obtain

$$p_i((I-A)^{-1}x - (I-A)^{-1}y) \leq (I-\mathcal{L}_i)^{-1}p_i(x-y).$$

Then, $(I-A)^{-1}$ is continuous. □

Now, we shall give a version of Krasnoselskii fixed point theorem in GLCSs.

Theorem 2.5. Let $(X, p_i)_{i \in I_X}$ be a Hausdorff complete GLCS. Let M be a closed convex subset of X and A, B be two operators from M into X such that:

(*i*) A is continuous,

(*ii*) B is $\mathcal{L}$-contraction,

(*iii*) $A(M)$ is contained in a compact set, and

(*iv*) $Bx + Ay \in M$, for each x, $y \in M$.
Then, the operator $A + B$ has a fixed point in M.

Proof. Let $y \in M$ and let $i \in I_X$. The mapping which assign for each $x \in M$ the values $Bx + Ay$ is $\mathcal{L}$-contraction . By Theorem 2.3 there exists a unique point $Ty \in M$ such that $Ty = BTy + Ay$, using assumption (*iv*), we get $Ty \in M$. Let y_1, $y_2 \in M$ so

$$p_i(Ty_1 - Ty_2) \ \leq \ (I-\mathcal{L}_i)^{-1}p_i(Ay_1 - Ay_2). \tag{7}$$

From the above estimation and assumption (*i*) we derive that the operator T is continuous. Now, let us prove that $T(M)$ is contained in a compact set. Let $(x_\alpha)_\alpha$ be a net in $T(M)$. From assumption (*iii*), the net $(Ax_\alpha)_\alpha$ has a convergent subnet $(Ax'_\alpha)_\alpha$, so $(Ax'_\alpha)_\alpha$ is a Cauchy net and by Eq. (7), $T(x_\alpha)_\alpha$ has a convergent subnet $(Tx'_\alpha)_\alpha$ is Cauchy net. Hence, $T(M)$ is contained in a compact set, then by Theorem 2.1 there exists $y \in M$ such that $y = Ty = By + Ay$. This completes the proof. □

3. Application 1

Differential equations with impulses are often used when modelling a variety of phenomena in engineering, physics and life sciences. This type of equations were considered for the first time by Milman and Myshkis [13] and then, followed by a period of active research which culminated with the monograph by Halanay and Wexler [10]. Impulsive problems arise also in various applications in communications, mechanics electrical engineering, medicine and biology. An introduction to the theory of impulsive differential equations can be found in the books [6, 9]. In this section, we shall use some fixed point theorem proved in the last section to study the existence and uniqueness results for the following system of impulsive differential equation.

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in J_* := [0, +\infty)\backslash\{t_0, t_1, \cdots t_k \cdots\}, \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)) \text{ for all } k \in \mathbb{N}, \\ y(0) = y_0. \end{cases} \tag{8}$$

Where $f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $I_k \in C(\mathbb{R},\mathbb{R})$ with $0 < t_0 < t_1 < \cdots < t_k < \cdots, \lim_{k\to\infty} t_k = \infty$. We denote by $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ the right and the left limits of the function y at $t = t_k$, respectively.
We define the space of piece wise continuous functions $PC([0,+\infty),\mathbb{R})$ (in short PC) by

$$PC = \Big\{y : [0,+\infty) \to \mathbb{R}, y \in C(J_*,\mathbb{R}) : \text{for all } k,\ y(t_k^+) \text{ and } \ y(t_k^-) \text{ exist and satisfy } \ y(t_k^-) = y(t_k)\Big\}.$$

We assume the following conditions

($\mathcal{H}_1$) $f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $t \longmapsto f(t,u)$ is measurable for each $u \in \mathbb{R}$.

($\mathcal{H}_2$) For each $c > 0$ there exists a function $l_c \in L^1_{loc}([0,c],\mathbb{R})$ such that:

$$|f(t,u) - f(t,\overline{u})| \le l_c(|u - \overline{u}|), t \in [0,c] \ \text{ for all } \ u,\overline{u} \in \mathbb{R}.$$

($\mathcal{H}_3$) For all $k \in \mathbb{N}$, there exists $a_k \ge 0$ such that

$$|I_k(u) - I_k(\overline{u})| \le a_k|u - \overline{u}|, \ \text{ for all } \ u,\overline{u} \in \mathbb{R},$$

with

$$\frac{1}{\sigma} + \sum_{k=1}^{\infty} a_k < 1, \ \text{ for } \sigma \ \text{ large enough.}$$

