Filomat 37:1 (2023), 251–264
https://doi.org/10.2298/FIL2301251G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Global asymptotic behavior of a discrete system of difference equations with delays

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Abstract. In the present paper, we mainly investigate the qualitative behavior of the solutions of a discrete system of difference equations

$$x_{n+1} = \alpha + \frac{\sum_{i=1}^{m} x_{n-i}}{y_n},\ y_{n+1} = \beta + \frac{\sum_{i=1}^{m} y_{n-i}}{x_n},\ n \in \mathbb{N}$$

where $\alpha, \beta \in (0, \infty)$, $m \in \mathbb{Z}^+$, x_{-i} and y_{-i} are non-negative real numbers for $i \in \{0, 1, \ldots, m\}$. Namely, we discuss the boundedness character and the asymptotic stability properties of steady states of the mentioned system. Finally, for this system, we give a rate of convergence result which has an important place in the discrete dynamical systems. Besides, some numerical simulations with graphs are given to emphasize the efficiency of our theoretical results in the article.

1. Introduction

The best way to overcome such questions as the causes of many problems encountered in real life, their reflections to the present, and their effects on the future is to determine the unknown of the problem as a function and then model the problem with an equation that includes this function. Determining the unknown function which is the solution of the model equation plays an important role in answering the questions mentioned above. On the other hand, the problems encountered in many scientific fields such as economy, population dynamics, defense and psychology are modeled mostly with difference equations (see [19, 26]). Therefore, difference equations have a significant role in many fields such as physics, chemistry, biology, medicine, engineering, economics and many others. In this sense, one-dimensional discrete dynamical systems (ordinary or partial difference equations) and high-dimensional systems (difference equations systems) have been very popular topics for the last twenty years (see [3, 4, 10, 15, 18, 21, 27]).

The investigations of difference equations have been progressing rapidly. The studies in this field are performed as quantitatively by determining the analytical solutions of the equation, qualitatively by examining the behavior of the solutions of the equation and numerically by determining the approximate values of the solution of the equation by various methods.

Therefore, this investigation can be seen as a qualitative investigation of the difference equation theory. Let us now provide a detailed background on the equation we are dealing with in this article;

2020 *Mathematics Subject Classification*. Primary 39A10, 39A23, 40A05.
Keywords. Competitive systems; Delays; Boundedness character; Stability; Convergence.
Received: 30 April 2021; Revised: 14 August 2022; Accepted: 01 December 2022
Communicated by Marko Petković
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In [6], Amleh *et al.* studied the asymptotic stability, the boundedness, the persistence and the periodicity of the solutions of the difference equation

$$u_{n+1} = \delta + \frac{u_{n-1}}{u_n},\ n \in \mathbb{N} \tag{1}$$

where the parameter δ is a non-negative real number, and the initial values are non-negative real numbers.

In [23], Papaschinopoulos and Schinas generalized some results concerning equation (1) to a discrete system of difference equations

$$u_{n+1} = \gamma + \frac{u_{n-1}}{v_n},\ v_{n+1} = \gamma + \frac{v_{n-1}}{u_n}, n \in \mathbb{N} \tag{2}$$

where γ is a non-negative constant and the initial conditions u_{-1}, u_0, v_{-1}, v_0 are non-negative real numbers.

Also, in [22], Papaschinopoulos and Schinas discussed the discrete system of difference equations

$$u_{n+1} = \varepsilon + \frac{v_n}{u_{n-k}}, v_{n+1} = \varepsilon + \frac{u_n}{v_{n-l}}, n \in \mathbb{N} \tag{3}$$

where $\varepsilon \in (0, \infty)$ and k, l are positive integers.

Taking reciprocals of the fractions (3) and letting $k = l$, the following discrete system

$$u_{n+1} = \varepsilon + \frac{u_{n-k}}{v_n}, v_{n+1} = \varepsilon + \frac{v_{n-k}}{u_n}, n \in \mathbb{N} \tag{4}$$

is obtained which was investigated in [28] by Zhang *et al.*.

In [30], Zhang *et al.* discussed the asymptotic properties of the discrete system of non-linear difference equations

$$u_{n+1} = \eta + \frac{v_{n-t}}{v_n}, v_{n+1} = \eta + \frac{u_{n-t}}{u_n}, n \in \mathbb{N} \tag{5}$$

where $\eta \in (0, \infty)$ and $u_{-i}, v_{-i} \in (0, \infty)$, $i = 0, 1 \ldots, t$. Also, in [11], Gumus gave supplementary conclusions about the global asymptotic behavior of the positive solutions of the discrete system (5). Namely, he studied the rate of convergence of the solutions of the system (5) and the stability analysis of the equilibrium points with a detailed investigation. So, the picture of the work to be done on this equation has been completed.

In [24], Papaschinopoulos and Papadopoulos studied some properties of the following system

$$u_{n+1} = \mu + \frac{u_n}{v_{n-\sigma}},\ v_{n+1} = \eta + \frac{v_n}{v_{n-\sigma}}, n \in \mathbb{N} \tag{6}$$

where $\mu, \eta \in (0, \infty)$ and the initial conditions $u_{-i}, v_{-i} \in (0, \infty)$, $i \in \{0, 1 \ldots, \sigma\}$; while in [9], Camouzis and Papaschinopoulos obtained some important results about the system (6) for the case $\mu = \eta = 1$, in [31], Zhang *et al.* investigated the system (6) taking reciprocals of the fractions and $\mu = \eta$.

