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Generalized growth of meromorphic functions and rational approximation

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Abstract. For an arbitrary compact set $E \subset \mathbb{C}$ we consider the Newton-Padé approximant and rational approximation error of meromorphic function f and relate these to the generalized order and generalized type of f. Our results generalize the various results of K. Reczek ([20],[21]) and Winiarski [28].

1. Introduction

Let f be a function holomorphic in a neighborhood of infinity except the point $z = \infty$. Following R.P. Boas [6], we define the order of f at infinity as

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}, 0 \le \rho(f) \le \infty.$$

If $0 < \rho(f) < \infty$, the type of f at infinity is defined by

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho}},$$

where $M(r,f) = \|f\|_{C(0,r)} = \sup\{|f(z)| : |z| = r\}$.

If $\rho(f) = 0$ or ∞, the above definition of type is not valid. To refine it Juneja et al. ([15],[16]) have introduced the concept of index-pair $(p,q), p \ge q \ge 1$ and studied the (p,q)-order and (p,q)-type in terms of coefficients occurring in the Maclaurin series expansion of f.

Let E be a compact subset of the complex plane $\mathbb{C}$ of positive logarithmic capacity, and f a complex function defined and bounded on E. For $k \in \mathbb{N}$, putting

$$E_n(E,f) = \|f - T_n\|_E,$$

where the norm $\|.\|_E$ is the maximum norm on E and T_n is the n^{th} Chebyshev polynomial of the best approximation to f on E. It is well known that the rate at which the sequence $\{[E_n(E,f)]^{\frac{1}{n}}\}_{n=0}^{\infty} \to 0$ depends on the growth parameters order and type of the entire function.

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Bernstein [5] and Varga [27] had given the characterization of growth parameters of an entire function f in terms of the sequence of polynomial approximation errors taken over $E = [-1, 1]$. Several authors like Batyrev [4], Winiarski [28], Ibragimov and Shikhaliev [14], Giroux [9], Kasana and Kumar [17], Kumar [18] and Ali et al. [2] and others considered the approximation on compact set $E \subset \mathbb{C}$. For related work on different approximation methods, we refer the readers to ([1],[10],[11]).

Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers and f is a function holomorphic in a neighborhood of the set $\{z_n : 1 \leq n < \infty\}$. Consider the Newton development of f with respect to the sequence (z_n):

$$f(z) = \sum_{n=0}^{\infty} C_n w_n(z).$$

Winiarski [28] had obtained the classical order and type in terms of Newton coefficients C_n. Reczek [21] generalized the results of Winiarski [28] for the functions $f \in M_m(\mathbb{C})$ (the class of meromorphic functions whose poles are not greater than m) by replacing the Newton coefficients C_n by the coefficients of the Newton-Padé approximants. Harfaoui et al. [12] studied (p, q)-growth of meromorphic functions in terms of the Newton-Padé approximants and rational approximation errors. In this paper we will study the growth properties of functions $f \in M_m(\mathbb{C})$ and generalize various results of Winiarski [28], Reczek ([20],[21]).

2. Generalized Growth

Let $\xi : [a, \infty) \to \mathbb{R}$ for some $a \geq 0$, such that $\xi(x)$ is positive, strictly increasing, differentiable and tends to ∞ as $x \to \infty$. Then ξ is said to belong to the class L^0 if for every real valued function $\phi(x)$ such that $\phi(x) \to 0$ as $x \to \infty, \xi$ satisfies

$$\lim_{x\to\infty} \frac{\xi[(1 + \phi(x))x]}{\xi(x)} = 1$$

and belongs to the class Λ if for all $c, 0 < c < \infty$, we have the stronger condition

$$\lim_{x\to\infty} \frac{\xi(cx)}{\xi(x)} = 1.$$

Using the generalized functions of the class L^0 and Λ, Seremeta [23], obtained the following characterizations for the entire function $f(z)$:

Theorem A. Let $\alpha(t) \in \Lambda, \beta(t) \in L^0$. Set $F(t, c) = \beta^{-1}[c\alpha(t)]$. If $\frac{dF(t,c)}{d(\log t)} = O(1)$, as $t \to \infty$ for all $c, 0 < c < \infty$, then for the entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$,

$$\rho = \limsup_{r\to\infty} \frac{\alpha(\log M(r, f))}{\beta(\log r)} = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta(\log |c_n|^{-\frac{1}{n}})}. \tag{2.1}$$