Theorem 3.1. Assume that ($\mathcal{H}_1$)-($\mathcal{H}_3$) are satisfied. Then, the problem (8) has a unique solution. ◊

Proof. We define the family of seminorms in PC, for all $n \in \mathbb{N}$ and σ is large enough.

$$|y|_n = \sup\Big\{\exp(-\sigma g_n(t))|y(t)|, \ 0 \le t \le t_n\Big\},$$

where $g_n(t) = \int_0^t l_n(s)ds$. From Example 1.6, we can define the generalized seminorm in PC by $p_n : PC \to \mathbb{R}^2$, $p_n(y) = \begin{pmatrix} |y|_n \\ |y|_n \end{pmatrix}$. Observe that $PC = \bigcap_{m\ge 1} PC_m$, where

$$PC_m = \Big\{y \in C(J_*,\mathbb{R}) : y(t_k^-) \text{ and } \ y(t_k^+) \text{ exist}, y(t_k^-) = y(t_k),\ k = 0,1,\cdots,m\Big\}.$$

We conclude that PC is a complete generalized locally convex space (we can call it generalized Frèchet space). Now, we define the operator $N : PC \to PC$ by

$$y \longmapsto Ny(t) = y_0 + \int_0^t f(s,y(s))ds + \sum_{0\le t_k\le t} I_k(y(t_k)).$$

We will prove that N is $\mathcal{L}$-contraction. For this purpose, let $u,\overline{u} \in \mathbb{R}$. Then,

$$\begin{aligned}
|Nu(t) - N\overline{u}(t)| &\leq \int_0^t |f(s,u(s)) - f(s,\overline{u}(s))|ds + \sum_{0\leq t_k\leq t} |I_k(u(t_k^-)) - I_k(\overline{u}(t_k^-))| \\
&\leq \int_0^t l_n(s)|u(s) - \overline{u}(s)|ds + \sum_{0\leq t_k\leq t} a_k|u(t_k^-) - \overline{u}(t_k^-)| \\
&\leq \int_0^t l_n(s)\exp(\sigma g_n(t))\exp(-\sigma g_n(t))|u(s) - \overline{u}(s)|ds \\
&+ \sum_{0\leq t_k\leq t} a_k \exp(\sigma g_n(t))\exp(-\sigma g_n(t))|u(t_k^-) - \overline{u}(t_k^-)| \\
&\leq \frac{1}{\sigma}\int_0^t \sigma l_n(s)\exp(\sigma g_n(s))ds\,|u - \overline{u}|_n + \sum_{0\leq t_k\leq t} a_k\exp(\sigma g_n(t))|u - \overline{u}|_n \\
&\leq \frac{1}{\sigma}\int_0^t \Big(\exp(\sigma g_n(t))\Big)' ds|u - \overline{u}|_n + \sum_{0\leq t_k\leq t} a_k\exp(\sigma g_n(t))|u - \overline{u}|_n \\
&\leq \frac{1}{\sigma}\exp(\sigma g_n(t))|u - \overline{u}|_n + \exp(\sigma g_n(t))\sum_{k=1}^{\infty} a_k|u - \overline{u}|_n,
\end{aligned}$$

then,

$$\begin{aligned}
\exp(\sigma g_n(t))|Nu(t) - N\overline{u}(t)| &\leq [\frac{1}{\sigma} + \sum_{0\leq t_k\leq t} a_k]|u - \overline{u}|_n, \\
|Nu(t) - N\overline{u}(t)|_n &\leq [\frac{1}{\sigma} + \sum_{0\leq t_k\leq t} a_k]|u - \overline{u}|_n
\end{aligned}$$

so,

$$p_n(Nu - N\overline{u}) \leq \mathcal{L}\, p_n(u - \overline{u}),$$

where

$$\mathcal{L} = \begin{pmatrix} \frac{1}{\sigma} & \sum_{k=1}^{\infty} a_k \\ \frac{1}{\sigma} & \sum_{k=1}^{\infty} a_k \end{pmatrix}.$$

From assumption $(\mathcal{H}_3)$ and Example 1.20 (iv), $\mathcal{L}$ is convergent to θ and so N is $\mathcal{L}$-contraction. By Theorem 2.3, N has a unique fixed point. Hence, the problem (8) has a unique solution. □

4. Application 2

Here, we deal with the following system of first order differential equation:

$$\begin{cases} x'(t) = f_1(t,x(t),y(t)), & t \in [0,+\infty), \\ y'(t) = f_2(t,x(t),y(t)), & t \in [0,+\infty), \\ x(t_k^+) - x(t_k^-) = I_{1k}(x(t_k^-), y(t_k^-)), \\ y(t_k^+) - y(t_k^-) = I_{2k}(x(t_k^-), y(t_k^-)), \\ x(0) = x_0, \\ y(0) = y_0. \end{cases} \tag{9}$$

where $f_i : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $I_{ik} \in C(\mathbb{R}\times\mathbb{R}, \ \mathbb{R})$ $i = 1, 2$. The notations $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k - h)$ stand for the right and the left limits of the function x at $t = t_k$, respectively.