In [7], Bao studied the global asymptotic analysis of positive solutions to the discrete system of the non-linear difference equations

$$u_{n+1} = \mu + \left(\frac{u_{n-1}}{v_n}\right)^{\gamma},\ v_{n+1} = \mu + \left(\frac{v_{n-1}}{u_n}\right)^{\gamma}, n \in \mathbb{N} \tag{7}$$

where $\mu \in (0, \infty)$, $\gamma \in [1, \infty)$, the initial values u_{-1}, u_0, v_{-1}, v_0 are non-negative real numbers.

In [29], Zhang *et al.* generalized the results related to the system (5) as the discrete system of non-linear difference equations

$$u_{n+1} = \gamma + \frac{u_n}{\sum_{j=1}^{t} v_{n-j}}, v_{n+1} = \delta + \frac{v_n}{\sum_{j=1}^{t} u_{n-j}},\ n \in \mathbb{N} \tag{8}$$

where γ, δ are non-negative numbers and the initial conditions are also non-negative real numbers.

Other relevant qualitative theory of difference equations or discrete systems can be obtained in references ([1, 2, 5, 12–14, 16, 17]). For more detailed information, refer to the books ([8, 19, 20, 26]).

In order to contribute to this field here, we aim to study the boundedness character, the persistence, the rate of convergence of the solutions, and the global asymptotic stability of the equilibrium points of the following discrete system

$$x_{n+1} = \alpha + \frac{\sum_{i=1}^{m} x_{n-i}}{y_n},\ y_{n+1} = \beta + \frac{\sum_{i=1}^{m} y_{n-i}}{x_n}\ n \in \mathbb{N} \tag{9}$$

where α, β are non-negative real numbers, $x_{-i}, y_{-i} \in (0, \infty)$ for $i \in \{0, 1, \dots, m\}$. The results obtained in this article generalize and complement the articles mentioned in the literature summary above.

We conclude this section by giving a definition and a famous result in the difference equation theory that we will use in this article.

Consider the high-dimensional discrete dynamical system

$$\begin{aligned} u_{n+1} &= \varphi(u_n, \dots, u_{n-m}, v_n, \dots, v_{n-m}),\ n \in \mathbb{N} \\ v_{n+1} &= \phi(u_n, \dots, u_{n-m}, v_n, \dots, v_{n-m}),\ n \in \mathbb{N} \end{aligned} \tag{10}$$

where φ and ϕ are continuously differentiable functions. The solution $\{(u_n, v_n)\}_{n=-m}^{\infty}$ of this system (10) is determined uniquely by certain initial values.

Definition 1.1. *([19]) Let φ and ϕ be continuously differentiable functions at the equilibrium $(\overline{u}, \overline{v})$ which is an equilibrium point of the map ψ. The linearized system of (10) related to $(\overline{u}, \overline{v})$ is*

$$\Phi_{n+1} = \Psi(\Phi_n) = J\Phi_n$$

with

$$\Phi_n = \begin{pmatrix} u_n \\ \vdots \\ u_{n-m} \\ v_n \\ \vdots \\ v_{n-m} \end{pmatrix}$$

and J is a Jacobian matrix of the system (10) related to the point $(\overline{u}, \overline{v})$.

Theorem 1.2. *([19]) Consider the system*

$$\Phi_{n+1} = \Psi(\Phi_n), n = 0, 1, ...,$$

of difference equations. Let ϖ be a fixed point of ψ. If all eigenvalues of the Jacobian matrix J about ϖ lie inside the open unit disk $|\mu| < 1$, then ϖ is called locally asymptotically stable equilibrium. If at least one eigenvalue of them has a modulus greater than 1, then ϖ is called an unstable equilibrium.

2. The boundedness character

In this section, we will give some inequalities about the solutions of the system (9). Using this result, we will constitute the boundedness result for the system (9).

Theorem 2.1. *Assume that*

$$\min\{\alpha,\beta\} > m \tag{11}$$

and consider a positive solution $\{x_n, y_n\}$ *of the system (9). Then,*

$$\begin{aligned} \frac{\alpha\beta + m(\alpha - m)}{\beta} &\le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le \frac{\alpha\beta}{\beta - m}, \\ \frac{\alpha\beta + m(\beta - m)}{\alpha} &\le \liminf_{n\to\infty} y_n \le \limsup_{n\to\infty} y_n \le \frac{\alpha\beta}{\alpha - m}. \end{aligned}$$

Thus, every solution of the system (9) is bounded.

