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On the existence and multiplicity of positive solutions for a p-Laplacian fractional boundary value problem with an integral boundary condition

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Abstract. Aim of this work is to investigate existence and multiplicity of positive solutions of a fractional boundary value problem with an integral boundary condition with p-Laplacian operator. Necessary and sufficient conditions are presented to obtain existence and multiplicity results. Main tools are Krasnoselskii, Schaefer and Leggett-Williams fixed point theorems. Two examples are given to illustrate our results.

1. Introduction

Fractional Calculus as an extension of integer order calculus to arbitrary order calculus, has attracted the attention of many scientists in recent years. With the development of fractional calculus, fractional differential equations have wide applications in modeling of different Physical phenomena and engineering, such as mechanics, chemistry, control system, etc. see [11, 21, 22, 25].
In the past decades, numerous articles have been published about the existence and uniqueness of solutions to the fractional initial and boundary value problems, the existence, uniqueness and multiplicity of positive solutions to the fractional initial and boundary value problems (see [1–7, 14, 23, 24, 30].
The study of boundary value problems, for fractional differential equations with p-Laplacian operators has attracted the attentions of mathematicians quite recently [8, 10, 12, 16–19, 26–28]. but only few paper can be found in the literature dealing with p-Laplacian fractional order boundary value problems with integral boundary condition.
Zhang et al. [29] discussed the eigenvalue problem for a class of singular p-Laplacian fractional differential equations involving the Riemann-Stieltjes integral boundary conditions

$$-D_t^{\beta}\left(\varphi_p(D_t^{\alpha}x)\right)(t) = \lambda f(t, x(t)), \quad t \in (0, 1)$$
$$x(0) = 0, \quad D_t^{\alpha}x(0) = 0, \quad x(1) = \int_0^1 x(s)dA(s)$$

where D_t^{β} and D_t^{α} are the standard Riemann-Liouville fractional derivatives with $1 < \alpha \leq 2, 0 < \beta \leq 1$, $\varphi_p(s) = |s|^{p-2}s$, A is a function of bounded variation and $\int_0^1 x(s)dA(s)$ denotes the Riemann-Stieltjes integral

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of x with respect to A, $f(t;x):(0;1)\times(0;1)\to[0;1)$ is continuous and may be singular at $t=0;1$ and $x=0$. their results are derived based on the method of upper and lower solutions and the Schauder fixed point theorem.

Yunhong Li and Guogang Li [16] by using the five functionals fixed point theorem, obtained the existence and multiple positive solutions for p-Laplacian fractional differential equations with integral boundary value conditions

$$\begin{aligned}
&D^{\beta}_{0^+}\left(\varphi_p(D^{\alpha}_{0^+}u(t))\right)+\lambda f(t,u(t))=0,\quad t\in(0,1)\\
&\varphi_p(D^{\alpha}_{0^+}u(0))^{(i)}=0,\quad i=1,2,\dots,l-1\\
&\varphi_p(D^{\alpha}_{0^+}u(1))=\int_0^1 h(t)\varphi_p(D^{\alpha}_{0^+}(u(t))dt,\\
&u^{(j)}=0,\quad j=1,2,\dots,n-1,\\
&u(0)=\int_0^1 k(t)u(t)dt,
\end{aligned}$$

where $D^{\beta}_{0^+}$ and $D^{\alpha}_{0^+}$ are the Caputo fractional derivative, $l-1<\beta\le l, n-1<\alpha\le n, l\ge1, n\ge1, \varphi_p(s)=|s|^{p-2}s$ and $l+n-1<\alpha+\beta\le l+n$.

Zhang, et al. by using Avery–Peterson fixed point theorem, obtained the existence of positive solutions for boundary value problem

$$\begin{aligned}
&D^{\beta}_{0^+}\varphi_p(D^{\alpha}_{0^+}x(t))+f(t,x(t),D^{\beta}_{0^+}x(t))=0,\quad t\in(0,1),\\
&\varphi_p(D^{\alpha}_{0^+}x(0))^{(i)}=\varphi_p(D^{\alpha}_{0^+}x(1))=0,\quad i=1,2,\dots,m-1,\\
&x(0)+x'(0)=\int_0^1 g_0(s)x(s)ds+a,\\
&x(1)+x'(1)=\int_0^1 g_1(s)x(s)ds+b,\\
&x^{(j)}(0)=0,\quad j=2,3,\dots,n-1,
\end{aligned}$$

where $1<n-1<\alpha<n, 1<m-1<\beta<m, \alpha-\beta>1, D^{\beta}_{0^+}$ are the Caputo fractional derivative, $g_0,g_1\in C([0,1],[0,+\infty))$, $f\in C([0,1]\times[0,+\infty),[0,+\infty))$ are given functions. a,b are disturbance parameters and $\varphi_p(s)=|s|^{p-2}s$.

Motivated by the aforementioned works, this paper discusses the existence of positive solutions for the fractional boundary value problem

$$\begin{aligned}
&\left(\varphi_p({}^cD^{\alpha}_{0^+}u(t))\right)'+a(t)f(t,u(t))=0,\quad t\in(0,1),\\
&{}^cD^{\alpha}_{0^+}u(0)=u'(0)=u''(0)=0, u(1)+u'(1)=\int_0^{\eta}u(t)dt.
\end{aligned}\tag{1}$$

where $2<\alpha\le3$, $0<\eta<1$, ${}^cD^{\alpha}_{0^+}$ is the Caputo fractional derivative, and $\varphi_p(s)=|s|^{p-2}s, p>1$. and $f(t,u(t))$ has some properties that will be presented in next sections. The organization of this paper is as follows. In section 2, some necessary definitions from fractional calculus theory will be presented. In section 3, Green function of the problem will be calculated and some properties of this function will be presented. In section 4, main results about the existence and uniqueness of positive solutions of the fractional boundary value problem (1) will be obtained and in section 5, some examples will be given to illustrate our main results.