Theorem B. Let $\alpha(t) \in L^0, \beta(t) \in L^0, \gamma(t) \in L^0$. Let ρ $(0 < \rho < \infty)$ be a fixed number. Set $F(t, \sigma, \rho) = \gamma^{-1}\{[\beta^{-1}(\sigma\alpha(t))]^{\frac{1}{\rho}}\}$. Suppose that for all $\sigma, 0 < \sigma < \infty$, F satisfies:
(a). If $\gamma(t) \in \Lambda$ and $\alpha(t) \in \Lambda$, then $\frac{dF(t,\sigma,\rho)}{d(\log t)} = O(1)$ as $t \to \infty$,
(b). If $\gamma(t) \in L^0 - \Lambda$ or $\alpha(t) \in L^0 - \Lambda$, then $\lim_{t\to\infty} \frac{d \log F(t,\sigma,\rho)}{d(\log t)} = \frac{1}{\rho}$.

Then we have

$$\sigma = \limsup_{r\to\infty} \frac{\alpha(\log M(r, f))}{\beta[(\gamma(r))^{\rho}]} = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta[\gamma(e^{\frac{1}{\rho}} |c_n|^{-\frac{1}{n}})]^{\rho}}.$$

3. Newton-Padé Approximants

Padé approximants are rational functions of the type (m, n) that interpolate a function element at a given point with order $m + n + 1$. These approximants were introduced for the exponential function by Hermite [13]. Padé approximants have been an effective device in analytic number theory ([13],[19],[24],[25]) and became an important tool in physical modelling and numerical analysis ([3],[7]). These also provide an important link between theory of rational approximation and the field of orthogonal polynomials [26].

Padé approximant of the type (m, n) is the best rational approximant to the functions at one point. With denominators of fixed degree Padé approximant converges uniformly to the approximated function in the disc of meromorphy and the degree of denominators matches the number of poles of the function [8].

We denote by $R_{n,m}$ the set of all rational functions, whose numerators and denominators are polynomials of degrees not exceeding n and m respectively. Let the function $f_{n,m}$ satisfy the following conditions:
(a) $f_{n,m} \in R_{n,m}$;
(b) $\frac{f-f_{n,m}}{w_{n+m+1}}$ is holomorphic at each point z_i for $1 \leq i < n + m + 1$,
where $w_n(z) = \Pi_{j=1}^{n}(z - \eta_{n_j}), n = 1, 2, \ldots.$ $\eta^n = (\eta_{n_0}, \eta_{n_1}, \ldots, \eta_{n_n})$ is the n^{th} extremal points system of compact set E of complex plane $\mathbb{C}$ (see [28], pp. 260).

For each couple (n, m) there exists at most one function satisfying the above conditions. It is called the (n, m)-th Newton-Padé approximants of the function f with respect to the sequence $\{z_n\}_{n=1}^{\infty}$. We shall consider the sequences of Newton-Padé approximants $(f_{n,m})$ with fixed m and $n \to \infty$. We denote

$$f_n = f_{n,m} = \frac{p_n}{q_n}, \tag{3.1}$$

where

$$p_n(z) = \sum_{i=0}^{n} p_{n_i} z^i \tag{3.2}$$

and

$$q_n(z) = (z - z_{n,1}).\ldots.(z - z_{n,m_n}), m_n \leq m, n \geq n_0, \tag{3.3}$$

where $z_{n,1}, \ldots, z_{n,m_n}$ are poles of the approximant f_n. Then the polynomials p_n and q_n have no common divisors of degree higher than zero. Also, assume that

$$|z_{n,1}| \leq \cdots \leq |z_{n,m_n}|. \tag{3.4}$$

Now we have the following lemma:

Lemma 3.1. Let $\{z_n\}_{n=1}^{\infty}$ be a bounded sequence of complex numbers and let f be a function meromorphic in $\mathbb{C}$, holomorphic in a neighborhood of the set $z_n : 1 \leq n < \infty$. Suppose that f has exactly m poles in $\mathbb{C}$, called with their multiplicities. Then
(1). for almost every n there exists the approximant f_n;
(2). the poles of f_n tends to the poles of f when $n \to \infty$;
(3). $\limsup_{n\to\infty} f_n = f(z)$ in $\mathbb{C}$, except for the poles of f;
(4). f can be extended to a function of the class $M_m(\mathbb{C})$.