We assume the following:

$(\mathcal{H}_4)$ For $i = 1, 2$ $f_i : [0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is such that $f_i(\cdot, x, y)$ is measurable for each $x, y \in \mathbb{R}$ and $f_i(t, \cdot, \cdot)$ is continuous for almost all $t \in [0, +\infty)$.

$(\mathcal{H}_5)$ For each $r > 0$ there exist functions $l_r^i \in L^1_{loc}([0, r], \mathbb{R})$ such that

$$|f_i(t, u, v) - f_i(t, \overline{u}, \overline{v})| \leq \ l_r^i(t)|u - \overline{u}| + l_r^i(t)|v - \overline{v}|.$$

$(\mathcal{H}_6)$ For $k \in \mathbb{N}$ the function $I_{ik} : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and there exist nonnegative constants a_{ik} and b_{ik} $i = 1, 2$ such that

$$|I_{ik}(u, v) - I_{ik}(\overline{u}, \overline{v})| \leq \ a_{ik}|u - \overline{u}| + b_{ik}|v - \overline{v}|,$$

and for $i = 1, 2$

$$\sum_{k=1}^{\infty} a_{ik} \leq \ \infty, \ \ \sum_{k=1}^{\infty} b_{ik} \leq \ \infty.$$

Theorem 4.1. Assume that $(\mathcal{H}_4)$- $(\mathcal{H}_6)$ are satisfied and the matrix

$$M = \begin{pmatrix} \frac{1}{\sigma} + \sum_{k=1}^{\infty} a_{1k} & \frac{1}{\sigma} + \sum_{k=1}^{\infty} b_{1k} \\ \\ \frac{1}{\sigma} + \sum_{k=1}^{\infty} a_{2k} & \frac{1}{\sigma} + \sum_{k=1}^{\infty} b_{2k} \end{pmatrix}, \tag{10}$$

is convergent to zero. Then, the problem (9) has a unique solution. ◊

Proof. We define the family of seminorms in $PC \times PC$ for all $n \in \mathbb{N}$. $P : PC \times PC \to \mathbb{R}^2$, $P(x, y) = \begin{pmatrix} |x|_n \\ |y|_n \end{pmatrix}$.
Let $y \in PC$, then y is a solution of the impulsive system (9) if, and only if, y is a solution of the following impulsive integral equation

$$\begin{cases} x(t) = x_0 + \int_0^t f_1(s, x(s), y(s))ds + \sum_{k=1}^{\infty} I_{1k}(x(t_k^-), y(t_k^-)) \\ y(t) = y_0 + \int_0^t f_2(s, x(s), y(s))ds + \sum_{k=1}^{\infty} I_{2k}(x(t_k^-), y(t_k^-)) \end{cases} \tag{11}$$

Consider the operator $N : PC \times PC \to PC \times PC$ defined by

$$N(x, y) = \begin{pmatrix} N_1(x, y) \\ N_2(x, y) \end{pmatrix} = \begin{pmatrix} x_0 + \int_0^t f_1(s, x(s), y(s))ds + \sum_{k=1}^{\infty} I_{1k}(x(t_k^-), y(t_k^-)) \\ y_0 + \int_0^t f_2(s, x(s), y(s))ds + \sum_{k=1}^{\infty} I_{2k}(x(t_k^-), y(t_k^-)) \end{pmatrix}.$$