2. Preliminaries

In this section, notations, definitions and preliminary facts which are used throughout this paper are introduced. At first, let us recall some basic definitions of fractional calculus which can be found in [11, 20–22, 25].

Definition 2.1. *The Reimann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{2}$$

provided that the right-hand side is point-wise defined on $(0,+\infty)$, where Γ is the Gamma function.

Definition 2.2. *The Reimann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is given by*

$$D^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s)ds, \tag{3}$$

where $n = [\alpha] + 1$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3. *The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is given by*

$${}^cD^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} f(s)ds, \tag{4}$$

where $n = [\alpha] + 1$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.4. *(See [25]) Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I^{\alpha}_{0^+}\, {}^cD^{\alpha}_{0^+} u(t) = u(t) + C_1 + C_2 t^2 + \cdots + C_n t^n,$$

where $n = [\alpha] + 1$.

Throughout this paper we let $\mathcal{B}$ equal to the Banach space $C([0,1],\mathbb{R})$ that equipped with the supremum norm

$$\|u\| = \sup_{t\in[0,1]} |u(t)|$$

Definition 2.5. *Let E be a real Banach space, A nonempty, closed, convex set $P \subset E$ is a cone if it satisfies the following conditions*

(i) If $x \in P$, $\lambda \geq 0$ then $\lambda x \in P$;

(ii) If $x \in P$ and $-x \in P$ then $x = 0$.

To prove our results, we need some fixed point theorems.

Theorem 2.6. *(Krasnoselskii's)[13] Let E be a Banach space and $P \subset E$, be a cone. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$ and let*

$$T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$$

be a completely continuous operator such that

(i) $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_2$; or

(ii) $\|Tu\| \geq \|u\|$, $u \in P \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in P \cap \partial\Omega_2$

Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$

Theorem 2.7. *(Schaefer)[9] Let E be a Banach space B a closed convex subset of E, U an open subset of B and $0 \in U$. Suppose that $T : \overline{U} \to B$ is a continuous, compact map. Then either*

(i) T has a fixed point in $\overline{U}$,or

(ii) There is a $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda A(u)$.

Theorem 2.8. *(Legget-Williams)[15]Let $T : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and ψ a nonnegative continuous concave functional on P such that $\psi(u) \le \|u\|$ for all u in $\overline{P}_c$. Suppose that there exists constant $0 < a < b < d \le c$ such that*

(i) $\{u \in P(\psi, b, d) : \psi(u) > b\} \neq 0$ and $\psi(Tu) > b$ if $u \in P(\psi, b, d)$,

(ii) $\|Tu\| < a$ if $u \in P_a$,

(iii) $\psi(Tu) > b$ for $u \in P(\psi, b, c)$ with $\|Tu\| > d$.

Then, T has at least three fixed points u_1, u_2 and u_3 such that $\|u_1\| < a$, $b < \psi(u_2)$ and $\|u_3\| > a$ with $\psi(u_3) < b$.

In order to apply the fixed point theorems, we need to calculate the Green function of the desired operator. In this section in addition to calculate Green function, we also outline some properties of it which is used throughout this paper.

Lemma 2.9. *Suppose $h : [0,1] \to [0,\infty)$ be a continuous function, then the unique solution of the fractional boundary value problem*

$$\begin{aligned} &\left(\varphi_p({}^cD^{\alpha}_{0^+}u(t))\right)' + h(t), \quad t \in (0,1), \quad 2 < \alpha \le 3 \\ &{}^cD^{\alpha}_{0^+}u(0) = u'(0) = u''(0) = 0, u(1) + u'(1) = \int_0^{\eta} u(t)dt, \quad 0 < \eta < 1. \end{aligned} \tag{5}$$

expressed by

$$u(t) = \int_0^1 G(t,s)\varphi\left(\int_0^s h(\tau)d\tau\right)ds \tag{6}$$

where

$$G(t,s) = G_1(t,s) + \frac{\eta}{1-\eta}G_2(\eta,s), \tag{7}$$

$$G_1(t,s) = \begin{cases} \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le t \le s \le 1, \end{cases} \tag{8}$$

$$G_2(t,s) \begin{cases} \frac{(1-s)^{\alpha-1}-\frac{1}{\alpha}(t-s)^{\alpha}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}, & 0 \le t \le s \le 1. \end{cases} \tag{9}$$

Proof. Integrating the first equation of (5), follows

$$\varphi_p({}^cD^{\alpha}_{0^+}u(t)) - \varphi_p({}^cD^{\alpha}_{0^+}u(0)) = -\int_0^t h(s)ds,$$

and so,

$${}^cD^{\alpha}_{0^+}u(t) = -\varphi_q\left(\int_0^t h(s)ds\right).$$

From Lemma 2.4, we get

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds+C_0+C_1t+C_2t^2.$$

Using the boundary conditions $u'(0)=u''(0)=0$, we have $C_1=C_2=0$. So,

$$u(t)=-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds+C_0, \tag{10}$$

$$u'(t)=-\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds.$$

By the boundary condition $u(1)+u'(1)=\int_0^\eta u(t)dt$, we have

$$\begin{aligned} C_0 &= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds+\int_0^\eta u(t)dt \\ &+\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds \end{aligned}$$