The above lemma is only a slight modification of the Saff theorem [22], so we omit the proof here.

The aim of the present paper is to obtain the coefficient characterization of generalized order and generalized type of functions $f \in M_m(\mathbb{C})$ in terms of coefficient of the development p_{nn} of the $(n, n)^{th}$ Newton-Padé approximants and the best rational approximation error on a compact subset of $\mathbb{C}$ in supnorm.

Theorem 3.1. Let $f \in M_m(\mathbb{C})$ and all the conditions of Lemma 3.1 satisfied with notations (3.1)-(3.4). Then f has generalized order ρ if, and only if

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta[\log |p_{nn}|^{-\frac{1}{n}}]} = \rho. \tag{3.5}$$

Proof. Let $f = \frac{\varphi}{Q}$, where φ is an entire function and Q is a polynomial of the form $Q(z) = (t - \xi_1) \ldots . (t - \xi_k), k$ is the number of poles of f, then f can be extended to a function of the class $M_m(\mathbb{C})$. The order and type of φ are equal to the order and type of f. Using the Hermite formula from (3.1), we get

$$p_{nn} = \frac{1}{2\pi i} \int_{C(0,r)} \frac{\varphi(z) q_n(z)}{w_{m+n+1}(z)} dz \tag{3.6}$$

for $r > s = \sup\{z_k : k = 1, 2, 3, \ldots\}$. We obtain the estimate

$$|p_{nn}| \leq \frac{r 2^m (r^m + |z_{n,m}|^m) M(r, \varphi)}{\min |w_{m+n+1}(z)|}, |z| = r, \tag{3.7}$$

where r is sufficiently large and $M(r, \varphi) = \|\varphi\|_{C(0,r)} = \sup\{|\varphi(z)| : |z| = r\}$. Using the definition (2.1) of generalized order ρ, for any given $\varepsilon > 0$ and $r > r_0(\varepsilon)$ we have

$$\alpha[M(r, f)] \leq \beta(\log r)(\rho + \varepsilon). \tag{3.8}$$

From (3.7) and (3.8) we get

$$|p_{nn}| \leq \frac{r 2^m (r^m + |z_{n,m}|^m) M(r, \varphi) \exp[\alpha^{-1}\{\overline{\rho} \beta(\log r)\}]}{\min |w_{m+n+1}(z)|}, \overline{\rho} = \rho + \varepsilon,$$

where $|w_{n+m+1}(z)| \geq (r - s)^{n+m+1}$.
Above inequality holds for all $r > r_0(\varepsilon)$, we can choose

$$r = r(n) = \exp[\beta^{-1}\{\frac{\alpha(n)}{\rho}\}] = \exp[F(n, \frac{1}{\overline{\rho}})],$$

substituting this value of r in the last inequality above, we obtain

$$\begin{aligned} |p_{nn}| &\leq \exp[-nF(n, \frac{1}{\overline{\rho}})] \exp[\alpha^{-1}\{\overline{\rho} \frac{\alpha(n)}{\overline{\rho}}\}] + O(1) \\ &\leq \exp[-n\{F(n, \frac{1}{\overline{\rho}}) - 1\}], \ \ as \ \ F(n, \frac{1}{\overline{\rho}}) \to \infty \ \ as \ \ n \to \infty. \end{aligned}$$

It gives

$$\log(|p_{nn}|)^{-\frac{1}{n}} > F(n, \frac{1}{\overline{\rho}}) - 1 = \beta^{-1}\{\overline{\rho} \frac{\alpha(n)}{\overline{\rho}}\}\{1 - (F(n, \frac{1}{\overline{\rho}}))^{-1}\}$$

or

$$\beta[\log(|p_{nn}|)^{-\frac{1}{n}}]\{1 - (F(n, \frac{1}{\overline{\rho}}))^{-1}\}^{-1} > \frac{\alpha(n)}{\overline{\rho}}.$$

Since $\beta \in L^0$ and $(F(n, \frac{1}{\overline{\rho}}))^{-1} \to 0$ as $n \to \infty$, we get

$$\limsup_{n \to \infty} \frac{\alpha(n)}{\beta[\log(|p_{nn}|)^{-\frac{1}{n}}]} \leq \overline{\rho} = \rho + \varepsilon.$$