Proof. Since $x_n > 0$ and $y_n > 0$ for all $n \ge 1$, for the system (9) we can get

$$x_n \ge \alpha \text{ and } y_n \ge \beta, \text{ for } n = 1, 2, \ldots. \tag{12}$$

So, solutions persist. Using (9) and (12), we obtain

$$x_n \le \alpha + \frac{\sum_{i=2}^{m+1} x_{n-i}}{\beta}, \; y_n \le \beta + \frac{\sum_{i=2}^{m+1} y_{n-i}}{\alpha}, \text{ for } n = m+1, m+2, \ldots. \tag{13}$$

Let u_n, v_n be a solution of the following system

$$u_n = \alpha + \frac{\sum_{i=2}^{m+1} u_{n-i}}{\beta}, \; v_n = \beta + \frac{\sum_{i=2}^{m+1} v_{n-i}}{\alpha}, \; n \ge m+1,$$

such that

$$u_i = x_i, \; v_i = y_i, \; i = 0, 1, 2, \ldots, m.$$

By mathematical induction we have

$$x_n \le u_n, \; y_n \le v_n, \; n \ge m+1. \tag{14}$$

Assume that (14) is satisfied for $n = k \ge m+1$. Then, from (13)

$$x_{k+1} \le \alpha + \frac{\sum_{i=1}^{m} x_{k-i}}{\beta} \le \alpha + \frac{\sum_{i=1}^{m} u_{k-i}}{\beta} = u_{k+1}.$$

$$y_{k+1} \le \beta + \frac{\sum_{i=1}^{m} y_{k-i}}{\alpha} \le \beta + \frac{\sum_{i=1}^{m} v_{k-i}}{\alpha} = v_{k+1}.$$

So as every solution of the difference equation

$$u_{k+1} = \alpha + \frac{\sum_{i=1}^{m} u_{k-i}}{\beta}, \; k = 0, 1, \ldots$$

converges to $\frac{\alpha\beta}{\beta - m}$, using the Comparison Theorem, it follows that

$$\limsup_{n\to\infty} x_n \le \frac{\alpha\beta}{\beta - m}.$$

Also, every solution of the difference equation

$$v_{k+1} = \beta + \frac{\sum_{i=1}^{m} v_{k-i}}{\alpha}, \; k = 0, 1, \ldots$$

converges to $\frac{\alpha\beta}{\alpha-m}$, using the Comparison Theorem, it follows that

$$\limsup_{n\to\infty} y_n \leq \frac{\alpha\beta}{\alpha-m}.$$

Let show that

$$(\alpha\beta + m(\beta - m))/\beta \leq \liminf_{n\to\infty} x_n$$

and

$$(\alpha\beta + m(\alpha - m))/\alpha \leq \liminf_{n\to\infty} y_n.$$

Let $\varepsilon > 0$. It is clear that there exists $n_0 \geq 0$ such that for all $n \geq n_0$,

$$x_n < \frac{\alpha\beta + \varepsilon}{\beta - m} \text{ and } y_n < \frac{\alpha\beta + \varepsilon}{\alpha - m},$$

then,

$$\begin{aligned} x_{n+1} &> \alpha + \frac{m\alpha(\alpha - m)}{\alpha\beta + \varepsilon} \\ &= \frac{\alpha^2\beta + \alpha\varepsilon + m\alpha^2 - m^2\alpha}{\alpha\beta + \varepsilon}, \\ y_{n+1} &> \beta + \frac{m\beta(\beta - m)}{\alpha\beta + \varepsilon} \\ &= \frac{\alpha\beta^2 + \beta\varepsilon + m\beta^2 - m^2\beta}{\alpha\beta + \varepsilon} \end{aligned}$$

and so

$$\begin{aligned} \liminf_{n\to\infty} x_n &\geq \frac{\alpha^2\beta + \alpha\varepsilon + m\alpha^2 - m^2\alpha}{\alpha\beta + \varepsilon}, \\ \liminf_{n\to\infty} y_n &\geq \frac{\alpha\beta^2 + \beta\varepsilon + m\beta^2 - m^2\beta}{\alpha\beta + \varepsilon}. \end{aligned}$$

Since ε is arbitrary, we have

$$\begin{aligned} \liminf_{n\to\infty} x_n &\geq \frac{\alpha\beta + m(\alpha - m)}{\beta}, \\ \liminf_{n\to\infty} y_n &\geq \frac{\alpha\beta + m(\beta - m)}{\alpha}. \end{aligned}$$

Thus, every solution of the system (9) is bounded. This completes the proof as desired. □

3. The stability nature of the system (9)

This section is devoted to first giving the equilibrium points of the system (9). Later, we will discuss the global attractivity and the local asymptotic stability of the equilibrium points of the system (9). Finally, using these results, we will give the global asymptotic result.

Lemma 3.1. *Consider the system (9). If the condition (11) is satisfied, then the system (9) has a unique positive equilibrium $(\overline{x}, \overline{y})$ given by*

$$\overline{x} = \frac{m^2 - \alpha\beta}{m - \beta}, \overline{y} = \frac{m^2 - \alpha\beta}{m - \alpha}. \tag{15}$$

Proof. It is clear from the equilibrium definition. □

Theorem 3.2. *If the condition (11) is satisfied, then every positive solution* (x_n, y_n) *of the system (9) converge to the positive equilibrium* $\left(\frac{m^2-\alpha\beta}{m-\beta}, \frac{m^2-\alpha\beta}{m-\alpha}\right)$.