Now by inserting C_0 into (10), we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &+\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds+\int_0^\eta u(t)dt \\ &+\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds. \end{aligned} \tag{11}$$

By integrating the above relation from 0 to η, we have

$$\begin{aligned} \int_0^\eta u(t)dt &= -\frac{1}{\Gamma(\alpha)}\int_0^\eta\int_0^t (t-s)^{\alpha-1}\varphi_q\left(\int_0^s h(\tau)d\tau\right)dsdt \\ &+\frac{1}{\Gamma(\alpha)}\int_0^\eta\int_0^1 (1-s)^{\alpha-1}\varphi_q\left(\int_0^s h(\tau)d\tau\right)dsdt+\int_0^\eta\int_0^\eta u(s)dsdt \\ &+\frac{1}{\Gamma(\alpha-1)}\int_0^\eta\int_0^1 (1-s)^{\alpha-2}\varphi_q\left(\int_0^s h(\tau)d\tau\right)dsdt \\ &= -\frac{1}{\alpha\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha}h(s)ds+\frac{\eta}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &+\frac{\eta}{\Gamma(\alpha-1)}\int_0^1 (1-s)^{\alpha-2}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds+\eta\int_0^\eta u(t)dt \end{aligned}$$

So

$$\begin{aligned} \int_0^t u(t)dt &= -\frac{\frac{1}{\alpha}}{(1-\eta)\Gamma(\alpha)}\int_0^\eta (\eta-s)^{\alpha}h(s)ds \\ &+\frac{\eta}{(1-\eta)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds \\ &+\eta(1-\eta)\Gamma(\alpha-1)\int_0^1 (1-s)^{\alpha-2}\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds \end{aligned}$$

By inserting $\int_0^\eta u(t)dt$ into (11), we get

$$\begin{aligned}
u(t) &= \int_0^t \left[\frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds\\
&+\int_t^1 \left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds\\
&+\int_0^\eta \frac{\eta}{1-\eta}\left[\frac{(1-s)^{\alpha-1}-\frac{1}{\alpha}(\eta-s)^{\alpha}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds\\
&+\int_\eta^1 \frac{\eta}{1-\eta}\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds\\
&= \int_0^1 \left(G_1(t,s)+\frac{\eta}{1-\eta}G_2(\eta,s)\right)\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds\\
&= \int_0^1 G(t,s)\varphi_q\left(\int_0^s h(\tau)d\tau\right)ds
\end{aligned}$$

□

Lemma 2.10. *The functions $G_1(t,s)$ and $G_2(t,s)$ defined by* (8) *and* (9) *are continuous on* $[0,1]\times[0,1]$ *and satisfy the following properties*

1. $G_1(t,s)\geq 0, G_2(t,s)\geq 0$ *for all* $t,s\in[0,1]$;
2. $(1-t^{\alpha-1})G_1(s,s)\leq G_1(t,s)\leq G_1(s,s)$ *for all* $(t,s)\in[0,1]\times[0,1]$;
3. $(1-\frac{t^\alpha}{\alpha})G_2(s,s)\leq G_2(t,s)\leq G_2(s,s)$ *for all* $(t,s)\in[0,1]\times[0,1]$;

Proof. It is clear that $G_1(t,s)$ and $G_2(t,s)$ are continuous. We prove statement (1). For $0\leq s\leq t\leq 1$, we have

$$\begin{aligned}
G_1(t,s) &= \frac{(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&\geq \frac{(1-s)^{\alpha-1}-(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&= \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&\geq 0.
\end{aligned}$$

For $s\geq t$, clearly

$$G_1(t,s)=\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\geq 0$$

So,

$$G_1(t,s)\geq 0,\quad \text{for all}\quad t,s\in[0,1].$$

On the other hand for $0\leq s\leq t\leq 1$, we have

$$\begin{aligned}
G_2(t,s) &= \frac{(1-s)^{\alpha-1}-\frac{1}{\alpha}(t-s)^{\alpha}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&\geq \frac{(1-s)^{\alpha-1}-\frac{1}{\alpha}(1-s)^{\alpha}}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&= \frac{(1-s)^{\alpha-1}(1-\frac{1-s}{\alpha})}{\Gamma(\alpha)}+\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\\
&\geq 0.
\end{aligned}$$

For $s \geq t$, clearly

$$G_2(t,s) = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \geq 0$$

So for all $t, s \in [0,1]$, we have $G_2(t,s) \geq 0$.
Next we prove statement (2). It is enough to show statement hold for $0 \leq s \leq t$. If $0 < s \leq t$, then

$$\begin{aligned}
G_1(t,s) &= \frac{(1-s)^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&= \frac{(1-s)^{\alpha-1} - t^{\alpha-1}(1-\frac{s}{t})^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\geq \frac{(1-s)^{\alpha-1} - t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\geq (1-t^{\alpha-1})\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\
&\geq (1-t^{\alpha-1})\left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right] \\
&= (1-t^{\alpha-1})G_1(s,s)
\end{aligned}$$

On the other hand, $G_1(t,s) \leq \left[\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right] = G_1(s,s)$.
Now If $s = 0$, we have

$$\begin{aligned}
G_1(t,0) &= \frac{1-t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} \\
&\leq \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} = G_1(0,0),
\end{aligned}$$

and

$$\begin{aligned}
G_1(t,0) &= \frac{1-t^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} \geq (1-t^{\alpha-1})\left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right] \\
&= (1-t^{\alpha-1})G_1(0,0)
\end{aligned}$$

