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Periodic nonuniform sinc-Gauss sampling

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Abstract. The periodic nonuniform sampling has attracted considerable attention both in mathematics and engineering although its convergence rate is slow. To improve the convergence rate, some authors incorporated a regularized multiplier into the truncated series. Recently, the authors of [18] have incorporated a Gaussian multiplier into the classical truncated series. This formula is valid for bandlimited functions and the error bound decays exponentially, i.e. $\sqrt{N}e^{-\beta N}$, where β is a positive number. The bound was established based on Fourier-analytic approach, so the condition that f belongs to $L^2(\mathbb{R})$ cannot be considerably relaxed. In this paper, we modify this formula based on localization truncated and with the help of complex-analytic approach. This formula is extended for wider classes of functions, the class of entire functions includes unbounded functions on $\mathbb{R}$ and the class of analytic functions in an infinite horizontal strip. The convergence rate is slightly better, of order $e^{-\beta N}/\sqrt{N}$. Some numerical experiments are presented to confirm the theoretical analysis.

1. Introduction

The Bernstein space B^p_Ω, $1 \le p < \infty$, contains all entire functions of exponential type Ω which belong to $L^p(\mathbb{R})$ when their domain is restricted to $\mathbb{R}$. According to Schwartz's theorem, cf. e.g. [7, 10],

$$B^p_\Omega = \left\{ f \in L^p(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-\Omega, \Omega] \right\},$$

where $\hat{f}$ is Fourier transform of f in the sense of generalized functions. Let $0 \le x_1 < x_2 < \ldots < x_J < Jh$ be arbitrary points that are not necessarily equidistant in $\mathbb{R}$. We define the sampling points to be

$$\tau_{j,n,h} := x_j + nJh, \quad n \in \mathbb{Z}, \; j = 1, \ldots, J, \tag{1}$$

where $h \in (0, \pi/\Omega]$, Ω is a positive number and J is a positive integer. In this form of sampling, the sampling points are divided into groups of J points. The periodic nonuniform sampling theorem states that if $f \in B^p_\Omega$, $1 \le p < \infty$, then f can be represented as, cf. [17],

$$f(z) = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{J} f(\tau_{j,n,\frac{\pi}{\Omega}})\, \psi_{j,n,\frac{\pi}{\Omega}}(z), \qquad z \in \mathbb{C}, \tag{2}$$

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where

$$\psi_{j,n,h}(z) := \operatorname{sinc}\left(\frac{\pi}{Jh}(z-\tau_{j,n,h})\right)\prod_{k=1,k\neq j}^{J}\frac{\sin\left(\frac{\pi}{Jh}(z-\tau_{k,n,h})\right)}{\sin\left(\frac{\pi}{Jh}(x_j-x_k)\right)}. \tag{3}$$

Note that we have defined the sampling points $\tau_{j,n,h}$ and the function $\psi_{j,n,h}(z)$, $h \in (0, \pi/\Omega)$ in the general form because they will be used in the sequel of the paper. The sinc function is defined by

$$\operatorname{sinc}(x) := \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

The series on the right-hand side of (2) converges absolutely and uniform on $\mathbb{R}$ and on any compact subset of $\mathbb{C}$, [1]. Unlike the sinc function, $\psi_{j,n,h}(t)$ does not take its maximum at the sampling points but it attains its maximum between nonuniform samples. The expression $\psi_{j,n,h}(t)$ turns to the simpler sinc function when $x_j = jh$, cf. [1]. Expansion (2) has been established for Paley-Wiener space, B^2_Ω, and it is not hard to check that it is also valid for all functions in the Bernstein space B^p_Ω, $1 \leq p < \infty$. It can be derived from the Lagrange interpolation, cf. [11, p. 130]. It has been extended to bandlimited functions in the fractional Fourier transform domains, see [9, p. 1755]. It has attracted a considerable attention both in mathematics and engineering, see e.g. [1, 8, 9, 11, 13] and references therein. Regrettably, the convergence rate, which is of the order $O(N^{-1/p})$, $p > 1$, is slow. While there are several algorithms in the literature that deal with the reconstruction of Bernstein's functions from periodic nonuniform samples, none of these algorithms provide a high order accuracy with respect to truncation error, for details see [13]. To improve the convergence rate, Strohmer and Tanner [13] have proposed a Gevrey regularized with (2) which led the truncation error of the order $O\left(e^{-\lambda N^{1/\alpha}}\right)$, where λ is a positive constant and $\alpha > 1$. However, the decay of the truncation error is a root-exponential and the most of Gevrey functions are hardly expressed explicitly. Recently, the authors of [18] have modified the series in (2) with a Gaussian function. They have introduced the following formula

$$\mathcal{S}_{J,N}[f](x) := \sum_{n=-N}^{N}\sum_{j=1}^{J} f(\tau_{j,n,1})\,\psi_{j,n,1}(x)\exp\left(-\frac{(\pi-\Omega)\left(x-\tau_{j,n,1}\right)^2}{2(N-1)J}\right), \qquad x \in \mathbb{R}, \tag{4}$$

where $\tau_{j,n,1}$ and $\psi_{j,n,1}(z)$ are defined in (1) and (3), respectively. For $\Omega < \pi$ and $f \in B^2_\Omega$, they have established the following estimate, [18, Theorem 1],

$$\sup_{x\in[-J,J]} |f(x) - \mathcal{S}_{J,N}[f](x)| \leq C\sqrt{N}e^{-J(\pi-\Omega)(N-1)/2}, \quad C > 0. \tag{5}$$

The bound in (5) is of exponential type but formula (4) needs a modification when we approximate the function f outside the interval $[-J, J]$ or the bandwidth $\Omega \geq \pi$. The proof in [18] is based on Fourier-analytic approach. Therefore, the condition that f belongs to $L^2(\mathbb{R})$ cannot be considerably relaxed.

In this paper, we show that the periodic nonuniform sampling with a Gaussian convergence factor can be conveniently studied by using a complex-analytic approach. This approach goes back to Schmeisser and Stenger (2007) in [12]. This approach has been successfully applied for various types of uniform sampling, one-dimensional [12], two-dimensional sampling [5], Hermite sampling [4], generalized sampling involving derivatives [3], generalized bivariate sampling [2] and the work of Tanaka et al. in [16]. Up to now, this approach has not been applied in any type of the nonuniform sampling expansions. As we see in the remaining of this paper, this approach with the periodic nonuniform sampling has the following advantages:

- It extends to classes of functions that do not need to belong to $L^p(\mathbb{R})$. They may even be unbounded on $\mathbb{R}$.

- It admits functions that are not restrictions of entire functions.
- It gives slightly better convergence rate. It will be of the order $e^{-(\pi-h\Omega)JN/2}/\sqrt{N}$.

This paper is designed as follows. In Section 2, we investigate the convergence rate of the periodic nonuniform sampling (2). Section 3 is devoted to establish the periodic nonuniform sinc-Gauss sampling formula which is a modification of the sampling series (2). Error bounds will be estimated in Section 4 for a class of entire functions and in Section 5 for a class of analytic functions on a strip. Numerical experiments are given in Section 6 to illustrate the accuracy of the proposed formulas. Lastly, Section 7 concludes the paper.