Since ε was arbitrary, it gives

$$\limsup_{n \to \infty} \frac{\alpha(n)}{\beta[\log(|p_{nn}|)^{-\frac{1}{n}}]} \leq \rho. \tag{3.9}$$

In order to prove the reverse inequality in (3.9), let us put

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta[\log(|p_{nn}|)^{-\frac{1}{n}}]} = \rho'.$$

Assume that $0 \le \rho' < \infty$. Then for a given $\varepsilon > 0$ and

$$|p_{nn}| < \exp[-nF(n, \frac{1}{\rho' + \varepsilon})]. \tag{3.10}$$

Let us define the set $D_\theta, \theta \in (0,1)$ as [18]:

$$D_\theta = \cup_{n=n_0}^{\infty} \cup_{i=1}^{m_n} B(z_{n_i}, \theta^n),$$

where $B(a, r) = \{z : |z - a| < r\}$.
Since $\lim_{n\to\infty} |p_{nn}|^{\frac{1}{n}} = 0$. Hence the sequence $\{f_n(z)\}_{n=n_0}^{\infty}$ converges if only $z \in \mathbb{C} \setminus D_\theta$. Let $z \in \mathbb{C} \setminus D_\theta$, assume that there exists a sequence $\{n_l\}$ and a neighborhood U of the point z such that for every l the function f_{n_l} has no poles in U. Then we have $\lim_{n\to\infty} f_n(z) = \lim_{l\to\infty} f_{n_l}(z) = f(z)$. This implies that $\lim_{n\to\infty} f_n(z) = f(z)$ in $\mathbb{C} \setminus D_\theta$ except for at most m points. So we can choose a number R_0 such that for every point

$$z \in (\mathbb{C} \setminus D_\theta) \setminus B(0, R_0), \lim_{n\to\infty} f_n(z) = f(z). \tag{3.11}$$

Let R_0 be sufficiently large such that $M(R, f)$ is an increasing function for $R > R_0$. Choose a number $R > s$. Then there exists $R_\theta, R \le R_\theta < R + d_\theta, d_\theta = \frac{2m}{1-\theta}$, such that the set D_θ does not intersect the circle $C(0, R_\theta)$. The sequence $\{f_n(z)\}_{n=n_0}^{\infty}$ is uniformly convergent on $C(0, R_\theta)$. Then

$$M(R, f) \le M(R_\theta, f) \le \|f_{n_0}\|_{C(0,R_\theta)} + \sum_{n=n_0+1}^{\infty} \|f_n - f_{n-1}\| C(0, R_\theta)$$

or

$$M(R, f) \le K_1(R_\theta)^n + \sum_{n=n_0+1}^{\infty} |p_{nn}| \theta^{-2mn} (R_\theta + s)^{m+n} \tag{3.12}$$

if R is large enough, then $(R_\theta + s) \le \theta^{-m} R$. Now using (3.10) in (3.12) for $n_1 \ge n_0$ we obtain

$$M(R, f) \le K_2 R^{n_1} + (\theta^{-m} R)^m \sum_{n=n_1}^{\infty} \exp[-nF(n, \frac{1}{\rho' + \varepsilon})](\theta^{-3m} R)^n, \tag{3.13}$$

where K_2 depends only on θ.
We now consider the function

$$h(z) = \sum_{n=n_1}^{\infty} \theta^{-3mn} z^n \exp[-nF(n, \frac{1}{\rho' + \varepsilon})].$$

From (3.13) we can conclude that the generalized order of $h(z) \ge$ the generalized order of $f(z)$. If ρ_1 denotes the generalized order of $h(z)$ then by (2.1) we have

$$\rho_1 = \limsup_{n\to\infty} \frac{\alpha(n)}{\beta[\log(\theta^{-3mn} \exp[-nF(n, \frac{1}{\rho'+\varepsilon})])^{-\frac{1}{n}}]},$$

since

$$-\frac{1}{n} \log(\theta^{-3m}) - \frac{1}{n} \log[\exp(nF(n, \frac{1}{\rho' + \varepsilon}))] = F(n, \frac{1}{\rho' + \varepsilon}) + O(1) \simeq \beta^{-1}[\frac{\alpha(n)}{\rho' + \varepsilon}].$$

Since $\alpha(x) \in \Delta$, we finally obtain

$$\rho_1 = \limsup_{n\to\infty} \frac{(\rho' + \varepsilon)\alpha(n)}{\alpha(n)} = \rho' + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we get $\rho' = \rho_1 \geq \rho$. Hence the proof is completed.