Thus for all $t, s \in [0,1]$ we have,

$$(1-t^{\alpha-1})G_1(s,s) \leq G_1(t,s) \leq G_1(s,s).$$

(3) If $0 < s \le t$, then

$$\begin{aligned} G_2(t,s) &= \frac{(1-s)^{\alpha-1} - \frac{1}{\alpha}(t-s)^{\alpha}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\geq \frac{(1-s)^{\alpha-1} - \frac{t^\alpha}{\alpha}(1-\frac{s}{t})^{\alpha}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\geq \frac{(1-s)^{\alpha-1} - \frac{t^\alpha}{\alpha}(1-s)^{\alpha}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\geq \frac{(1-s)^{\alpha-1} - \frac{t^\alpha}{\alpha}(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &= \left(1-\frac{t^\alpha}{\alpha}\right)\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \\ &\geq \left(1-\frac{t^\alpha}{\alpha}\right)\left(\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)}\right) \\ &= \left(1-\frac{t^\alpha}{\alpha}\right)G_2(s,s) \end{aligned}$$

On the other hand, for $0 \le t \le s$, we have

$$G_2(t,s) \le \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} = G_2(s,s).$$

Now, if $s = 0$, then

$$\begin{aligned} G_2(t,0) &= \frac{1-\frac{t^\alpha}{\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} \\ &\leq \frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} = G_2(0,0). \end{aligned}$$

and

$$\begin{aligned} G_2(t,0) &= \frac{1-\frac{t^\alpha}{\alpha}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} \\ &\geq \left(1-\frac{t^\alpha}{\alpha}\right)\left[\frac{1}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)}\right] \\ &= \left(1-\frac{t^\alpha}{\alpha}\right)G_2(0,0) \end{aligned}$$

Hence we can conclude that, for $t, s \in [0,1]$,

$$G_2(s,s) \ge G_2(t,s) \ge (1-\frac{t^\alpha}{\alpha})G_2(s,s),$$

□

Lemma 2.11. *Let $\xi \in (0,1)$ be a fixed. Then for $G(t,s)$ we have*

1. $G(t,s) \ge 0$, *for all* $t, s \in [0,1]$,
2. $\left(1-\frac{\eta^\alpha}{\alpha}\right)(1-t^{\alpha-1})H_\eta(s) \le G(t,s) \le H_\eta(s)$ *for all* $0 \le t, s \le 1$,
3. $\left(1-\frac{\eta^\alpha}{\alpha}\right)(1-\xi^{\alpha-1})H_\eta(s) \le G(t,s) \le H_\eta(s)$, *for all* $(t,s) \in [0,\xi]\times[0,1]$,

where $H_\eta(s) = G_1(s,s) + \frac{\eta}{1-\eta}G_2(s,s) = \frac{(\alpha-s)(1-s)^{\alpha-2}}{(1-\eta)\Gamma(\alpha)}$

Proof. It is clear that (1) holds and (3) is the direct result of (2). So We prove only statement (2). From Lemma 2.10 and relation (7), it is concluded that

$$\begin{aligned} G(t,s) &= G_1(t,s) + \frac{\eta}{1-\eta} G_2(\eta,s) \\ &\leq G_1(s,s) + \frac{\eta}{1-\eta} G_1(s,s) = H_\eta(s) \end{aligned}$$

On the other hand from Lemma 2.10, we obtain

$$\begin{aligned} G(t,s) &= G_1(t,s) + \frac{\eta}{1-\eta} G_2(t,s) \\ &\geq (1-t^{\alpha-1})G_1(s,s) + (1-\eta^\alpha/\alpha)\frac{\eta}{1-\eta} G_1(s,s) \\ &\geq (1-\eta^\alpha/\alpha)(1-t^{\alpha-1})G_1(s,s) + (1-\eta^\alpha/\alpha)(1-t^{\alpha-1})\frac{\eta}{1-\eta} G_2(s,s) \\ &\geq (1-\eta^\alpha/\alpha)(1-t^{\alpha-1})[G_1(s,s) + \frac{\eta}{1-\eta} G_2(s,s)] \\ &= (1-\eta^\alpha/\alpha)(1-t^{\alpha-1})H_\eta(s) \end{aligned}$$

□

Lemma 2.12. *Let $0 < \xi < 1$ be an arbitrary and fixed number. If $h : [0,1] \to [0,\infty)$ be a continuous function then fractional boundary value problem* (5) *has a unique non-negative solution like u such that*

$$\min_{t\in(0,\xi)} u(t) \geq \rho\|u\|,$$

where $\rho = (1-\eta^\alpha/\alpha)(1-\xi^{\alpha-1})$.