2. Convergence rate

This section is devoted to investigate the convergence rate of the sampling series (2). We show that the series in (2) has a convergence rate of the order $O(1/N^p)$, $p > 1$. To get this order, we have used the technique of Ye and Song in [15] for the truncation based on localized sampling without any decay condition that f satisfies. Let us start with the following result which will be used in the sequel.

Lemma 2.1. *If $f \in B^p_\Omega$, then we have*

$$\sum_{j=1}^{J}\left(\sum_{n=-\infty}^{\infty}\left|f(\tau_{j,n,\frac{\pi}{\Omega}})\right|^p\right)^{1/p} \leq A_{p,\Omega,J}\|f\|_p, \tag{6}$$

where $A_{p,\Omega,J}$ is some constant which depends on p, Ω and J only.

Proof. Since $f \in B^p_\Omega$, for any increasing sequence λ_n such that $\lambda_{n+1} - \lambda_n \geq \delta > 0$, we have [6, Theorem 6.7.15]

$$\left(\sum_{n=-\infty}^{\infty}\left|f(\lambda_n\right|^p\right)^{1/p} \leq a_{p,\Omega,\delta}\|f\|_p, \tag{7}$$

where $a_{p,\Omega,\delta}$ is some constant which depends on p, Ω and δ only. Letting $\lambda_n := \tau_{j,n,\frac{\pi}{\Omega}}$ in (7) and taking the sum over the j imply (6). □

Let $Z_{j,N}(x)$ to be

$$Z_{j,N}(x) := \left\{n \in \mathbb{Z} : -N < (x - x_j)\Omega/J\pi - n \leq N\right\}. \tag{8}$$

We truncate the sampling series (2), after the change of order of summations, as follows

$$T_{J,N}[f](x) := \sum_{j=1}^{J}\sum_{n\in Z_{j,N}(x)} f(\tau_{j,n,\frac{\pi}{\Omega}})\,\psi_{j,n,\frac{\pi}{\Omega}}(x), \qquad x \in \mathbb{R},\ N \in \mathbb{N}. \tag{9}$$

Note that the change of order of summations in (2) is justified from the absolutely convergence on $\mathbb{R}$. In the following theorem, we estimate the truncation error $\left|f(x) - T_{J,N}[f](x)\right|$.

Theorem 2.2. *Let $f \in B^p_\Omega$, $1 < p < \infty$. Then, for any $x \in \mathbb{R}$, we have*

$$\left|f(x) - T_{J,N}[f](x)\right| \leq B_{p,\Omega,J}\max_{1\leq j\leq J}|\rho_j|\|f\|_p\, N^{-1/p}, \tag{10}$$

where $B_{p,\Omega,J}$ is some constant which depends on p, Ω and J only. The function ρ_j is given by

$$\rho_j := \prod_{k=1,k\neq j}^{J}\frac{1}{\sin\left(\frac{\Omega}{J}(x_j - x_k)\right)}. \tag{11}$$

Proof. Since $f \in B^p_\Omega$, the expansion (2) holds. Together with (9) and the triangle's inequality, we obtain

$$\left|f(x) - T_{J,N}[f](x)\right| \le \sum_{j=1}^{J} \sum_{n\in\mathbb{Z}\setminus Z_{j,N}(x)} \left|f(\tau_{j,n,\frac{\pi}{\Omega}})\, \psi_{j,n,\frac{\pi}{\Omega}}(x)\right|. \tag{12}$$

Applying Hölder's inequality, we obtain

$$\left|f(x) - T_{J,N}[f](x)\right| \le \sum_{j=1}^{J} \left(\sum_{n\in\mathbb{Z}\setminus Z_{j,N}(x)} \left|f(\tau_{j,n,\frac{\pi}{\Omega}})\right|^p\right)^{1/p} \left(\sum_{n\in\mathbb{Z}\setminus Z_{j,N}(x)} \left|\psi_{j,n,\frac{\pi}{\Omega}}(x)\right|^q\right)^{1/q}, \tag{13}$$

with $p, q > 1$ and $1/p + 1/q = 1$. In view of (3), we have

$$\left(\sum_{n\in\mathbb{Z}\setminus Z_{j,N}(x)} \left|\psi_{j,n,\frac{\pi}{\Omega}}(x)\right|^q\right)^{1/q} \le \rho_j \left(\sum_{n\in\mathbb{Z}\setminus Z_{j,N}(x)} \left|\operatorname{sinc}\left(\frac{\Omega}{J}(x - \tau_{j,n,\frac{\pi}{\Omega}})\right)\right|^q\right)^{1/q}. \tag{14}$$

From the definition of $Z_{j,N}(x)$ and combining the inequality, [15, p. 415],

$$\left(\sum_{|\sigma x/\pi - n|>N} |\operatorname{sinc}\,(\sigma x - n\pi))|^q\right)^{1/q} \le C\, N^{-1/p}, \quad \sigma > 0, \tag{15}$$

with (14) implies

$$\left(\sum_{n\in\mathbb{Z}\setminus Z_{j,N}(x)} \left|\psi_{j,n,\frac{\pi}{\Omega}}(x)\right|^q\right)^{1/q} \le C\rho_j\, N^{-1/p}. \tag{16}$$

Combining (16), (6) with (13), we get (10) and the proof is completed. □

In view of the bound in (10), the convergence rate of the sampling series (2) couldn't be faster than $O(1/N)$.

3. Sinc-Gauss formula

In this section, we modify the sampling formula (4) based on localization truncated and with help of the complex-analytic approach. This formula will be verified for wider classes of functions, the class of entire functions includes unbounded functions on $\mathbb{R}$ and the class of analytic functions in an infinite horizontal strip which satisfy a certain growth condition. Let $E_\Omega(\varphi)$, $\Omega > 0$, be the class

$$E_\Omega(\varphi) := \left\{f : \mathbb{C} \to \mathbb{C} \mid \text{ is entire and } |f(z)| \le \varphi\left(\left|\Re z\right|\right)\, \exp\left(\Omega|\Im z|\right),\ z \in \mathbb{C}\right\}, \tag{17}$$

where φ is a continuous, non-decreasing and non-negative function on $\mathbb{R}^+$. The space $E_\Omega(\varphi)$ is introduced in [12] and it is larger than the Bernstein space B^p_Ω. It is clear that $B^p_\Omega = E_\Omega(C) \bigcap L^p(\mathbb{R})$ such that the intersection is understood as $f \in E_\Omega(C)$ and $f_\mathbb{R} \in L^p(\mathbb{R})$ and C is a positive constant. The class $E_\Omega(C)$ contains all entire functions of an exponential type Ω which do not necessary belong to $L^p(\mathbb{R})$ when restricted to $\mathbb{R}$. Let $\mathcal{E}_{L^p(\mathbb{R})}$ be the class of all entire functions which belong to $L^p(\mathbb{R})$ when restricted to $\mathbb{R}$. We define the nonuniform sinc-Gauss localization operator $\mathcal{G}_{h,J,N} : E_\Omega(\varphi) \to \mathcal{E}_{L^p(\mathbb{R})}$ for every $1 \le p \le \infty$ as follows

$$\mathcal{G}_{h,J,N}[f](z) = \sum_{n\in\mathbb{Z}_N(z)} \sum_{j=1}^{J} f(\tau_{j,n,h})\, \psi_{j,n,h}(z)\, \exp\left(-\frac{\alpha}{NJh^2}\left(z - \tau_{j,n,h}\right)^2\right), \tag{18}$$

where the function $\psi_{j,n,h}$ is defined in (3), $\alpha := (\pi - h\Omega)/2$, $x_J/J < h \leq \pi/\Omega$ and

$$\mathbb{Z}_N(z) := \left\{n \in \mathbb{Z} : |\lfloor J^{-1}h^{-1}\Re z + 1/2\rfloor - n| \leq N\right\}.$$