We now show the existence of solutions for the impulsive problem (9) from Theorem 2.3. We have to prove that N is M-contractive. Let (x, y), $(\overline{x}, \overline{y}) \in \mathbb{R}^2$

$$\begin{aligned}|N_i(x,y)-N_i(\overline{x},\overline{y})| \le\ & \int_0^t |f_i(s,x(s),y(s))-f_i(s,\overline{x}(s),\overline{y}(s))| \\ & +\sum_{k=1}^{\infty}|I_{ik}(x(t_k^-),y(t_k^-))-I_{ik}(\overline{x}(t_k^-),\overline{y}(t_k^-))| \\ \le\ & \int_0^t l_n^i(s)|x(s)-\overline{x}(s)|+l_n^i|y(s)-\overline{y}(s)| \\ & +\sum_{k=1}^{\infty}a_{ik}|x(s)-\overline{x}(s)|+\sum_{k=1}^{\infty}b_{ik}|y(s)-\overline{y}(s)| \\ \le\ & \int_0^t l_n^i(s)\exp(\sigma g_n(t))\exp(-\sigma g_n(t))\Big[|x(s)-\overline{x}(s)|+|y(s)-\overline{y}(s)|\Big]ds \\ & +\sum_{k=1}^{\infty}\exp(\sigma g_n(t))\exp(-\sigma g_n(t))\Big[a_{ik}|x(t_k^-)-\overline{x}(t_k^-)|+b_{ik}|y(t_k^-)-\overline{y}(t_k^-)|\Big] \\ \le\ & \frac{1}{\sigma}\int_0^t \sigma l_n^i(s)\exp(\sigma g_n(t))\Big[|x-\overline{x}|_n+|y-\overline{y}|_n\Big]ds \\ & +\sum_{k=1}^{\infty}\exp(\sigma g_n(t))\Big[a_{ik}|x-\overline{x}|_n+|y-\overline{y}|_n\Big] \\ \le\ & \frac{1}{\sigma}\Big[|x-\overline{x}|_n+|y-\overline{y}|_n\Big]+\sum_{k=1}^{\infty}\exp(\sigma g_n(t))\Big[a_{ik}|x-\overline{x}|_n+b_{ik}|y-\overline{y}|_n\Big],\end{aligned}$$

hence,

$$|N_i(x,y)-N_i(\overline{x},\overline{y})|_n \le \frac{1}{\sigma}\Big[|x-\overline{x}|_n+|y-\overline{y}|_n\Big]+\sum_{k=1}^{\infty}a_{ik}|x-\overline{x}|_n+\sum_{k=1}^{\infty}b_{ik}|y-\overline{y}|_n,$$

thus,

$$P(N(x,y)-N(\overline{x},\overline{y}))=\begin{pmatrix}|N_1(x,y)-N_1(\overline{x},\overline{y})|\\|N_2(x,y)-N_2(\overline{x},\overline{y})|\end{pmatrix}\le M\begin{pmatrix}|x-\overline{x}|_n\\|y-\overline{y}|_n\end{pmatrix}=M\ P((x,y)-(\overline{x},\overline{y})).$$

where M is given by (10) and converges to zero. The result follows now from Theorem 2.3. □

Example 4.2. Consider the following problem

$$\begin{cases} x'(t)=\dfrac{1}{(t+1)(t+2)}\Big(x^2(t)+y^2(t)\Big)\equiv f_1(t,x,y),\\ y'(t)=\dfrac{1}{(t+1)(t+2)}\Big(|x(t)+|y(t)|\Big)\equiv f_2(t,x,y),\\ I_{1k}(x,y)=a_kx(k), a_k>0,\\ I_{2k}(x,y)=b_ky(k), b_k>0,\\ x(0)=0,\\ y(0)=0.\end{cases} \tag{12}$$

Let $r>0$ and $(x,y),\ (\overline{x},\overline{y})\in\mathbb{R}^2$ such that $|x|\le\ r$.

$$|f_1(t,x,y)-f_1(t,\overline{x},\overline{y})|\le \frac{2r}{(t+1)(t+2)}\Big[|x-\overline{x}|+|y-\overline{y}|\Big],$$

and,

$$|f_2(t,x,y) - f_2(t,\overline{x},\overline{y})| \leq \frac{2r}{(t+1)(t+2)}\Big[|x-\overline{x}| + |y-\overline{y}|\Big].$$

Let $l_r(t) = \dfrac{2r}{(t+1)(t+2)} \in L^1_{loc}([0,+\infty),\mathbb{R})$. Also,

$$|I_{1k}(x,y) - I_{1k}(\overline{x},\overline{y}) \leq a_k|x-\overline{x}|.$$

$$|I_{2k}(x,y) - I_{2k}(\overline{x},\overline{y}) \leq b_k|y-\overline{y}|.$$

If

$$\sum_{k=1}^{\infty} a_k \leq 1 \text{ and } \sum_{k=1}^{\infty} b_k \leq 1,$$

then by Theorem 4.1 the impulsive problem (12) has a unique solution. ◇

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References

[1] T. Abdeljawad, E. Karapinar, *A common fixed point theorem of a gregus type on convex cone metric spaces*, J. Comput. Anal. Appl. **13(4)** (2011), 609–621.