Proof. The positiveness of $u(t)$ is concluded directly from Lemma 2.9 and Lemma 2.11. For all $t \in [0,1]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^s g(\tau)d\tau\right)ds \\ &\leq \int_0^1 H_\eta(s)\varphi_q\left(\int_0^s g(\tau)d\tau\right)ds \end{aligned}$$

Then

$$\|u\| \leq \int_0^1 H_\eta(s)\varphi_q\left(\int_0^s g(\tau)d\tau\right)ds \tag{12}$$

On the other hand, from Lemma 2.9, Lemma 2.11 and relation (12), for any $t \in [0,\xi]$, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^s g(\tau)d\tau\right)ds \\ &\geq \gamma(\eta)(1-\xi^{\alpha-1})\int_0^1 H_\eta(s)\varphi_q\left(\int_0^s g(\tau)d\tau\right)ds \\ &= \rho\int_0^1 H_\eta(s)\varphi_q\left(\int_0^s g(\tau)d\tau\right)ds \\ &\geq \rho\|u\|. \end{aligned}$$

Therefor

$$\min_{t\in[0,\xi]} u(t) \geq \rho\|u\|$$

□

3. Existence Results

In this section we present the existence results about the fractional boundary value problem (1). Let $\xi \in (0,1)$ be fixed. Introduce the cone that we shall use in the sequel.

$$P = \{u \in \mathcal{B} : u(t) \geq 0, t \in [0,1], \min_{t\in[0,\xi]} u(t) \geq \rho\|u\|\}$$

and define the operator $T : P \to P$ by

$$Tu(t) = \int_0^1 G(t,s)\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))\right)ds \tag{13}$$

where $G(t,s)$ defined by (13). It is clear that , the fixed points of the operator T in P is the non-negative solutions of the fractional boundary value problem (1). Let, we list some assumptions about the right hand side of the differential equation in (1) that we will use them throughout this section.

(A1) $f \in C([0,1] \times [0,\infty), [0,\infty))$, and $f(t,0) \not\equiv 0$ on $[0,1]$;

(A2) $a \in C([0,1], [0,\infty)$ and $a(t) \not\equiv 0$ on any subinterval of $[0,1]$;

At first we show that the operator T is completely continuous

Theorem 3.1. *Assume (A1) and (A2) hold, then the operator T, defined by* (13) *is completely continuous and satisfies $TP \subset P$.*

Proof. Since f is a continuous function(in view of (A1)), Lemma 2.12 implies that $TP \subset P$. From (A1) and non-negativeness and continuity of $G(t,s)$ and by applying Lebesgue's dominated convergence theorem, it is concluded that $T : P \to P$. Let Ω be an arbitrary bounded set in P. Then, there exists $M > 0$ such that $\Omega \subset \{u \in P : \|u\| < M\}$. Set

$$\gamma = \max\{f(t,u) : t \in [0,1], u \in \Omega\}$$

From Lemma 2.9 and 2.10, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds \\
&\leq \int_0^1 G(t,s)\varphi_q\left(\int_0^1 a(\tau)\gamma d\tau\right)ds \\
&\leq \gamma^{q-1}\left(\int_0^1 a(\tau)d\tau\right)\int_0^1 H_\eta(s)ds,
\end{aligned}$$

Hence, $T(\Omega)$ is uniformly bounded. Now for each $u \in \Omega$ and for all $t_1, t_2 \in [0,1]$ that satisfy $t_1 < t_2$, from Lemma 2.9 and 2.11 we have

$$\begin{aligned}
|Tu(t_1) - Tu(t_2)| &= \left|\int_0^1 G(t_1,s)\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds\right. \\
&\quad \left. - \int_0^1 G(t_2,s)\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds\right| \\
&= \left|\int_0^1 [G(t_1,s) - G_1(t_2,s)]\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds\right| \\
&= \left|\int_0^1 [G_1(t_1,s) - G_1(t_2,s)]\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds\right| \\
&\leq \gamma^{q-1}\int_0^1 |G_1(t_1,s) - G_1(t_2,s)|\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds.
\end{aligned}$$

Since G_1 is continuous for all $0 \le t, s \le 1$, we conclude that the right hand side of the above inequality tends to zero if $t_2 \to t_1$. That is, $T(\Omega)$ is equicontinuous. Thus by using of the Arzela-Ascoli theorem, we conclude that $T : P \to P$ is completely continuous. □

For the convenience we set

$$M = \left(\varphi_q\left(\int_0^1 a(\tau)d\tau\right)\int_0^1 H_\eta(s)ds\right)^{-1}, N = \left(\rho\int_0^\xi H_\eta(s)\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds\right)^{-1}$$

Now we assert that $0 < M < N$. In fact we have

$$\begin{aligned}
N^{-1} &= \rho\int_0^\xi H_\eta(s)\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds \\
&< \int_0^\xi H_\eta(s)\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds \\
&\le \varphi_q\left(\int_0^1 a(\tau)d\tau\right)\int_0^1 H_\eta(s)ds \\
&= M^{-1}
\end{aligned}$$

Theorem 3.2. *Assume that (A1)-(A2) hold, and there exist constants $r_1 > 0$, $r_2 > 0$, $\lambda_1 \in (0, M]$, and $\lambda_2 \in [N, \infty)$, such that $r_1 < r_2$ and $\lambda_2 r_1 < \lambda_1 r_2$. Furthermore f satisfies in*

1. *$f(t, u) \le \varphi_p(\lambda_1 r_2)$ for all $u \in [0, r_2]$ and $t \in [0, 1]$, and*
2. *$f(t, u) \ge \varphi_p(\lambda_2 r_1)$ for all $u \in [\rho r_1, r_1]$ and $t \in [0, \xi]$.*

then the fractional boundary value problem (1) *has at least one positive solution $u \in P$ satisfying $r_1 < \|u\| < r_2$.*