Proof. Since every positive solution of the system (9) is bounded from Theorem 2.1, there exist

$$\left.\begin{array}{l}\limsup x_n = \Lambda, \liminf x_n = \lambda \\ \limsup y_n = M, \liminf y_n = \mu\end{array}\right\} \tag{16}$$

where $\Lambda, \lambda, M, \mu \in (0, \infty)$. Then, from (9) and (16) we can derive the following inequalities

$$\begin{aligned}\Lambda &\leq \alpha + \frac{m\Lambda}{\mu}, \lambda \geq \alpha + \frac{m\lambda}{M}, \\ M &\leq \beta + \frac{mM}{\lambda}, \mu \geq \beta + \frac{m\mu}{\Lambda},\end{aligned}$$

from which we can get

$$\begin{aligned}\beta\Lambda + m\mu &\leq \mu\Lambda \leq \alpha\mu + m\Lambda \\ \alpha M + m\lambda &\leq \lambda M \leq \beta\lambda + mM.\end{aligned}$$

Thus, we have

$$\Lambda(\beta - m) \leq \mu(\alpha - m), \; M(\alpha - m) \leq \lambda(\beta - m). \tag{17}$$

Then, from the relations (11) and (17), we obtain that

$$\Lambda M \leq \lambda\mu$$

from which it follows that

$$\Lambda M = \lambda\mu. \tag{18}$$

We claim that

$$\Lambda = \lambda \text{ and } M = \mu. \tag{19}$$

Assume on the contrary that $\lambda < \Lambda$. Then, from (18) we obtain $\Lambda M = \lambda\mu < \Lambda\mu$ and thus $M < \mu$ which is a contradiction with the assumption. So, it must be $\Lambda = \lambda$. In a similar way, it can be proven that $M = \mu$. So, (19) is satisfied. Therefore, from (9) and (19), there exist the finite limits

$$\lim x_n \text{ as } n \to \infty$$

and

$$\lim y_n \text{ as } n \to \infty.$$

So,

$$\lim x_n = \overline{x} \text{ and } \lim y_n = \overline{y}$$

where $(\overline{x}, \overline{y})$ the unique positive equilibrium of the system (9). The proof is completed as desired. □

Theorem 3.3. *If the condition (11) is satisfied and the following conditions*

$$m(2m-(\alpha+\beta))(m-\beta) < (m^2-\alpha\beta)(m-\alpha) \tag{20}$$

and

$$m(2m-(\alpha+\beta))(m-\alpha) < (m^2-\alpha\beta)(m-\beta) \tag{21}$$

hold, then the unique positive equilibrium $\left(\frac{m^2-\alpha\beta}{m-\beta}, \frac{m^2-\alpha\beta}{m-\alpha}\right)$ *of the system (9) is locally asymptotically stable.*

Proof. From Lemma 3.1, the system (9) has a unique positive equilibrium $\left(\frac{m^2-\alpha\beta}{m-\beta}, \frac{m^2-\alpha\beta}{m-\alpha}\right)$. The linearized equation of the system (9) associated the equilibrium point $\left(\frac{m^2-\alpha\beta}{m-\beta}, \frac{m^2-\alpha\beta}{m-\alpha}\right)$ is

$$X_{n+1} = JX_n$$

with $X_n = (x_n, x_{n-1}, ..., x_{n-m+1}, x_{n-m}, y_n, y_{n-1}, ..., y_{n-m+1}, y_{n-m})^T$ and the matrix $J = (\tau_{ij})$ is a high-dimensional with $1 \le i, j \le 2m+2$ such that

$$J = \begin{pmatrix} 0 & \frac{1}{\overline{y}} & ... & \frac{1}{\overline{y}} & \frac{1}{\overline{y}} & -\frac{m\overline{x}}{\overline{y}^2} & 0 & ... & 0 & 0 \\ 1 & 0 & ... & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & ... & \vdots & \vdots \\ 0 & 0 & ... & 1 & 0 & 0 & 0 & ... & 0 & 0 \\ -\frac{m\overline{y}}{\overline{x}^2} & 0 & ... & 0 & 0 & 0 & \frac{1}{\overline{x}} & ... & \frac{1}{\overline{x}} & \frac{1}{\overline{x}} \\ 0 & 0 & ... & 0 & 0 & 1 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & 0 & 0 & 0 & 0 & ... & 1 & 0 \end{pmatrix}_{(2m+2)\times(2m+2)}.$$

Consider $\zeta_1, \zeta_2, \ldots, \zeta_{2m+2}$ indicate the eigenvalues of matrix J and consider the diagonal matrix

$$W = diag(w_1, w_2, \ldots, w_{2m+2})$$

with $w_1 = w_{m+2} = 1$, $w_i = w_{m+1+i} = 1 - i\varepsilon$ with $i = 1, 2, \ldots, m+1$. From the relations (20) and (21), we can choose a small positive number ε as

$$0 < \varepsilon < \min\left\{\frac{1}{m+1}\left(1 - \frac{m(\overline{x}+\overline{y})}{\overline{y}^2}\right), \frac{1}{m+1}\left(1 - \frac{m(\overline{x}+\overline{y})}{\overline{x}^2}\right)\right\}. \tag{22}$$