We denote by $\lfloor x \rfloor$, where $x \in \mathbb{R}$, the largest integer does not exceed x. The uniform sinc-Gauss operator, established in [12, Eq. (11)], is included in our operator (18) as special case when we specify $J = 1$ and $x_j = 0$ in (1).

Now, let us denote the nonuniform sampling expansion (2) by $\mathcal{L}_{J,\Omega}[f](z)$ such that $\mathcal{L}_{J,\Omega} : B^p_\Omega \to B^p_\Omega$. The question is what the relationship between the operators $\mathcal{L}_{J,\Omega}$ and $\mathcal{G}_{h,J,N}$ is? In the following lemma, we see that the operator $\mathcal{L}_{J,\Omega}$ is a limit for the operators $\mathcal{G}_{h,J,N}$ as $N \to \infty$. This result is only satisfied on the Bernstein space $B^p_\Omega \subset E_\Omega(\varphi)$.

Lemma 3.1. *Assume that φ is a constant function and $h = \pi/\Omega$. For any $f \in B^p_\Omega$ we have*

$$\lim_{N\to\infty} \mathcal{G}_{h,J,N} f = \mathcal{L}_{J,\Omega} f = f.$$

Proof. Letting $h = \pi/\Omega$ gives $\alpha = 0$. Setting $\alpha = 0$ in (18) and taking $N \to \infty$ yields the right-hand side of (2) which equals f because $f \in B^p_\Omega$. □

Consider the following kernel function

$$\mathcal{K}_z(\zeta) = \frac{\displaystyle\prod_{k=1}^{J} \sin\left(\frac{\pi}{Jh}(z - \tau_{k,n})\right) \exp\left(-\frac{\alpha}{NJh^2}(z-\zeta)^2\right)}{\displaystyle(\zeta - z)\prod_{k=1}^{J} \sin\left(\frac{\pi}{Jh}(\zeta - \tau_{j,n,h})\right)}, \quad z \in \mathbb{C}\backslash\{\tau_{k,n}\}, \tag{19}$$

where the points $\tau_{k,n}$ are defined in (1). From the description of the points x_k, $j = 1,\ldots,J$, we have $\sin\left(\frac{\pi}{Jh}(x_j - x_k)\right) \neq 0$ for all $k \neq j$. Therefore, the kernel $\mathcal{K}_z$, as a function of variable ζ, has a simple pole at $\zeta = z$ and at the sampling points $\zeta = \tau_{j,n,h}$ where $j = 1,\ldots,J$ and $n \in \mathbb{N}$. As we see in the following result, the error of approximation of functions from $E_\Omega(\varphi)$ by the operator $\mathcal{G}_{h,J,N}$ can be written as the integral of the kernel $\mathcal{K}_z$ over a positively oriented simple closed curve encloses the poles.

Lemma 3.2. *For $f \in E_\Omega(\varphi)$, we have*

$$f(z) - \mathcal{G}_{h,J,N}[f](z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\oint_C \mathcal{K}_z(\zeta) f(\zeta) d\zeta, & z \in \mathbb{C}\backslash\{\tau_{j,n,h}\}, \\ 0, & z = \tau_{j,n,h}, \end{cases} \tag{20}$$

where C is a positively oriented simple closed curve encloses the poles $\zeta = z$ and $\zeta = \tau_{j,n,h}$, $n \in \mathbb{Z}_N(z)$, $k = 1,\ldots,J$.

Proof. From the definition of $\psi_{j,n,h}$ in (3), we have

$$\psi_{j,n,h}(\tau_{j,n,h}) := \begin{cases} 0, & k \neq j, \\ 1, & k = j. \end{cases} \tag{21}$$

Setting $z = \tau_{j,n,h}$ in (18) and using (21) yields $\mathcal{G}_{h,J,N}[f](\tau_{j,n,h}) = f(\tau_{j,n,h})$, $n \in \mathbb{Z}_N(z)$, $j = 1,\ldots,J$. Therefore, the second part of equality (20) is valid. To establish the first part of the equality in (20), we apply the residue theorem to obtain

$$\frac{1}{2\pi i}\oint_C \mathcal{K}_z(\zeta) f(\zeta) d\zeta = \text{Res}\,(\mathcal{K}_z f; z) + \sum_{n\in\mathbb{Z}_N(z)} \sum_{j=1}^{J} \text{Res}\left(\mathcal{K}_z f; \tau_{j,n,h}\right), \quad z \in \mathbb{C}\backslash\{\tau_{j,n,h}\}. \tag{22}$$

The residue at each point satisfies

$$\text{Res}\,(\mathcal{K}_z f; z) = f(z), \tag{23}$$

and

$$\sum_{n\in\mathbb{Z}_N(z)} \sum_{j=1}^{J} \text{Res}\left(\mathcal{K}_z f; \tau_{j,n,h}\right) = -\mathcal{G}_{h,J,N}[f](z). \tag{24}$$

The first part of the equality in (20) comes from the combining (22)-(24). $\square$

Remark 3.3. *The condition $f \in E_\Omega(\varphi)$ in Lemma 3.2 is sufficient but not necessary. We can relax this condition to be f belongs to a class of analytic functions in an infinite horizontal strip $\mathcal{D}_d := \left\{z \in \mathbb{C} : \left|\Im z\right| < d\right\}$. Indeed, let $A_d(\varphi)$ be the class*

$$A_d(\varphi) := \left\{f : \mathcal{D}_d \to \mathbb{C} \mid \text{ is analytic in } \mathcal{D}_d \text{ and } |f(z)| \le \varphi\left(\left|\Re z\right|\right), \quad z \in \mathcal{D}_d\right\}, \tag{25}$$

where φ is a continuous, non-decreasing and non-negative function on $\mathbb{R}^+$. This class is firstly introduced in [12] and used in some studies, cf. e.g. [3, 4]. It is easy to check the validity of (20) for all $f \in A_d(\varphi)$.