Proof. We define the open set

$$\Omega_2 = \{u \in \mathcal{B}; \|u\| < r_2\}$$

Let $u \in P \cap \partial\Omega_2$. Then from (1) and Lemma 2.11, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t, s)\left(\int_0^s a(\tau)f(\tau, u(\tau))d\tau\right)ds \\
&\le \int_0^1 G(t, s)\varphi_q\left(\int_0^s a(\tau)\varphi_p(\lambda_1 r_2)d\tau\right)ds \\
&\le \lambda_1 r_2\int_0^1 G(t, s)\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds \\
&\le \lambda_1 r_2\varphi_q\left(\int_0^1 a(\tau)d\tau\right)\int_0^1 H_\eta(s)ds \\
&\le MM^{-1}r_2 = r_2.
\end{aligned}$$

So for all $u \in P \cap \partial\Omega_2$ we have $\|Tu\| \le \|u\|$.
Now we define the open set $\Omega_1 = \{u \in \mathcal{B} : \|u\| < r_1\}$. For each u in $P \cap \partial\Omega_1$, by using (A1)-(A2), assumption(2) and Lemma 2.11, for all $t \in [0, \xi]$, it is concluded that

$$
\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau)d\tau\right)ds \\
&\geq \int_0^\xi G(t,s)\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau)d\tau\right)ds \\
&\geq \rho\int_0^\xi H_\eta(s)ds\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau)d\tau\right)ds \\
&\geq r_1N\rho\int_0^\xi H_\eta(s)\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds, \\
&= r_1NN^{-1} = r_1
\end{aligned}
$$

so for all $u \in P \cap \partial\omega_1$, we have $\|Tu\| \geq \|u\|$. Hence by applying (ii) in Theorem 2.6, we conclude that T has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$, which it is the solution of fractional boundary value problem (1). In the other words, fractional boundary value problem (1) has at least one positive solution like u such that $r_1 < \|u\| < r_2$. □

by the analogous way, one can obtain the following result.

Theorem 3.3. *Assume (A1)-(A2) hold. If there exists constants $r_1 > 0, r_2 > 0$, $\lambda_1 \in (0, M]$, and $\lambda_2 \in [N, \infty)$, where $\rho r_2 < r_1$, and $\lambda_1 r_1 > N r_2$, such that f satisfies*

1. *$f(t,u) \geq \varphi_p(\lambda_2 r_2)$ for all $u \in [\rho r_2, r_2]$, and*
2. *$f(t,u) \leq \varphi_p(\lambda_1 r_1)$ for all $u \in [0, r_1]$ and $t \in [0,1]$,*

then the problem (1) *has at least one positive solution $u \in P$ satisfying $r_1 < \|u\| < r_2$.*

Theorem 3.4. *Under assumptions (A1)-(A2) hold and existence constant $\mu > 0$ such that*

$$
\mu > \gamma^{q-1}\varphi_q\left(\int_0^1 a(\tau)d\tau\right)\int_0^1 H_\eta(s)ds \tag{14}
$$

where $\gamma = \max\{f(t,u) : (t,u) \in [0,1] \times [0,\mu]\}$, fractional boundary value problem (1) *has at least one positive solution.*

Proof. Let

$$
\mathcal{U} = \{u \in P : |u\| < \mu\}.
$$

In view of Theorem 3.1, the operator $T : \overline{\mathcal{U}} \to P$ is completely continuous. Assume that there exists $u \in \overline{\mathcal{U}}$ and $\lambda \in (0,1)$ such that $u = \lambda Tu$. We have

$$
\begin{aligned}
|u(t)| = |\lambda Tu(t)| &= \left|\lambda\int_0^1 G(t,s)\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau))d\tau\right)ds\right| \\
&\leq \int_0^1 G(t,s)\varphi_q\left(\int_0^s a(\tau)\gamma d\tau\right)ds \\
&\leq \gamma^{q-1}\varphi_q\left(\int_0^1 a(\tau)d\tau\right)\int_0^1 H_\eta(s)ds
\end{aligned}
$$

So

$$
\|u\| \leq \gamma^{q-1}\varphi_q\left(\int_0^1 a(\tau)d\tau\right)\int_0^1 H_\eta(s)ds.
$$

Now (3.4), implies that $\|u\| < \mu$,that is $u \notin \partial\mathcal{U}$. Hence it is concluded that there is no $u \in \partial\mathcal{U}$ such that $u = \lambda Tu$ for $\lambda \in (0, 1)$. Therefor by Theorem 2.7, it is concluded that the fractional boundary value problem (1) has at least one positive solution. □

Our next result is based on the Leggett-Williams fixed point theorem. In fact we present such conditions that fractional boundary value problem (1) has at least three positive solutions. To do this we define the following subsets of a cone P.

$$P_c = \{u \in K : \|u\| < c\}, \quad P(\psi, b, d) = \{u \in P : b \leq \psi(u), \quad \|u\| \leq d\}.$$

Theorem 3.5. *Assume (A1)-(A2) hold and there exist constants a, b, c with $0 < a < \rho b < b \leq c$ such that*

(B1) $f(t, u(t)) < \varphi_p(Ma)$ for all $(t, u) \in [0, 1] \times [0, a]$,

(B2) $f(t, u(t)) \geq \varphi_p(\rho Nb)$ for all $(t, u) \in [0, \xi] \times [\rho b, b]$,

(B3) $f(t, u(t)) \leq \varphi_p(Mc)$ for all $(t, u) \in [0, \xi] \times [0, c]$.