It is obvious W is an invertible matrix. If we compute the matrix WJW^{-1}, then we can obtain the following high-dimensional matrix that

$$WJW^{-1} = \begin{pmatrix} 0 & \frac{1}{\overline{y}}\frac{w_1}{w_2} & \cdots & \frac{1}{\overline{y}}\frac{w_1}{w_m} & \frac{1}{\overline{y}}\frac{w_1}{w_{m+1}} & -\frac{m\overline{x}}{\overline{y}^2}\frac{w_1}{w_{m+2}} & 0 & \cdots & 0 & 0 \\ \frac{w_2}{w_1} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & & \vdots & \vdots & & \vdots & & \vdots \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{w_{m+1}}{w_m} & 0 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{m\overline{y}}{\overline{x}^2}\frac{w_{m+2}}{w_1} & 0 & \cdots & 0 & 0 & 0 & \frac{1}{\overline{x}}\frac{w_{m+2}}{w_{m+3}} & \cdots & \frac{1}{\overline{x}}\frac{w_{m+2}}{w_{2m+1}} & \frac{1}{\overline{x}}\frac{w_{m+2}}{w_{2m+2}} \\ 0 & 0 & \cdots & 0 & 0 & \frac{w_{m+3}}{w_{m+2}} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \frac{w_{2m+2}}{w_{2m+1}} & 0 \end{pmatrix}.$$

Based on

$$w_1 > w_2 > \ldots > w_{m+1} > 0$$

and

$$w_{m+2} > w_{m+3} > \ldots > w_{2m+2} > 0,$$

we have

$$\begin{array}{c} w_2 w_1^{-1} < 1, \\ w_3 w_2^{-1} < 1, \\ \vdots \\ w_{m+1} w_m^{-1} < 1, \\ w_{m+3} w_{m+2}^{-1} < 1, \\ \vdots \\ w_{2m+2} w_{2m+1}^{-1} < 1. \end{array}$$

Also, based on (11), (20), (21), and (22) we can obtain

$$\begin{aligned} \frac{1}{\overline{y}}\left(\frac{1}{1-2\varepsilon} + \ldots + \frac{1}{1-(m+1)\varepsilon}\right) + \frac{m\overline{x}}{\overline{y}^2} &< \frac{1}{\overline{y}}\frac{m}{1-(m+1)\varepsilon} + \frac{m\overline{x}}{\overline{y}^2} \\ &< \frac{1}{1-(m+1)\varepsilon}\left(\frac{m}{\overline{y}} + \frac{m\overline{x}}{\overline{y}^2}\right) \\ &< 1, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{\overline{x}}\left(\frac{1}{1-2\varepsilon} + \ldots + \frac{1}{1-(m+1)\varepsilon}\right) + \frac{m\overline{y}}{\overline{x}^2} &< \frac{1}{\overline{x}}\frac{m}{1-(m+1)\varepsilon} + \frac{m\overline{y}}{\overline{x}^2} \\ &< \frac{1}{1-(m+1)\varepsilon}\left(\frac{m}{\overline{x}} + \frac{m\overline{y}}{\overline{x}^2}\right) \\ &< 1. \end{aligned}$$

It is obvious J has the same eigenvalues as WJW^{-1}; so, we obtain

$$\begin{aligned} \max|\zeta_i| &\leq \left\|WJW^{-1}\right\|_\infty \\ &= \max\left\{ \begin{array}{c} w_2 w_1^{-1}, \ldots, w_{m+1} w_m^{-1}, w_{m+3} w_{m+2}^{-1}, \ldots, w_{2m+2} w_{2m+1}^{-1}, \\ \frac{1}{\overline{y}}\left(\frac{1}{1-2\varepsilon} + \ldots + \frac{1}{1-(m+1)\varepsilon}\right) + \frac{m\overline{x}}{\overline{y}^2}, \\ \frac{1}{\overline{x}}\left(\frac{1}{1-2\varepsilon} + \ldots + \frac{1}{1-(m+1)\varepsilon}\right) + \frac{m\overline{y}}{\overline{x}^2} \end{array} \right\} \\ &< 1. \end{aligned}$$

Therefore, we obtain that all eigenvalues of J lie inside the unit disk $|\zeta| < 1$. Thus, thanks to Theorem 1.2, we can say that the unique positive equilibrium point $\left(\frac{m^2-\alpha\beta}{m-\beta}, \frac{m^2-\alpha\beta}{m-\alpha}\right)$ is locally asymptotically stable. This completes the proof, as desired. □

If we combine Theorem 3.2 and Theorem 3.3, then we will create the following important result which is seen as the main result for this article.

Theorem 3.4. *If relations (11), (20), and (21) are satisfied, then the unique positive equilibrium* $\left(\frac{m^2-\alpha\beta}{m-\beta}, \frac{m^2-\alpha\beta}{m-\alpha}\right)$ *of the system (9) is globally asymptotically stable.*

4. Rate of Convergence Result

Here, we will discuss the rate of convergence result of the solution that tends to the equilibrium point $\left(\frac{m^2-\alpha\beta}{m-\beta}, \frac{m^2-\alpha\beta}{m-\alpha}\right)$ of the system (9). Consider the system

$$\Psi_{n+1} = [M + N(n)]\Psi_n \tag{23}$$

where Ψ_n is a high-dimensional vector, $M \in C^{(2m+2)\times(2m+2)}$ is a constant matrix, also,

$$N : \mathbb{Z}^+ \to C^{(2m+2)\times(2m+2)}$$

is a matrix function with

$$\|N(n)\| \to 0, \text{ when } n \to \infty, \tag{24}$$

where $\|.\|$ denotes any matrix norm which is associated with the vector norm. The following famous result which is called Perron's Theorem gives the rate of convergence of positive solutions of a discrete system of difference equations.