The operator $\mathcal{G}_{h,J,N}$ approximates the functions from the class $E_\sigma(\varphi)$ or the class $A_d(\varphi)$ on each of the vertical strips

$$\left\{z \in \mathbb{C} : \left(k - \frac{1}{2}\right)hJ \le \Re z \le \left(k + \frac{1}{2}\right)hJ\right\}, \qquad k \in \mathbb{Z}, \tag{26}$$

as piecewise analytic approximation. In the next two sections, we will estimate bounds for the error $\left|f(z) - \mathcal{G}_{h,J,N}[f](z)\right|$ where $f \in E_\sigma(\varphi)$ and $f \in A_d(\varphi)$, respectively. Those bounds will be of exponential order.

4. Error bound for $E_\Omega(\varphi)$-functions

In this section, we estimate a bound for the error $\left|f(z) - \mathcal{G}_{h,J,N}[f](z)\right|$ when f belongs to the class $E_\Omega(\varphi)$, see (17) for definition. To do that, we will use the complex analytic technique which is given in [12]. Some advantageous results will be presented.

Theorem 4.1. *Let $f \in E_\Omega(\varphi)$. The following estimate holds true*

$$\left|f(z) - \mathcal{G}_{h,J,N}[f](z)\right| \le 2^{J-1} \left|\prod_{k=1}^{J} \sin\left(\frac{\pi}{Jh}(z - x_k)\right)\right| \varphi\left(|\Re z| + Jh(N+2)\right) \beta_{N,J}\left(\Im z\right) \frac{e^{-\alpha JN}}{\sqrt{\pi\alpha JN}}, \tag{27}$$

where $|\Im z| < JhN$, J is a positive integer, $h \in (0, \pi/\Omega]$ and $0 \le x_1 < x_2 < \ldots < x_J < Jh$ are arbitrary points that are not necessarily equidistant in $\mathbb{R}$. The function $\beta_{J,N}$ is given by

$$\begin{aligned}\beta_{N,J}(t) &:= \frac{2e^{\alpha t^2/N}}{h\sqrt{\pi\alpha JN}\left(1 - (t/JhN)^2\right)} + \frac{e^{-2\alpha t}}{(1 - e^{-\frac{2\pi}{Jh}(JhN+t)})^J} + \frac{e^{2\alpha t}}{(1 - e^{-\frac{2\pi}{Jh}(JhN-t)})^J} \\ &= 2\cosh(2\alpha t) + O(N^{-1/2}), \quad \textit{as} \quad N \to \infty.\end{aligned} \tag{28}$$

Proof. In view of (19), we have

$$\oint_C \mathcal{K}_z(\zeta) f(\zeta) d\zeta = (-1)^{nJ} \prod_{k=1}^{J} \sin\left(\frac{\pi}{Jh}(z - x_k)\right) \oint_C \frac{f(\zeta)\exp\left(-\frac{\alpha}{NJh^2}(z-\zeta)^2\right)}{(\zeta - z)\prod_{k=1}^{J}\sin\left(\frac{\pi}{Jh}(\zeta - \tau_{k,n,h})\right)} d\zeta, \tag{29}$$

where the curve C is defined in Lemma 3.2. Let $\mathcal{R}_z$ be the positively oriented rectangle with vertices at $\pm Jh(N+3/2)+JhN_{z/Jh}+\mathrm{i}(\Im z \pm JhN)$, where $N_z := \lfloor \Re z + 1/2 \rfloor$. Clearly that the rectangle $\mathcal{R}_z$ encloses the poles of the kernel $\mathcal{K}_z$, $\zeta = z$ and $\zeta = \tau_{j,n,h}$ where $n \in \mathbb{Z}_N(z)$ and $j = 1, \dots, J$. Note that $\mathcal{R}_z$ is not fixed but it depends on the point z. The concept here is to estimate the integral in the right-hand side of (29) with $C := \mathcal{R}_z$. Since $f \in E_\Omega(\varphi)$, f satisfies the decay in (17) and for any point $\zeta \in \mathcal{R}_z$ we have

$$\left|f(\zeta)\right| \le \varphi\left(hJ(N+3/2)+hJ|N_{z/Jh}|\right)\mathrm{e}^{\Omega|\Im\zeta|} \le \varphi\left(|\Re z| + hJ(N+2)\right)\mathrm{e}^{\Omega|\Im\zeta|}. \tag{30}$$

Combining (30), (29) with (20) implies

$$\begin{aligned}\left|f(z)-\mathcal{G}_{h,J,N}[f](z)\right| \quad &\le \quad \frac{1}{2\pi}\left|\prod_{k=1}^{J}\sin\left(\frac{\pi}{Jh}(z-x_k)\right)\right|\varphi\left(|\Re z| + Jh(N+2)\right)\\ &\times \quad \oint_{\mathcal{R}_z}\left|\frac{\mathrm{e}^{\Omega|\Im\zeta|}\exp\left(-\frac{\alpha}{NJh^2}(z-\zeta)^2\right)}{(\zeta-z)\displaystyle\prod_{k=1}^{J}\sin\left(\frac{\pi}{Jh}(\zeta-\tau_{k,n,h})\right)}\right||d\zeta|.\end{aligned} \tag{31}$$

The integral of (31) is estimated by Schmeisser and Stenger, over the boundary of the rectangle $\mathcal{R}_z$, for the especial case $J = 1$ and $x_1 = 0$ in [12, pp. 203-205]. Applying similar technique of [12], the integral of (31) can be estimated as

$$\oint_{\mathcal{R}_z}\left|\frac{\mathrm{e}^{\Omega|\Im\zeta|}\exp\left(-\frac{\alpha}{NJh^2}(z-\zeta)^2\right)}{(\zeta-z)\displaystyle\prod_{k=1}^{J}\sin\left(\frac{\pi}{Jh}(\zeta-\tau_{k,n,h})\right)}\right||d\zeta| \le 2^J\pi\beta_{N,J}(\Im z)\,\frac{\mathrm{e}^{-\alpha JN}}{\sqrt{\pi\alpha JN}}, \tag{32}$$

where $\beta_{N,J}$ is given in (28). The proof of the estimate (32) is omitted because it is not short and basically similar to [12]. Substituting (32) into (31), we obtain (27) and the proof is completed. □

Remark 4.2. *The bound in (27) is zero at every sampling point $\tau_{k,n,h}$ and the absolute error finishes at these points because $\mathcal{G}_{h,J,N}[f](\tau_{j,n,h}) = f(\tau_{j,n,h})$, cf. Lemma 3.2.*

Remark 4.3. *The classical sampling series of Whittaker, Kotelnikov and Shannon is a special case of expansion (2) when we specify $J = 1$ and $x_1 = 0$ in (1). Furthermore, letting $J = 1$ and $x_1 = 0$ in (27) yields the bound in [12, Theorem 2.1].*

Theorem 4.1 shows that the convergence rate of the nonuniform sinc-Gauss sampling formula (18) is of an exponential order. Since $\alpha = (\pi - h\Omega)/2$, the bound in (27) decreases when h decreases which means that the precision increases when N is fixed but, h decreases. The growth of the bound in (27) depends on the growth of the function φ. The order of the exponential convergence rate for the formula (18) is affected by the exponential growth of the function φ on the real line, see Corollary 4.6, while the polynomial growth of φ on $\mathbb{R}$ does not affect the order of exponential of formula (18), see Corollary 4.5.