Theorem 3.2. Let $\alpha, \beta, \gamma \in L^0$ and $0 < \rho < \infty$. Denote by $F(x, \sigma, \rho) = \gamma^{-1}\{[\beta^{-1}(\sigma\alpha(x))]^{\frac{1}{\rho}}\}$. For $0 < \sigma < \infty$, suppose that the function F satisfies the conditions:
(i) If $\gamma(x) \in \Delta$ and $\alpha(x) \in \Delta$, then $\frac{dF(x,\sigma,\rho)}{d\log x} = O(1)$ as $x \to \infty$;
(ii) If $\gamma(x) \in L^0 - \Delta$ or $\alpha(x) \in L^0 - \Delta$, then $\lim_{x\to\infty} \frac{d\log F(x,\sigma,\rho)}{d\log x} = \frac{1}{\rho}$.
Then the entire function f is of generalized type σ if, and only if

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{[\gamma(e^{\frac{1}{\rho}}(|p_{nn}|)^{-\frac{1}{n}})]^{\rho}\}} = \sigma.$$

Proof. Let $\sigma < \infty$. Using the definition of generalized type for arbitrary $\varepsilon > 0$ and $r > r'(\varepsilon)$, we have

$$M(r, f) < \exp[\alpha^{-1}\{(\sigma + \varepsilon)\beta((\gamma(r))^{\rho})\}]. \tag{3.14}$$

Taking into account (3.7) and (3.14) we obtain

$$|p_{nn}| \leq \frac{r2^m(r^m + |z_{n,m}|^m)\exp[\alpha^{-1}\{(\sigma + \varepsilon)\beta((\gamma(r))^{\rho})\}]}{\min|w_{m+n+1}(z)|}. \tag{3.15}$$

Now choose $r = r(n) = F(\frac{n}{\rho}, \frac{1}{\sigma+\varepsilon}, \rho)$, then for all sufficiently large values of n, we have

$$\exp[\alpha^{-1}\{(\sigma + \varepsilon)\beta((\gamma(r))^{\rho})\}] = \exp[\alpha^{-1}\{\alpha(\frac{n}{\rho})\}].$$

From (3.15) we have

$$|p_{nn}| < \exp(\frac{n}{\rho})\{F(\frac{n}{\rho}, \frac{1}{\sigma + \varepsilon}, \rho)\}^{-n} + O(1)$$

or

$$F(\frac{n}{\rho}, \frac{1}{\sigma + \varepsilon}, \rho) < |p_{nn}|^{-\frac{1}{n}} e^{\frac{1}{\rho}}$$

or

$$\frac{1}{\sigma + \varepsilon}\alpha(\frac{n}{\rho}) < \beta\{[\gamma(e^{\frac{1}{\rho}}(|p_{nn}|)^{-\frac{1}{n}})]^{\rho}\}.$$

Since $\varepsilon > 0$ is arbitrary, proceeding to limits we get

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{[\gamma(e^{\frac{1}{\rho}}(|p_{nn}|)^{-\frac{1}{n}})]^{\rho}\}} \leq \sigma. \tag{3.16}$$

Conversely, suppose that

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{[\gamma(e^{\frac{1}{\rho}}(|p_{nn}|)^{-\frac{1}{n}})]^{\rho}\}} \leq \sigma_1.$$

Then for a given $\varepsilon > 0$ and all $n > n_1$,

$$|p_{nn}| < \exp(\frac{n}{\rho})\{F(\frac{n}{\rho}, \frac{1}{\sigma_1 + \varepsilon}, \rho)\}^{-n}. \tag{3.17}$$

Using (3.12) with (3.17) we get

$$M(R,f) \le K_2 R^{n_1} + (\theta^{-m}R)^m) \sum_{n=n_1+1}^{\infty} \exp(\frac{n}{\rho})\{F(\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho)\}^{-n}(\theta^{-3m}R)^n. \tag{3.18}$$

Now we consider the function $H(z)$ defined by

$$H(z) = \sum_{n=n_1+1}^{\infty} \exp(\frac{n}{\rho})\{F(\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho)\}^{-n}(\theta^{-3mn})z^n = \sum_{n=n_1+1}^{\infty} b_n z^n.$$