Proposition 4.1. *(see, [25]) Consider the system (23) and assume the condition (24) is provided. If Ψ_n is a solution of (23), then either*

$$\Psi_n = 0$$

for all large n or

$$\vartheta = \lim_{n\to\infty} \sqrt[n]{\|\Psi_n\|}$$

or

$$\vartheta = \lim_{n\to\infty} \frac{\|\Psi_{n+1}\|}{\|\Psi_n\|}$$

exists and ϑ equals to the modulus of one of the eigenvalues of the matrix M.

Theorem 4.2. *Suppose that the solution $\{(x_n, y_n)\}_{n=-m}^{\infty}$ of the system (9) tends to the global asymptotic stable equilibrium point $\left(\frac{m^2-\alpha\beta}{m-\beta}, \frac{m^2-\alpha\beta}{m-\alpha}\right) = (\overline{x}, \overline{y})$. The error vector*

$$e_n = \begin{pmatrix} e_n^1 \\ e_{n-1}^1 \\ \vdots \\ e_{n-m}^1 \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-m}^2 \end{pmatrix} = \begin{pmatrix} x_n - \overline{x} \\ x_{n-1} - \overline{x} \\ \vdots \\ x_{n-m} - \overline{x} \\ y_n - \overline{y} \\ y_{n-1} - \overline{y} \\ \vdots \\ y_{n-m} - \overline{y} \end{pmatrix}$$

of all solutions of (9) hold the following asymptotic connections:

$$\lim_{n\to\infty} \sqrt[n]{\|e_n\|} = \left|\lambda_i B(\overline{x}, \overline{y})\right|$$

and

$$\lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = \left|\lambda_i B(\overline{x}, \overline{y})\right|$$

with $i \in \{1, 2, \ldots, m\}$, also, $\left|\lambda_i B(\overline{x}, \overline{y})\right|$ equals to the modulus of one the eigenvalues of the Jacobian matrix calculated at the equilibrium $\left(\frac{m^2-\alpha\beta}{m-\beta}, \frac{m^2-\alpha\beta}{m-\alpha}\right) = (\overline{x}, \overline{y})$.

Proof. Firstly, for the system (9), we will investigate a system of limiting equations. The error terms of the system are obtained that

$$\left.\begin{array}{l} x_{n+1}-\overline{x}=\sum_{i=0}^{m}A_i(x_{n-i}-\overline{x})+\sum_{i=0}^{m}B_i(y_{n-i}-\overline{y}) \\ y_{n+1}-\overline{y}=\sum_{i=0}^{m}C_i(x_{n-i}-\overline{x})+\sum_{i=0}^{m}D_i(y_{n-i}-\overline{y}). \end{array}\right\}. \tag{25}$$

Set

$$e_n^1=x_n-\overline{x},\ e_n^2=y_n-\overline{y}.$$

So, we can describe the system (25) as

$$\begin{array}{l} e_{n+1}^1=\sum_{i=0}^{m}A_ie_{n-i}^1+\sum_{i=0}^{m}B_ie_{n-i}^2 \\ e_{n+1}^2=\sum_{i=0}^{m}C_ie_{n-i}^1+\sum_{i=0}^{m}D_ie_{n-i}^2 \end{array}$$

with

$$\begin{array}{rcl} A_0 & = & 0, A_i=\dfrac{1}{y_n}, \\ B_0 & = & -\dfrac{\sum_{i=1}^{m}x_{n-i}}{y_n^2}, B_i=0, \\ C_0 & = & -\dfrac{\sum_{i=1}^{m}y_{n-i}}{x_n^2}, C_i=0, \\ D_0 & = & 0,\ D_i=\dfrac{1}{x_n}, \end{array}$$

with $i\in\{1,2,\ldots,m\}$. If we transition the limiting case, then it is obvious that

$$\begin{array}{rcl} \lim\limits_{n\to\infty}A_0 & = & 0,\ \lim\limits_{n\to\infty}A_i=\dfrac{1}{\overline{y}}, \\ \lim\limits_{n\to\infty}B_0 & = & -\dfrac{m\overline{x}}{\overline{y}^2},\ \lim\limits_{n\to\infty}B_i=0, \\ \lim\limits_{n\to\infty}C_0 & = & -\dfrac{m\overline{y}}{\overline{x}^2},\ \lim\limits_{n\to\infty}C_i=0, \\ \lim\limits_{n\to\infty}D_0 & = & 0,\ \lim\limits_{n\to\infty}D_i=\dfrac{1}{\overline{x}} \end{array}$$

with $i\in\{1,2,\ldots,m\}$. That is,

$$\begin{array}{rcl} A_i & = & \dfrac{1}{\overline{y}}+a_{n_i} \text{ for } i=1,2,\ldots,m,\ B_0=-\dfrac{m\overline{x}}{\overline{y}^2}+b_n \\ D_i & = & \dfrac{1}{\overline{x}}+c_{n_i} \text{ for } i=1,2,\ldots,m,\ C_0=-\dfrac{m\overline{y}}{\overline{x}^2}+d_n \end{array}$$

where $a_n\to 0, b_n\to 0, c_n\to 0, d_n\to 0$.