Then fractional boundary value problem (1) has at least three positive solution u_1, u_2 and u_3, such that

$$\|u_1\| < a, \quad \rho b < \psi(u_2), \quad \|u_3\| > a \quad with \quad \psi(u_3) < \rho b.$$

Proof. By Theorem 3.1 $T : P \to P$ is a completely continuous operator. Let

$$\psi(u) = \min_{0 \leq t \leq \xi} u(t)$$

It is clear that ψ is a nonnegative continuous concave functional on P with $\psi(u) \leq \|u\|$, for $u \in \overline{P}_c$. Now we assert that the conditions of Theorem 2.8 are satisfied. For this, let $u \in \overline{K}_c$, that is $\|u\| \leq c$. For $t \in [0, 1]$ from definition of the operator (13) and (B3), we have

$$\begin{aligned}
\|Tu(t)\| &= \max_{0 \leq t \leq 1} \int_0^1 G(t, s)\varphi_q\left(\int_0^s a(\tau)f(\tau, u(\tau))d\tau\right)ds \\
&\leq \int_0^1 H_\eta(s)\varphi_q\left(\int_0^s a(\tau)\varphi_p(Mc)d\tau\right)ds \\
&= Mc\int_0^1 H_\eta(s)\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds \\
&\leq Mc\int_0^1 H_\eta(s)\varphi_q\left(\int_0^1 a(\tau)d\tau\right)ds \\
&= McM^{-1} = c.
\end{aligned}$$

This implies that $T : \overline{P}_c \to P_c$. By the same method, if $u \in \overline{P}_a$, then, we can get $\|Tu\| < a$ and therefor condition (ii) of Theorem 2.8 is satisfied. Since the constant function $\frac{\rho b+b}{2} \in \{u \in P(\psi, \rho b, b) : \psi(u) > \rho b\}$, we conclude that $\{u \in P(\psi, \rho b, b) : \psi(u) > \rho b\} = \emptyset$. On the other hand, for $u \in P(\psi, \rho b, b)$, we have

$$\rho b \leq \psi(u) = \min_{0 \leq t \leq \xi} \leq u(t) \leq \|u\| \leq b, \quad t \in [0, \xi].$$

That is $\psi(Tu) > \rho b$ for all $u \in P(\psi, \rho b, b)$. Thus in view of the assumption (2) of Lemma 2.11 and (B2), we have

$$\begin{aligned}\psi(Tu) &= \min_{0\le t\le\xi}\int_0^1 G(t,s)\varphi_q\left(\int_0^s a(\tau)f(t,u(\tau))d\tau\right)ds\\ &\ge \int_0^1 \rho H_\eta(s)\varphi_q\left(\int_0^s a(\tau)\varphi_p(Nb)d\tau\right)ds\\ &= Nb\int_0^1 \rho H_\eta(s)\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds\\ &= b > \rho b.\end{aligned}$$

Thus, condition (i) of Theorem 2.8 is satisfied. Finally, we show that if $u \in P(\psi, \rho b, c)$ with $\|Tu\| > b$ then $\psi(Tu) > \rho b$. To see this, suppose that $u \in P(\psi, \rho b, c)$ with $\|Tu\| > b$, then by Lemma 2.12, we have

$$\psi(Tu) = \min_{0\le t\le 1}(Tu)(t) \ge \rho\|Tu\| > \rho b.$$

Thus, condition (iii) of Theorem 2.8 is satisfied too.
Hence, an application of Theorem 2.8completes the proof. □

4. Examples

Example 4.1. *Consider the following fractional boundary value problem*

$$\begin{cases}\left(\varphi_{\frac{9}{2}}({}^cD_{0^+}^{\frac{5}{2}}u(t)\right)' + a(t)f(t,u(t)) = 0, \quad t\in(0,1),\\ {}^cD_{0^+}^{\frac{5}{2}}u(0) = u'(0) = u''(0) = 0, u(1) + u'(1) = \int_0^{\frac{1}{4}} u(t)dt\end{cases} \tag{15}$$

where $f(t,u) = \frac{1}{200}(14 + 45\sqrt{u} + t)$, $a(t) = \frac{7}{2}t^2\sqrt{t}$, $\alpha = \frac{5}{2}$, $p = \frac{9}{2}$, $\eta = \frac{1}{4}$, $q = \frac{9}{7}$.
We let

$$I_1 := (1-\eta)\Gamma(\alpha)\int_0^1 H_\eta(s)ds = \int_0^1 (\alpha - s)(1-s)^{\frac{1}{2}}ds = \int_0^1 (\frac{3}{2} + t^2)t(2t)dt = \frac{7}{5}, \tag{16}$$

$$\begin{aligned}I_2 &:= (1-\eta)\Gamma(\alpha)\int_0^1 sH_\eta(s)ds = \int_0^\xi s(\alpha - s)(1-s)^{\frac{1}{2}}ds\\ &= \frac{1}{35}(18 - \sqrt{1-\xi}(\xi-1)(10\xi^2 - 27\xi - 18)\end{aligned} \tag{17}$$

So

$$\begin{aligned}M &= \left(\varphi_q\left(\int_0^1 a(\tau)d\tau\right)\int_0^1 H_\eta(s)ds\right)^{-1}\\ &= \left(\varphi_{\frac{9}{7}}\left(\int_0^1 \frac{7}{2}\tau^2\sqrt{\tau}d\tau\right)\frac{4}{3}\frac{I_1}{\frac{3}{4}\sqrt{\pi}}\right)^{-1}\\ &= \frac{45\sqrt{\pi}}{112} = 0.7120\end{aligned}$$

and

$$\begin{aligned}N^{-1} &= \rho\int_0^\xi H_\eta(s)\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds\\ &= \rho\int_0^\xi H_\eta(s)\varphi_{\frac{9}{7}}\left(\int_0^s \frac{7}{2}\tau^2\sqrt{\tau}d\tau\right)ds = \rho\int_0^\xi sH_\eta(s)ds\end{aligned} \tag{18}$$