The following results show three advantageous special cases of Theorem 4.1. The first result describes the case when the function f be in the Bernstein space B^p_Ω, $1 \le p \le \infty$. In the second result, we choose the function which has a polynomial growth on the real line while the third result is devoted to the case when f has an exponential growth on $\mathbb{R}$.

Corollary 4.4. *It holds for $f \in B^p_\Omega$, $1 \le p \le \infty$, and $|\Im z| < JhN$*

$$\left|f(z) - \mathcal{G}_{h,J,N}[f](z)\right| \le 2^{J-1}\|f\|_\infty \left|\prod_{k=1}^{J} \sin\left(\frac{\pi}{Jh}(z - x_k)\right)\right| \beta_{N,J}(\Im z)\, \frac{e^{-\alpha JN}}{\sqrt{\pi\alpha JN}}, \tag{33}$$

where $\beta_{J,N}$ is defined in (28).

Proof. In the case $f \in B^\infty_\Omega$, we have for all $z \in \mathbb{C}$

$$|f(z)| \le \|f\|_\infty e^{\Omega|\Im z|}. \tag{34}$$

If we choose φ as $\|f\|_\infty$, we get (33) in the case $p = \infty$. It is well known that $B^p_\Omega \subset B^\infty_\Omega$ for all $1 \le p < \infty$, cf. e.g [10, p. 48]. Therefore, estimation (33) is valid for all $f \in B^p_\Omega$, $1 \le p \le \infty$. □

Corollary 4.5. *Suppose $g \in B^\infty_{\Omega'}$ and define f to be $f(z) = (1 + z)^\nu g(z)$ where $\nu \in \mathbb{N}$. Then $f \in E_\Omega(\varphi)$ for all $\Omega > \Omega'$ and the following estimate holds*

$$\left|f(z) - \mathcal{G}_{h,J,N}[f](z)\right| \le 2^{J-1}M \left|\prod_{k=1}^{J} \sin\left(\frac{\pi}{Jh}(z - x_k)\right)\right| \mathcal{B}_{\nu,N}(\Re z)\beta_{N,J}(\Im z)\, \frac{N^{\nu-1/2}e^{-\alpha JN}}{\sqrt{\pi\alpha J}}, \tag{35}$$

where $M > 0$, $\beta_{J,N}$ is defined in (28). The function $\mathcal{B}_{\nu,N}$ is given by

$$\mathcal{B}_{\nu,N}(t) := \left(\frac{|t| + 2Jh + 1}{N} + Jh\right)^\nu = O(1) \quad \text{as} \quad N \to \infty.$$

Proof. Since $f(z) = (1 + z)^\nu g(z)$ and $g \in B^\infty_{\Omega'}$, there exists $M > 0$ such that

$$|f(z)| \le (1 + |z|)^\nu \|g\|_\infty e^{\Omega'|\Im z|} \le (1 + |\Re z|)^\nu (1 + |\Im z|)^\nu \|g\|_\infty e^{\Omega'|\Im z|} \le M(1 + |\Re z|)^\nu \|g\|_\infty e^{\Omega|\Im z|},$$

for all $\Omega > \Omega'$. Therefore, f is an entire function of a polynomial growth on the real axis and $f \in E_\Omega(\varphi)$ with $\varphi(t) = (1 + t)^\nu$. Letting $\varphi(t) = (1 + t)^\nu$ in (27), we get (35). □

Corollary 4.6. *Assume that f is an entire function satisfies $|f(z)| \le M e^{\kappa|\Re z| + \sigma|\Im z|}$, $z \in \mathbb{C}$ such that $M > 0$ and $\sigma, \kappa \ge 0$ satisfy $\sigma + \kappa \ne 0$. For $h \in (0, \pi/(\Omega + 2\kappa))$ and $|\Im z| < JhN$, the following estimate holds*

$$\left|f(z) - \mathcal{G}_{h,J,N}[f](z)\right| \le 2^{J-1}Me^{\kappa(|\Re z| + 2Jh)} \left|\prod_{k=1}^{J} \sin\left(\frac{\pi}{Jh}(z - x_k)\right)\right| \beta_{N,J}(\Im z)\, \frac{e^{-(\alpha-\kappa h)JN}}{\sqrt{\pi\alpha JN}}, \tag{36}$$

$\beta_{J,N}$ is defined in (28).

Proof. Setting $\varphi(x) = Me^{\kappa x}$, $x \in \mathbb{R}^+$ in Theorem 4.1, immediately implies (36) after we restrict h to the interval $(0, \pi/(\Omega + 2\kappa))$. □

5. Error bound for $A_d(\varphi)$-functions

An estimation for the error $\left|f(z) - \mathcal{G}_{h,J,N}[f](z)\right|$ when f belongs to the class $A_d(\varphi)$, see the definition in (25), is derived in this section. On the class $A_d(\varphi)$, the operator (18) will be defined as

$$\mathcal{G}_{\frac{d}{N},J,N}[f](z) = \sum_{n \in \mathbb{Z}_N(z)} \sum_{j=1}^{J} f\left(\tau_{j,n,h}\right) \psi_{j,n,h}(z) \exp\left(-\frac{\pi}{2NJh^2}\left(z - \tau_{j,n,h}\right)^2\right), \tag{37}$$

where $h := d/JN$. In the general operator $\mathcal{G}_{h,J,N}$, the numbers N and h can be chosen independently while in the special case of this operator, $\mathcal{G}_{\frac{d}{N},J,N}$, they are correlated.

Theorem 5.1. *Let $f \in A_d(\varphi)$. The following estimate holds true*

$$\left|f(z) - \mathcal{G}_{\frac{d}{N},J,N}[f](z)\right| \leq 2^{J+1/2}\left|\prod_{k=1}^{J}\sin\left(\frac{\pi N}{Jd}(z-x_k)\right)\right|\varphi\left(|\Re z| + \rho_N\right)\gamma_N\left(\Im z/d\right)\frac{e^{-\frac{\pi}{2}\left(JN-\frac{2|\Im z|}{d}\right)}}{\pi\sqrt{N}}, \tag{38}$$

where $z \in \mathcal{D}_{d/4}$ and $\rho_N := Jd(1+\frac{2}{N})$. The function γ_N is given by

$$\begin{aligned}\gamma_N(t) &:= \frac{1}{1-t}\left[\frac{1}{(1-e^{-2\pi N})^J} + \frac{2\sqrt{2}}{\pi\sqrt{JN}(1+t)}\right]\\ &= \frac{1}{1-t}\left[1 + O\left(N^{-1/2}\right)\right], \quad as \quad N\to\infty. \end{aligned}\tag{39}$$