Now, one can obtain the following error system of (23):

$$e_{n+1}=(M+N(n))e_n,$$

where $e_n = (e^1_n, e^1_{n-1}, \ldots, e^1_{n-m}, e^2_n, e^2_{n-1}, \ldots, e^2_{n-m})^T$ and

$$M = B(\overline{x}, \overline{y}) = \begin{pmatrix} 0 & \frac{1}{\overline{y}} & \ldots & \frac{1}{\overline{y}} & \frac{1}{\overline{y}} & -\frac{m\overline{x}}{\overline{y}^2} & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ -\frac{m\overline{y}}{\overline{x}^2} & 0 & \ldots & 0 & 0 & 0 & \frac{1}{\overline{x}} & \ldots & \frac{1}{\overline{x}} & \frac{1}{\overline{x}} \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

$$N(n) = \begin{pmatrix} 0 & a_{n_1} & \ldots & a_{n_{m-1}} & a_{n_m} & b_n & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ c_n & 0 & \ldots & 0 & 0 & 0 & d_{n_1} & \ldots & d_{n_{m-1}} & d_{n_m} \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

$\|N(n)\| \to 0$, when $n \to \infty$. Thus, the limiting system of error terms can be obtained that

$$e_{n+1} = \begin{pmatrix} 0 & \frac{1}{\overline{y}} & \ldots & \frac{1}{\overline{y}} & \frac{1}{\overline{y}} & -\frac{m\overline{x}}{\overline{y}^2} & 0 & \ldots & 0 & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ -\frac{m\overline{y}}{\overline{x}^2} & 0 & \ldots & 0 & 0 & 0 & \frac{1}{\overline{x}} & \ldots & \frac{1}{\overline{x}} & \frac{1}{\overline{x}} \\ 0 & 0 & \ldots & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix} \begin{pmatrix} e^1_n \\ e^1_{n-1} \\ \vdots \\ e^1_{n-m} \\ e^2_n \\ e^2_{n-1} \\ \vdots \\ e^2_{n-m} \end{pmatrix}.$$

For positive equilibrium obtained system is completely the linearized system of (9). Thanks to Theorem 4.1, the proof is completed as desired. □

5. Numerical Simulations

In this section, some numerical simulations with graphs will be given to emphasize the efficiency of our analytical results in the article. To affirm our analytical results, we deal with various effective numerical simulations. These simulations describe diverse types for the global and asymptotic behavior of the solutions of the system (9). Also, we state that all schemes are drawn via Mathematica.

Example (1) Consider the following discrete system of difference equations

$$u_{n+1} = \alpha + \frac{\sum_{i=1}^{4} u_{n-i}}{v_n},\ v_{n+1} = \beta + \frac{\sum_{i=1}^{4} v_{n-i}}{u_n},\ n \in \mathbb{N} \tag{26}$$

with the parameters $\alpha = 12 > 4 = m$, $\beta = 10 > 4 = m$ and the initial values $u_{-4} = 15$, $u_{-3} = 15$, $u_{-2} = 6$, $u_{-1} = 0.4$, $u_0 = 0.1$, $v_{-4} = 1.3$, $v_{-3} = 1.3$, $v_{-2} = 0.2$, $v_{-1} = 0.1$, $v_0 = 0.8$, then every positive solution of the system (26) is bounded and the positive equilibrium point $(\overline{u}, \overline{v}) = (17.33, 13)$ of the system (26) is global attractor (see Figure 1).

Example (2) Consider the following discrete system of non-linear difference equations

$$u_{n+1} = \alpha + \frac{\sum_{i=1}^{5} u_{n-i}}{v_n},\ v_{n+1} = \beta + \frac{\sum_{i=1}^{5} v_{n-i}}{u_n},\ n \in \mathbb{N} \tag{27}$$

with the parameters $\alpha = 5.1 > 5 = m$, $\beta = 4.9 < 5 = m$ and the initial values $u_{-5} = 1.3$, $u_{-4} = 1.3$, $u_{-3} = 0.5$, $u_{-2} = 0.6$, $u_{-1} = 0.4$, $u_0 = 0.1$, $v_{-5} = 2.3$, $v_{-4} = 3.3$, $v_{-3} = 0.7$, $v_{-2} = 1.2$, $v_{-1} = 4.1$, $v_0 = 0.8$, then Eq.(27) has unbounded solutions (see Figure 2).

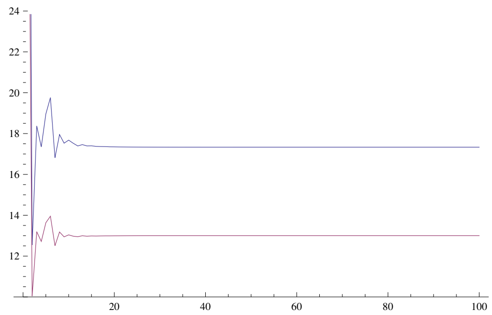

Figure 1: The behaviors of solutions of system (5.1)

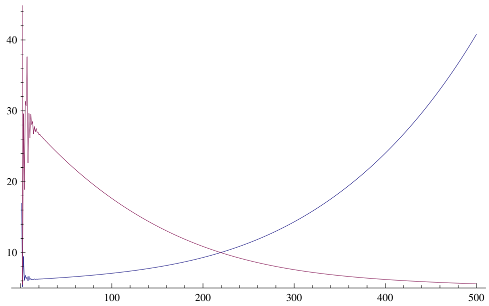

Figure 2: The behaviors of solutions of system (5.2)