Since $F(t,\sigma,\rho) \to \infty$ as $t \to \infty$, we have

$$\lim_{n\to\infty}[\exp(\frac{n}{\rho})\{F(\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho)\}^{-n}(\theta^{-3mn})]^{\frac{1}{n}} = 0,$$

therefore, $H(z)$ represents an entire function. From (3.18) we see that $H(z)$ is an entire function of generalized type $\sigma' \ge \sigma$. Applying Theorem B to the entire function $H(z)$, we get

$$\sigma' = \limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{[\gamma(e^{\frac{1}{\rho}}(|b_n|)^{-\frac{1}{n}})]^{\rho}\}}. \tag{3.19}$$

Now

$$e^{\frac{1}{\rho}}|b_n|^{-\frac{1}{n}} = (1+0(1))F(\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho),$$

since $\gamma(n) \in L^0, \gamma(e^{\frac{1}{\rho}}|b_n|^{-\frac{1}{n}}) \simeq \gamma[F(\frac{n}{\rho}, \frac{1}{\sigma_1+\varepsilon}, \rho)]$. Using above estimates in (3.19) we get $\sigma' = \sigma + \varepsilon$, or $\sigma' \ge \sigma$. Hence for arbitrary $\varepsilon > 0, \sigma \le \sigma_1 + \varepsilon$ i.e., $\sigma \le \sigma_1 + \varepsilon$ i.e., $\sigma \le \sigma_1$. This completes the proof of the theorem.

4. Best Rational Approximation

Let E be a compact subset of the complex plane $\mathbb{C}$ such that $Cap(E) > 0$, and f a complex function defined and bounded on E. For $n \in \mathbb{N}$, the best rational approximation error is defined as

$$e_n(E,f) = \inf_{r_n} \|f - r_n\|_E,$$

where $r_n \in R_{n,m}$. It is known [20] that if f is entire function then

$$\lim_{n\to\infty}[e_n(E,f)]^{\frac{1}{n}} = 0.$$

We now prove the following theorems.

Theorem 4.1. Let $f \in M_m(\mathbb{C})$ and all the conditions of Lemma 3.1 satisfied with notations (3.1)-(3.4). Then f has generalized order ρ if, and only if

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta[\log\{e_n(E,f)\}^{-\frac{1}{n}}]} = \rho. \tag{4.1}$$

Proof. Taking into account the Poisson- Jensen formula, we have

$$\begin{aligned} \log|\frac{(fQ_n - P_n)(z)}{z^{2n+1}}| &= \frac{1}{2\pi}\int_0^{2\pi} \log|\frac{(fQ_n - P_n)(Re^{i\theta})}{(Re^{i\theta})^{2n+1}}|\frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\theta - \phi)}d\theta \\ &+ \sum_{|b_v|<R} \log|\frac{R^2 - \overline{b_v}z}{R(z - b_v)}| - \sum_{|a_v|<R} \log|\frac{R^2 - \overline{a_v}z}{R(z - a_v)}|, \end{aligned} \tag{4.2}$$

where $z = re^{i\phi}$ and a_ν and b_ν are the zeros and poles of $fQ_n - P_n$, respectively. Let $Q_n(z) = \Pi(z - z_\nu)$. Since P_n is the n^{th} Taylor polynomial to fQ_n, and hence majorized by a constant times $|Q_n|$ near the origin. We have on $|w| = R : |P_n(w) \leq \Pi(1 + |z_\nu|)|(RK^*)^n$ by the Walsh-Bernstein lemma. Following on the lines of Proof of Theorem 4.3 of [12] for $r < R$ and sufficiently large R we have

$$\begin{aligned}\log|\frac{(fQ_n - P_n)(z)}{z^{2n+1}}| \leq \log(\frac{r}{R})^{2n+1} + n\log K^* + \log R^n \Pi(1 + |z_\nu|) + K^{**}R^{\frac{1}{\alpha}} \\ + \sum_{|b_\nu|\leq 2r} \log 2R - \sum_{|b_\nu|\leq 2r} \log \varepsilon,\end{aligned} \tag{4.3}$$

where K^*, K^{**} are constants and $\alpha < \rho^{-1}$.
Let us assume $R = R(n) = \exp[\beta^{-1}\{\frac{\alpha(n)}{\overline{\rho}}\}]$ for sufficiently small r, subtracting $\log|Q_n|$ from both sides we have