By a simple calculation we obtain

$$N = \left[\frac{16}{9\sqrt{\pi}}\left(1-\frac{1}{80}\right)(1-\xi^{\frac{3}{2}})\frac{1}{35}(18-\sqrt{1-\xi}(\xi-1)(10\xi^2-27\xi-18)\right]^{-1}$$

We choose $\lambda_1 = M$, $\lambda_2 = N$, $r_1 = \frac{1}{40} = 0.025$ *and* $r_2 = 1$. *It is easy to check that* $8.36 \le N \le 28.05$ *if* $0.2 \le \xi \le 0.8$. *Hence* $\lambda_2 r_1 = \frac{N}{40} = 0.70125 < \lambda_1 r_2 = \frac{45\sqrt{\pi}}{112} = 0.7120$. *Now by a simple calculation, we get* $\lambda_1^{\frac{7}{2}} \approx 0.3045$, *and* $\left(\frac{\lambda_2}{40}\right)^{\frac{7}{2}} \le 0.2887$. *On the other hand* f *satisfies the following relations*

(i) $f(t,u) = \frac{1}{200}(14+45\sqrt{u}+t) \le 0.3 < \varphi_{\frac{9}{2}}(\lambda_1 r_2) \approx 0.3045, (t,u) \in [0,1]\times[0,1]$

(ii) $f(t,u) = \frac{1}{200}(14+45\sqrt{u}+t) > 0.7 > \varphi_{\frac{9}{2}}(\lambda_2 r_1), \quad (t,u) \in [0,\xi]\times[0.025\rho, 0.025]$

Hence, all conditions of Theorem 3.2 are satisfied, consequently fractional boundary value problem (15) *has at least one positive solution u such that* $0.1 \le |u\| \le 1$

Example 4.2. *Consider the following fractional boundary value problem*

$$\begin{cases} \left(\varphi_{\frac{5}{2}}({}^cD_{0^+}^{\frac{5}{2}}u(t)\right)' + a(t)f(t,u(t)) = 0, \quad t \in (0,1), \\ {}^cD_{0^+}^{\frac{5}{2}}u(0) = u'(0) = u''(0) = 0, u(1)+u'(1) = \int_0^{\frac{1}{2}} u(t)dt \end{cases} \tag{19}$$

where $\alpha = p = \frac{5}{2}$, $q = \frac{5}{3}$, $\eta = \frac{1}{2}$, $a(t) = \frac{3}{2}\sqrt{t}$ *and*

$$f(t,u) = \begin{cases} 125u^6 + \frac{\sin^2 \pi t}{20}, & (t,u) \in [0,1]\times[0,1] \\ 124 + u^{\frac{1}{4}} + \frac{\sin^2 \pi t}{20}, & (t,u) \in [0,1]\times(1,\infty) \end{cases}$$

From relation (16), *we have* $M = \left(\frac{I_1}{\frac{1}{2}\Gamma(\frac{5}{2})}\right)^{-1} = \frac{15\sqrt{\pi}}{56} = 0.47476$, *and from* (18) *we obtain*

$$\begin{aligned} N &= \left(\rho\int_0^{\xi}(s)\varphi_q\left(\int_0^s a(\tau)d\tau\right)ds\right)^{-1} \\ &= \left(\rho\int_0^{\xi}H_\eta(s)\varphi_{\frac{5}{3}}\left(\int_0^s \frac{3}{2}\sqrt{\tau}d\tau\right)ds\right)^{-1} = \left(\rho\int_0^{\xi}sH_\eta(s)ds\right)^{-1} \\ &= \left(\frac{4}{3\sqrt{\pi}}\left(1-\frac{1}{10\sqrt{2}}\right)(1-\xi^{\frac{3}{2}})\frac{1}{35}(18-\sqrt{1-\xi}(\xi-1)(10\xi^2-27\xi-18)\right)^{-1}. \end{aligned}$$

Let $\xi = 0.5$ *then* $N \approx 5.03$. *Chosing* $a = \frac{1}{5}$, $b = 5$ *and* $c = 64$, *we obtain*

(i) $f(t,u) \le 0.028 < \varphi_{\frac{5}{2}}(Ma) = (0.2\times 0.47476)^{\frac{3}{2}} \approx 0.02926, \quad (t,u) \in [0,1]\times[0,\frac{1}{5}]$;

(ii) $f(t,u) \ge 125.36 > \varphi_{\frac{5}{2}}(\rho Nb) = \varphi_{\frac{5}{2}}(15.09) \approx 58.6183, \quad (t,u) \in [0,0.5]\times[3,5]$

(iii) $f(t,u) \le 126 \le \varphi_{\frac{5}{2}}(Mc) = (30.38464)^{\frac{3}{2}} \approx 167.487, \quad (t,u) \in [0,0.5]\times[0,64]$.

Thus all conditions of the Theorem 3.5 are satisfied. Therefore, the fractional boundary value problem (19) *has at least three positive solution* u_1, u_2 *and* u_3 *such that*

$$\|u\| \le \frac{1}{5}, \quad 3 < \psi(u_2), \|u_3\| > \frac{1}{5} \quad \textit{with} \quad \psi(u_3) > 3.$$

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