Example (3) Consider the following discrete system of rational difference equations

$$u_{n+1} = \alpha + \frac{\sum_{i=1}^{4} u_{n-i}}{v_n},\ v_{n+1} = \beta + \frac{\sum_{i=1}^{4} v_{n-i}}{u_n},\ n \in \mathbb{N} \tag{28}$$

with the parameters $\alpha = 3.3 < 4 = m$, $\beta = 3.4 < 4 = m$ and the initial values $u_{-4} = 0.1$, $u_{-3} = 1.5$, $u_{-2} = 0.6$, $u_{-1} = 0.4$, $u_0 = 0.1$, $v_{-4} = 3.3$, $v_{-3} = 0.3$, $v_{-2} = 0.2$, $v_{-1} = 0.1$, $v_0 = 1.8$, then Eq.(28) has unbounded solutions (see Figure 3).

Example (4) Consider the following discrete system of non-linear difference equations

$$u_{n+1} = \alpha + \frac{\sum_{i=1}^{7} u_{n-i}}{v_n},\ v_{n+1} = \beta + \frac{\sum_{i=1}^{7} v_{n-i}}{u_n},\ n \in \mathbb{N} \tag{29}$$

with the parameters $\alpha = 100 > 7 = m$, $\beta = 120 > 7 = m$ and the initial values $u_{-7} = 2.5$, $u_{-6} = 1.5$, $u_{-5} = 0.5$, $u_{-4} = 0.6$, $u_{-3} = 2.3$, $u_{-2} = 0.6$, $u_{-1} = 1.4$, $u_0 = 1.1$, $v_{-7} = 2.3$, $v_{-6} = 2.2$, $v_{-5} = 0.9$, $v_{-4} = 1.3$, $v_{-3} = 1.6$, $v_{-2} = 1.2$, $v_{-1} = 0.1$, $v_0 = 1.8$. Since the relations (11), (20), and (21) are satisfied, then the unique positive equilibrium $(\overline{u}, \overline{v})$ of the system (29) is globally asymptotically stable (see Figure 4).

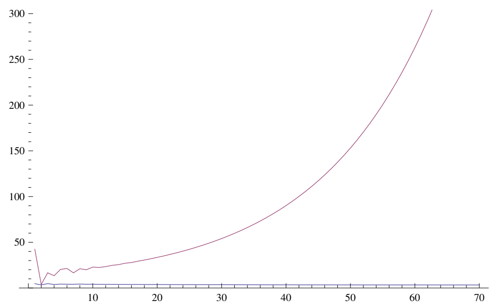

Figure 3: The behaviors of solutions of system (5.3)

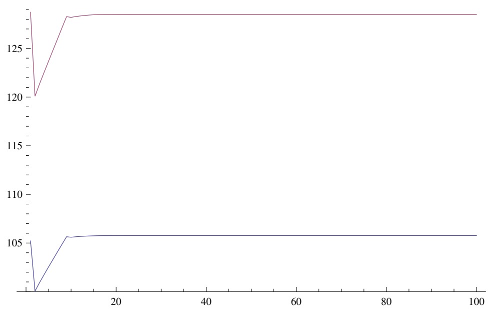

Figure 4: The behaviors of solutions of system (5.4)

The examples in this section characterize the long-term behavior of the solutions for every system.

6. Conclusions and Suggestions

In this paper, we studied the qualitative behavior of a discrete system which is seen as an extension of [3, 4, 6, 23, 28]. Here, we obtained that the positive equilibrium point of the system (9) is globally asymptotically stable if (11), (20), and (21) are satisfied. Every solution of the system (9) is bounded and persists if the condition (11) is satisfied. Finally, the rate of convergence result is given. It is our future work to investigate the asymptotic behavior of the system (9) when $\min\{\alpha, \beta\} \leq m$.

Finally, we approach that this work can be generalized to a system that has non-autonomous parameters. Also, one can study a discrete system with arbitrary parameters and arbitrary powers. Therefore, for interested researchers in this field, some interesting open problems are presented as follows;

Open Problem 1: Investigate the asymptotic behavior of the discrete system of non-autonomous difference equations

$$u_{n+1} = \mu_n + \frac{\sum_{i=1}^{m} u_{n-i}}{v_n}, v_{n+1} = \eta_n + \frac{\sum_{i=1}^{m} v_{n-i}}{u_n} \; n \in \mathbb{N}$$

where μ_n, η_n are some sequences that can be taken, respectively, as convergent sequences, periodic sequences or bounded sequences, the initial values are non-negative real numbers and m is a positive integer.

Open Problem 2: Investigate the asymptotic behavior of the discrete system of non-linear difference equations

$$u_{n+1} = \alpha + \frac{\sum_{i=1}^{m} (u_{n-i})^{p_i}}{v_n^q}, v_{n+1} = \beta + \frac{\sum_{i=1}^{m} (v_{n-i})^{q_i}}{u_n^p} \; n \in \mathbb{N}$$

where the parameters $\alpha, \beta, p, q, p_i, q_i$ are non-negative for $i = 1, ..., m$, initial values are non-negative real numbers and m is a positive integer.

7. Competing Interests

The author declares that they have no competing interests concerning the publication of the manuscript.

8. Data Availability Statement

The author declares that data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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