$$\log|(f - \frac{P_n}{Q_n})(z)| \leq \log \frac{r^{2n+1}}{[\exp[\beta^{-1}\{\frac{\alpha(n)}{\overline{\rho}}\}]]^{2n+1}} + n\log K^* - \log \delta^n$$

when $|Q_n(z)| \leq \delta^n$. From above inequality we get

$$-\frac{1}{n}\log(e_n(E, f)) \geq 2\log r - \frac{1}{n}\log r + (2 + \frac{1}{n})\beta^{-1}\{\frac{\alpha(n)}{\overline{\rho}}\},$$

or

$$\beta[\log(e_n(E, f))^{-\frac{1}{n}}] > \frac{\alpha(n)}{\overline{\rho}}.$$

Now proceeding to limits we get

$$\limsup_{n\to\infty} \frac{\alpha(n)}{\beta[\log\{e_n(E, f)\}^{-\frac{1}{n}}]} \leq \overline{\rho}.$$

Now assume that the opposite inequality is not true. Then we can choose a number $0 < \rho' < \rho$ such that

$$e_n(E, f) < \exp[-nF(n, \frac{1}{\rho'})]$$

for almost every n. Then

$$\|f_{n+1} - f_n\|_E < 2\exp[-nF(n, \frac{1}{\rho'})], n \geq N_0. \tag{4.4}$$

Let $\eta > 0$ be smaller than the distance from $z_{n,j}$ to E for each j. Set

$$D = \{z \in \mathbb{C} : |z - z_{n,j}| \geq \delta, 1 \leq j \leq m\}.$$

There exists an integer N_1 and a number $C > 0$ such that $|q_n(z)| \geq C^{-1}$ if $z \in D$ and $n \geq N_1$. Hence, we have

$$|f_n(z) - f_{n-1}(z)| \leq C^2|p_n(z)q_{n-1}(z) - p_{n-1}(z)q_n(z)|. \tag{4.5}$$

Since the sequence q_n is convergent, therefore we have

$$\|q_n\|_E \leq M \;\; for \;\; every \;\; n, \tag{4.6}$$

where M is a positive constant. Then from (4.4) and (4.6) we get

$$\|p_n q_{n-1} - p_{n-1}q_n\|_E \leq 2M^2 \exp[-nF(n, \frac{1}{\rho'})], n \geq N_0.$$

Let $\rho' < \rho'' < \rho$. Then there exists an integer N_2 such that

$$\|p_n q_{n-1} - p_{n-1} q_n\|_E \le \exp[-(n+m)F(n+m, \frac{1}{\rho''})], n \ge N_2. \tag{4.7}$$

Now from (4.5) we obtain

$$|f(z)| \le |f_{N_1}(z)| + C^2 \sum_{n=N_1+1}^{\infty} |p_n(z)q_{n-1}(z) - p_{n-1}(z)q_n(z)|. \tag{4.8}$$

for $z \in D$.
In view of inequalities (4.7) and (4.8) we conclude that the generalized order of f is smaller than ρ, which is a contradiction of the assumptions. Hence the proof is completed.

Theorem 4.2. If all conditions of Theorem 3.2 are satisfied. Then the entire function f is of generalized type σ if, and only if

$$\limsup_{n\to\infty} \frac{\alpha(\frac{n}{\rho})}{\beta\{[\gamma(e^{\frac{1}{\rho}}(e_n(E,f))^{-\frac{1}{n}})]^{\rho}\}} = \sigma.$$

Proof. Following the same reasoning as in the proof of Theorem 3.1 we can easily obtain the required proof.

Remark 4.1. Taking $\alpha(t) = \beta(t) = \log t$ in Theorems 3.1 and 4.1, we obtain the coefficient characterization for the classical order of entire function $f \in M_m(\mathbb{C})$ in terms of Newton-*padé* approximant and rational approximation error respectively.

Remark 4.2. Taking $\alpha(t) = \log^{[p-2]} t, \gamma(t) = \log^{[q-1]} t$ and $\beta(t) = t, p > q > 1$ in Theorems 3.2 and 4.2 where $\log^{[n]} t = \log\log \ldots \log(t)(n - times)$. Then we obtain coefficient characterization for the (p, q)-type of entire function $f \in M_m(\mathbb{C})$ in terms of Newton-*padè* approximant and rational approximation error respectively.

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