Proof. Denoted by $\mathfrak{R}_z$ the positively oriented rectangle with vertices at $\pm Jh(N+3/2) + JhN_{z/Jh} + \mathrm{i}Jd$ and $\pm Jh(N+3/2) + JhN_{z/Jh} + \mathrm{i}(dJ-y)$, where $N_z := \lfloor \Re z + 1/2\rfloor$ and $h := \frac{d}{N}$. The rectangle $\mathfrak{R}_z$ depends on the point z and encloses the poles of the kernel $\mathcal{K}_z$. From (20) and the definition of the kernel (19), we obtain

$$f(z) - \mathcal{G}_{\frac{d}{N},J,N}[f](z) = \frac{(-1)^{nJ}}{2\pi\mathrm{i}}\prod_{k=1}^{J}\sin\left(\frac{\pi N}{Jd}(z-x_k)\right)\oint_{\mathfrak{R}_z}\frac{f(\zeta)\exp\left(-\frac{\pi N}{2Jd^2}(z-\zeta)^2\right)}{(\zeta-z)\prod_{k=1}^{J}\sin\left(\frac{\pi N}{Jd}(\zeta-\tau_{k,n,h})\right)}d\zeta. \tag{40}$$

Since $f \in A_d(\varphi)$, for every $\zeta \in \mathfrak{R}_z$ it holds

$$|f(\zeta)| \leq \varphi\left(|\Re z| + Jd + 2Jd/N\right). \tag{41}$$

Combining (41) with (40), we obtain

$$\left|f(z) - \mathcal{G}_{\frac{d}{N},J,N}[f](z)\right| \leq \frac{1}{2\pi}\left|\prod_{k=1}^{J}\sin\left(\frac{\pi N}{Jd}(z-x_k)\right)\right|\varphi\left(|\Re z| + Jd + 2Jd/N\right) \tag{42}$$

$$\times \oint_{\mathfrak{R}_z}\left|\frac{\exp\left(-\frac{\pi N}{2Jd^2}(z-\zeta)^2\right)}{(\zeta-z)\prod_{k=1}^{J}\sin\left(\frac{\pi N}{Jd}(\zeta-\tau_{k,n,h})\right)}\right||d\zeta|. \tag{43}$$

The integral in the right-hand side of (42) is also bounded by Schmeisser and Stenger for $J = 1$ and $x_1 = 0$ in [12, pp. 209-211]. Applying a similar technique of [12], the integral of (42) can be estimated as

$$\oint_{\mathfrak{R}_z}\left|\frac{\exp\left(-\frac{\pi N}{2Jd^2}(z-\zeta)^2\right)}{(\zeta-z)\prod_{k=1}^{J}\sin\left(\frac{\pi N}{Jd}(\zeta-\tau_{k,n})\right)}\right||d\zeta| \leq 2^{J+3/2}\gamma_N\left(\Im z/d\right)\frac{e^{-\frac{\pi}{2}\left(JN-\frac{2|\Im z|}{d}\right)}}{\sqrt{N}}. \tag{44}$$

The proof of the estimate (44) is omitted because it is not short and basically similar to [12]. The proof is completed by combining (44) and (42). □

The behaviour of bound in (38) depends on the growth of the function φ and the imaginary part of the variable z. For example, when φ is a constant function and $z \in \mathbb{R}$, then the bound (38) will be of the order $N^{-1/2}\mathrm{e}^{-\pi JN/2}$. As we have done at the end of last section, we can establish three advantageous special cases of Theorem 5.1. These cases deal with f bounded on $\mathbb{R}$, has polynomial and exponential growths on the real line, respectively. It is not hard to calculate these cases so we have left this task for the reader.

6. Numerical Experiments

We restrict ourselves to examples which are not accessible by the results in [18]. The functions in the first to third examples are chosen from the space $E_\sigma(\varphi)$ while the final example f belongs to $A_d(\varphi)$. The functions in examples 1-3 have constant, polynomial and exponential growths on the real line, respectively. We would like to mention that the accuracy of our formula increases when N is fixed but h decreases without any additional cost except that the step size hJ will be smaller. All computations were carried out using *Mathematica 12* on a personal computer. During this section, we let $x_{k,h} := \left(k - \frac{1}{2}\right)hJ$ where $k \in \mathbb{Z}^+$ and $x_{[J]} := (x_1, x_2, \dots, x_J)$ where x_j is defined in (1).

Example 6.1. *Consider the function $f(z) = \sin(z)$, $z \in \mathbb{C}$. Obviously $|\sin(z)| \le e^{|\Im z|}$ for every $z \in \mathbb{C}$. Thus the function f has a constant growth on $\mathbb{R}$ and it belongs to the Bernstein space B_1^∞. Therefore, we apply Corollary 4.4 with $\Omega = 1$, $N = 6$ and $x_{[3]} = (0.1, 0.6, 1.3)$. In this example, we choose $h = 1$ and $h = 0.5$ to show the increasing of the accuracy when N is fixed but h decreases. Let B_N be the real-valued bound in (33) with $J = 3$, i.e.,*

$$B_N(x) := 4\|f\|_\infty \left| \prod_{k=1}^{3} \sin\left(\frac{\pi}{3h}(x - x_k)\right) \right| \beta_{N,3}(0) \frac{e^{-3\alpha N}}{\sqrt{3\pi\alpha N}},$$

where $\beta_{N,3}$ is defined in (28). Accidentally, the values of the bound B_N are identical at the points $x_{k,h}$ when h is fixed, see Table 1. This gives us an unfaithful impression on the bound. Therefore we show in Figure 3 the correct behavior of the bound. Figures 1 and 2 show the graphs of the error of the periodic nonuniform sinc-Gauss sampling formula on the interval $[0, 10]$ for $N = 6$ and $h = 1, 1/2$.

Table 1: Approximation of the function at $x_{k,h}$

k	Absolute error and bound for $\mathcal{G}_{h,3,6}$ with $x_{[3]} = (0.1, 0.6, 1.3)$ and $h = 3/2, 1$			
	$\lvert(f - \mathcal{G}_{\frac{3}{2},3,6}[f])(x_{k,\frac{3}{2}})\rvert$	$B_N(x_{k,\frac{3}{2}})$	$\lvert(f - \mathcal{G}_{1,3,6}[f])(x_{k,1})\rvert$	$B_N(x_{k,1})$
1	5.86754×10^{-8}	2.77273×10^{-7}	1.36506×10^{-10}	8.26503×10^{-10}
2	1.11766×10^{-7}	2.77273×10^{-7}	8.68571×10^{-11}	8.26503×10^{-10}
3	1.15557×10^{-8}	2.77273×10^{-7}	3.54703×10^{-11}	8.26503×10^{-10}
4	1.16638×10^{-7}	2.77273×10^{-7}	1.66257×10^{-11}	8.26503×10^{-10}
5	3.76179×10^{-8}	2.77273×10^{-7}	6.83905×10^{-11}	8.26503×10^{-10}

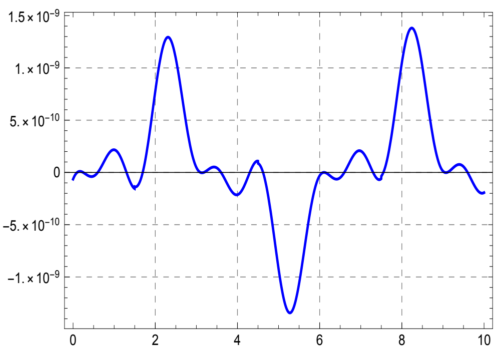

Figure 1: $f - \mathcal{G}_{1,3,6}[f]$.