[2] M. Aliyari, M. Gabeleh, E. Karapinar, *Mann and ishikawa iterative processes for cyclic relatively nonexpansive mappings in uniformly convex Banach spaces*, J. Nonlinear Convex Anal. **22(4)** (2021), 699–713.

[3] M. Asadi, *Some results of fixed point theorems in convex metric spaces*, Nonlinear Funct. Anal. Appl. **19(2)** (2014), 171–175.

[4] M. Asadi, *On ekeland's variational principle in m-metric spaces*, J. Nonlinear Convex Anal. **17(6)** (2016), 1151–1158.

[5] M. Asadi, *Discontinuity of control function in the (f, φ, θ)-contraction in metric spaces*, Filomat **31(17)** (2017), 5427–5433.

[6] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive differential equations and inclusions*, Hindawi Publishing Corporation, New York, 2006.

[7] H. Berrezoug, J. Henderson, A. Ouahab, *Existence and uniqueness of solutions for a system of impulsive differential equations on the halfline*, J. Nonlinear Funct. Anal. (2017), 1–16.

[8] L. Gholizadeh, E. Karapinar, M. Roohi, *Some fixed point theorems in locally p-convex spaces*, Fixed Point Theory Appl. **1** (2013), 1–10.

[9] J. R. Graef, J. Henderson, A. Ouahab, *Topological methods for differential equations and inclusions*, CRC Press, 2018.

[10] A. Halany and D. Wexler, *Teoria calitativa a systeme cu impulduri*, Editura Republicii Socialiste Romania, Bucharest, 1968.

[11] A. Jeribi, B. Krichen, *Nonlinear functional analysis in Banach spaces and Banach algebras: fixed point theory under weak topology for nonlinear operators and block operator matrices with applications*, volume 12, CRC Press, 2015.

[12] E. Karapinar, A. Fulga, *Fixed point on convex b-metric space via admissible mappings*, TWMS J. Pure. Appl. Math. **12(2)** (2021), 254–264.

[13] V. D. Milman, A. D. Myshkis, *On the stability of motion in the presence of impulses*, Sib. Math. J. **1(2)** (1960), 233–237.

[14] J. J. Nieto, A. Ouahab, R. Rodriguez-Lopez, *Fixed point theorems in generalized* Banach *algerbras and applications*, Fixed Point Theory **19(2)** (2018), 707–732.

[15] A. Ouahab, *Some* P*ervo's and* K*rasnoselskii type fixed point results and application*, Comm. Appl. Nonlinear Anal. **19** (2015), 623–642.

[16] A. I. Perov, *The* C*auchy problem for systems of ordinary differential equations*, 1964.

[17] A. I. Perov, A. Kibenko, *On a certain general method for investigation of boundary value problems*, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya **30(2)** (1966), 249–264.

[18] I. R. Petre, *A multivalued version of* K*rasnoselskii's theorem in generalized* B*anach spaces*, Analele Universitatii" Ovidius" Constanta-Seria Matematica, **22(2)** (2014), 177–192.

[19] I. R. Petre, A. Petrusel, K*rasnoselskii's theorem in generalized* B*anach spaces and application*, Electronic Journal of Qualitative Theory of Differential Equations **85** (2012). 1–20.

[20] A. Petruşel, G. Petruşel, C. Urs, *Vector-valued metrics, fixed points and coupled fixed points for nonlinear operators*, Fixed Point Th. Appl. **1** (2013), 218.

[21] J. Simon, *Espaces de Banach, Fréchet, Hilbert et Neumann*, volume 1, ISTE Group, 2017.

[22] M. L. Sinacer, J. J. Nieto, A. Ouahab, *Random fixed point theorem in generalized* B*anach space and applications*, Random Operators and Stochastic Equations, **24(2)** (2016), 93–112.

[23] D. R. Smart, *Fixed point theorems*, volume 66, CUP Archive, 1980.

[24] A. Viorel, *Contributions to the study of nonlinear evolution equations*, PhD thesis, 2011.

[25] M. Younis, D. Singh, M. Asadi, V. Joshi, *Results on contractions of* R*eich type in graphical b-metric spaces with applications*, Filomat **33(17)** (2019). 5723–5735.