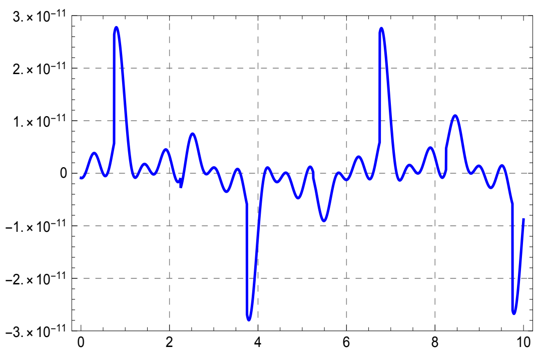

Figure 2: $f - \mathcal{G}_{\frac{1}{2},3,6}[f]$.

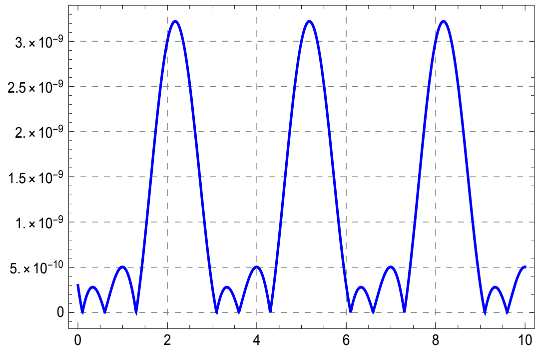

Figure 3: The bound B_N, $N = 6, h = 1$.

Example 6.2. *In this example, we approximate the function* $f(z) = (1+z)^3 \sin(z)$, $z \in \mathbb{C}$. *This function has a polynomial growth on* $\mathbb{R}$, *so we apply Corollary 4.5 with* $\nu = 3$, $\Omega = 1$, $h = 1$ *and* $x_{[3]} = (0.4, 1, 1.7)$. *The bound (35) is written in the real line as*

$$B_N(x) := 4\left|\prod_{k=1}^{3} \sin\left(\frac{\pi}{3h}(x - x_k)\right)\right| \left(\frac{|x| + 6h + 1}{N} + 3h\right)^3 \beta_{N,3}(0) \frac{N^{3/2} e^{-3\alpha N}}{\sqrt{3\pi\alpha}},$$

where $\beta_{N,3}$ *is given in (28). Unlike the bound of last example, the values of the bound* B_N *slightly increase when* N *is fixed but* k *increases. This comes from the polynomial growth of the bound with respect to* x. *The results are summarized in Table 2 and Figs. 4 and 5.*

Table 2: Approximation of the function at $x_{k,h}$

k	Absolute error and bound for $\mathcal{G}_{1,3,N}$ with $N = 7, 9$ and $x_{[3]} = (0.4, 1, 1.7)$			
	$\lvert(f - \mathcal{G}_{1,3,7}[f])(x_{k,1})\rvert$	$B_7(x_{k,1})$	$\lvert(f - \mathcal{G}_{1,3,9}[f])(x_{k,1})\rvert$	$B_9(x_{k,1})$
1	7.39870×10^{-8}	4.45203×10^{-7}	2.29916×10^{-10}	1.09515×10^{-9}
2	6.53499×10^{-8}	5.95309×10^{-7}	2.15870×10^{-10}	1.39691×10^{-9}
3	5.62771×10^{-8}	7.75850×10^{-7}	2.03670×10^{-10}	1.74957×10^{-9}
4	4.65337×10^{-8}	9.89634×10^{-7}	1.92813×10^{-10}	2.15709×10^{-9}
5	3.59701×10^{-8}	1.23947×10^{-6}	1.84173×10^{-10}	2.62342×10^{-9}

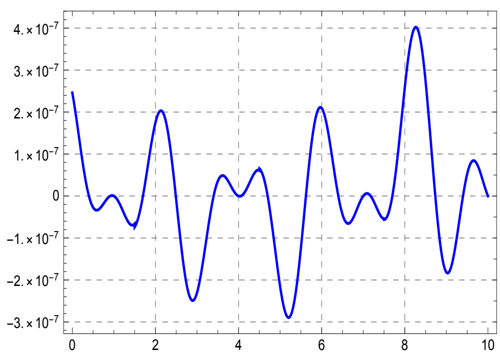

Figure 4: $f - \mathcal{G}_{1,3,7}[f]$.

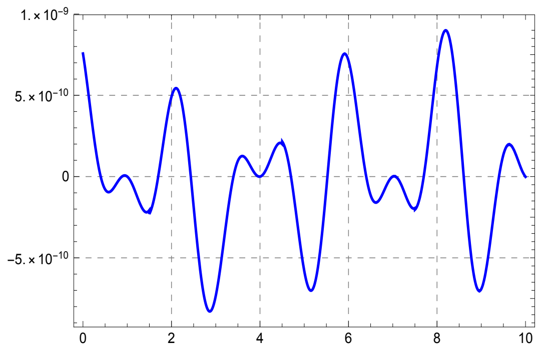

Figure 5: $f - \mathcal{G}_{1,3,9}[f]$.

Example 6.3. *Let $f(z) = \cosh(z)$, $z \in \mathbb{C}$. This function satisfies the inequality $|f(x)| \le e^{|x|}$ for all $x \in \mathbb{R}$. Here we apply Corollary 4.6 with $\Omega = 0$, $\kappa = 1$, $M = 1$, $J = 2$, $h = 1$ and $x_{[2]} = (0.3, 0.7)$. The function f has an exponential growth on $\mathbb{R}$ and consequently, the absolute errors increase with k. For this reason, we use the relative errors rather than the absolute errors. In Table 3, we summarize the numerical results and the graphs of the relative errors $1 - \mathcal{G}_{h,J,N}[f](x)/f(x)$ are given in Figs. 6 and 7 with $N = 5, 7$. Let R_N be the relative bound associated with the real-valued bound in (36), i.e.,*

$$R_N(x) := 2\, e^{\kappa(|x|+4h)} \left| \prod_{k=1}^{2} \sin\left(\frac{\pi}{2h}(x - x_k)\right) \right| \beta_{N,2}(0) \, \frac{e^{-(\alpha-\kappa h)2N}}{\sqrt{2\pi\alpha N}} / f(x),$$

where $\beta_{N,2}$ is given in (28).

Table 3: Approximation of the function at $x_{k,h}$

k	Relative error for $\mathcal{G}_{\frac{1}{2},2,N}$ with $N = 6, 8$ and $x_{[2]} = (0.3, 0.7)$			
	$\lvert(1 - \mathcal{G}_{\frac{1}{2},2,6}[f]/f)(x_{k,\frac{1}{2}})\rvert$	$R_6(x_{k,\frac{1}{2}})$	$\lvert(1 - \mathcal{G}_{\frac{1}{2},2,8}[f]/f)(x_{k,\frac{1}{2}})\rvert$	$R_8(x_{k,\frac{1}{2}})$
2	4.42618×10^{-9}	2.04979×10^{-5}	6.51457×10^{-11}	2.75026×10^{-7}
4	1.68149×10^{-9}	2.14988×10^{-5}	2.25989×10^{-12}	2.88456×10^{-7}
6	1.79893×10^{-9}	2.15181×10^{-5}	1.05082×10^{-12}	2.88714×10^{-7}
8	1.80108×10^{-9}	2.15184×10^{-5}	1.02831×10^{-12}	2.88719×10^{-7}
10	1.80112×10^{-9}	2.15184×10^{-5}	1.02674×10^{-12}	2.88719×10^{-7}

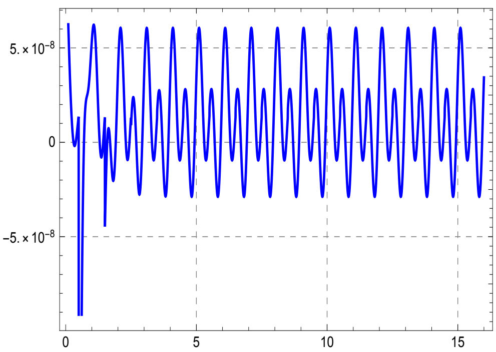

Figure 6: $1 - \mathcal{G}_{\frac{1}{2},2,5}[f]/f$.

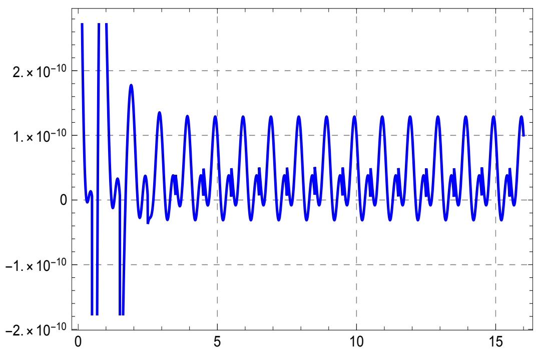

Figure 7: $1 - \mathcal{G}_{\frac{1}{2},2,7}[f]/f$.

Example 6.4. *Consider the analytic function $f(z) = \frac{1}{z^2+4}$ on the strip $\mathcal{D}_2$. For this function, we apply Theorem 5.1 with $J = 2$, $d = 2$ and $x_{[2]} = (0.1, 0.2)$. We denote by B_N the real-valued bound in (38), i.e.*

$$B_N(x) := 4\sqrt{2} \left| \prod_{k=1}^{2} \sin\left(\frac{\pi N}{4}(x - x_k)\right) \right| \gamma_N(0) \, \frac{e^{-\pi N}}{\pi\sqrt{N}},$$

where γ_N is defined in (39). The results are summarized in Table 4 and Figs. 8, 9.

Table 4: Approximation of the function at $x_{k,h}$

k	Absolute error and bound for $\mathcal{G}_{h,2,N}$ with $N = 4,6$ and $x_{[2]} = (0.1, 0.2)$			
	$\lvert(1-\mathcal{G}_{\frac{1}{4},2,4}[f]/f)(x_{k,\frac{1}{4}})\rvert$	$B_4(x_{k,\frac{1}{4}})$	$\lvert(1-\mathcal{G}_{\frac{1}{6},2,6}[f]/f)(x_{k,\frac{1}{6}})\rvert$	$R_6(x_{k,\frac{1}{6}})$
1	1.43393×10^{-6}	4.20379×10^{-6}	2.59179×10^{-9}	4.41259×10^{-9}
2	1.14083×10^{-6}	4.72118×10^{-6}	1.21058×10^{-9}	4.65604×10^{-9}
3	5.92563×10^{-7}	5.75595×10^{-6}	3.74721×10^{-10}	5.14295×10^{-9}
4	4.38022×10^{-7}	7.30812×10^{-6}	2.10412×10^{-10}	5.87331×10^{-9}
5	3.73748×10^{-7}	9.37768×10^{-6}	1.61743×10^{-10}	6.84712×10^{-9}

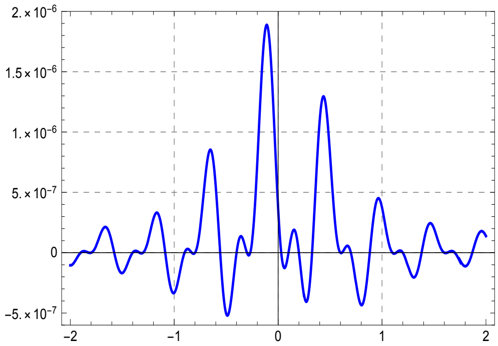

Figure 8: $f - \mathcal{G}_{\frac{1}{4},2,4}[f]$.

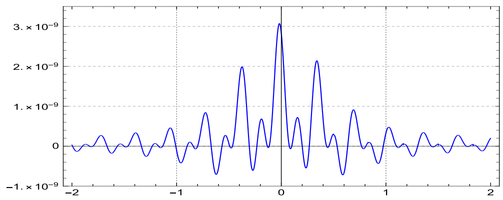

Figure 9: $f - \mathcal{G}_{\frac{1}{6},2,6}[f]$.

7. Conclusions

There are many studies that improve the convergence rate of the various types of uniform sampling using Gauss regularized factor. Despite the interest of the mathematicians and engineers in the periodic nonuniform sampling, the convergence rate of this type of sampling did not accelerate using Gauss regularized except Wang's et al. paper, cf. [18]. They established the periodic nonuniform version of sinc-Gauss sampling formula for Bernstein's functions based on Fourier-analytic approach. In this paper, we investigate a modification and extension for Wang et al.'s sampling formula based on complex-analytic approach for wider classes of functions, the class of entire functions including unbounded functions on $\mathbb{R}$ and the class of analytic functions in an infinite horizontal strip. This technique gives us slightly better convergence rate, it is of the order $e^{-\alpha NJ}/\sqrt{N}$, while in [18] it was of the order $\sqrt{N}e^{-\alpha NJ}$, where $\alpha > 0$ and N, J are positive integers. The numerical experiments show a quite good agreement with the theoretical analysis. The complex-analytic approach has been used successfully for various types of uniform sampling, one-dimensional and two-dimensional sampling, Hermite sampling and generalized sampling involves derivatives. We expect that this technique will succeed to speed up various types of nonuniform sampling series, like sampling expansion for linear canonical transform and its various generalized. In the future work, we use this technique to speed up the generalization of the nonuniform sampling associated with fractional Fourier transform. This type of sampling has attracted considerable attention for both mathematics and engineering although its convergence rate is slow.

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