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Shapiro's uncertainty principles and scalogram associated with the Riemann-Liouville wavelet transform

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Abstract. The Riemann-Liouville operator has been extensively investigated and has witnessed a remarkable development in numerous fields of harmonic analysis over a couple of decades. The aim of this article is to explore two more aspects of the time-frequency analysis associated with the Riemann-Liouville wavelet transform, including the Shapiro uncertainty principle and the scalogram.

1. Introduction

The spherical mean operators constitute a vital class of operators in harmonic analysis in the sense that all the harmonic functions are characterized by the fact that they coincide with their spherical mean values. These operators can also be viewed as the generalized Radon transform that is self dual in the context of Helgason's double fibration. In the classical work of John [13], the spherical means have been successfully applied to diverse problems in the theory of partial differential equations. Subsequently, they paved the way into the Fourier analysis with the celebrated theorem of Stein on spherical analogue of the Lebesgue differentiation theorem. A recent addition to the theory of spherical mean operators on $\mathbb{R}^2$ appeared with the work of Trimèche [23], wherein the author generalized the spherical mean operators on $\mathbb{R}^2$ by introducing the permutation operator which commutes with some partial differential operators. Besides, Trimèche also studied the harmonic analysis associated with this permutation operator, which is being widely employed in literature under the name Riemann-Liouville operator [3–6, 12, 15–18]. As of now, these operators have found numerous applications in image processing of synthetic aperture, radar data and acoustics [9, 11].

On the other hand, the wavelet transform is a multi-scale integral transform, which serves as one of the corner stones of non-stationary signal processing. It can be used in time-frequency analysis wherein the scale and frequency are inverse to each other. The wavelet transform decomposes a signal into components determined by the translations and dilations of a single function known as the mother wavelet. By applying these local decomposition filters, the wavelet transform has proved to be of substantial importance in capturing the local characteristics of non-stationary signals and has paved its way to a number of fields including signal and image processing, sampling theory, geophysics, astrophysics, quantum mechanics and so on [7, 8, 24]. Recently, Rachdi and Herch [20] introduced the notion of Riemann-Liouville

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wavelet transform by using the generalized scale-translation procedure and the singular partial differential operators.

As the harmonic analysis associated to the Riemann-Liouville operator has been extensively investigated and has witnessed a remarkable development, it is natural to study several aspects of the time-frequency analysis associated with the Riemann-Liouville wavelet transform. The aim of this article is to explore two subjects of the time-frequency analysis associated with the Riemann-Liouville wavelet transform, viz, the Shapiro uncertainty principle and the scalogram. It is worth mentioning that the scalogram plays a vital role in the applications of the wavelet transform to different aspects of signal processing. For example, Addison et al.[2] employed the Morlet wavelet scalograms to detected a previously unknown co-ordinated contractility behaviour of the atrium during ventricular fibrillation, a phenomenon which is not captured in a normal electrocardiogram. Besides, Sukiennik and Bialasiewicz [22] applied the scalogram to biomedical signals to detect their short-lived temporal interactions.

The remainder of this paper is arranged as follows: In §2, we present a gentle exposition regarding the Riemann-Liouville operator. In §3, we formulate both the quantitative Shapiro's dispersion uncertainty principle and umbrella theorem associated with the Riemann-Liouville wavelet transform. In §4, we study the eigenvalues and eigenfunctions of the time-frequency localization operator. Besides, we also study the scalogram associated with the Riemann-Liouville wavelet transform.

2. Preliminaries

The aim of this section is to present a healthy overview of the prerequisites circumscribing the Riemann-Liouville operators, Schatten-von Neumann classes, and the localization operators associated with the continuous wavelet transform. For a detailed perspective regarding the content of the section, we refer to [4, 20, 23, 25]. For the sake of distinction, we sub-divide the section into three sub-sections.

2.1. *Harmonic analysis associated with the Riemann-Liouville operator*

Prior to starting the formal aspects of this sub-section, we fix some notations as under:

- $C_*(\mathbb{R}^2)$ denotes the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable.
- $C_{*,c}(\mathbb{R}^2)$ denotes the subspace of $C_*(\mathbb{R}^2)$ formed by functions with compact support.
- $\mathcal{E}_*(\mathbb{R}^2)$ is the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable.
- $\mathcal{S}_*(\mathbb{R}^2)$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^2$, even with respect to the first variable.
- S^1 is the unit sphere in $\mathbb{R}^2$, $S^1 = \Big\{(\eta, \xi) \in \mathbb{R}^2 : \eta^2 + \xi^2 = 1\Big\}$.
- $\mathbb{R}^2_+ = \Big\{(r, x) \in \mathbb{R}^2 : \ r \geq 0\Big\}$.

Note that, for all $(\mu, \lambda) \in \mathbb{C}^2$, the system

$$\begin{cases} \Delta_1 u(r,x) &= -i\lambda u(r,x), \\ \Delta_2 u(r,x) &= -\mu^2 u(r,x) \\ u(0,0) &= 1, \quad \frac{\partial u}{\partial r}(0,x) = 0, \ \forall\, x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\mu,\lambda}$, given by [4, 23]

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha(r\sqrt{\mu^2 + \lambda^2})e^{-i\lambda x},$$

where Δ_1 and Δ_2 denote the singular partial differential operators, given by

$$\begin{aligned}\Delta_1 &= \frac{\partial}{\partial x},\\ \Delta_2 &= \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \qquad (r,x) \in (0,\infty)\times\mathbb{R}, \qquad \alpha \geqslant 0,\end{aligned}$$

and j_α is the normalized Bessel function defined as

$$\forall\, z \in \mathbb{C}, \quad j_\alpha(z) = \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+1+\alpha)}(z/2)^{2k}.$$

Definition 2.1. *For any $(r,x) \in \mathbb{R}^2_+$, the Riemann-Liouville operator on $C_*(\mathbb{R}^2)$ is defined by:*

$$\mathcal{R}_\alpha f(r,x) = \begin{cases} \frac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds & \text{if} \quad \alpha > 0\\ \frac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)(1-t^2)^{-\frac{1}{2}}dt & \text{if} \quad \alpha = 0.\end{cases}$$

Remark 2.1. *(i) The function $\varphi_{\mu,\lambda}$, $(\mu,\lambda) \in \mathbb{C}^2$, can be expressed as*

$$\forall (r,x) \in \mathbb{R}^2_+, \quad \varphi_{\mu,\lambda}(r,x) = \mathcal{R}_\alpha(\cos(\mu.)e^{-i\lambda.})(r,x).$$

(ii) For all $\nu \in \mathbb{N}^2$, $(r,x) \in \mathbb{R}^2_+$ and $z = (\mu,\lambda) \in \mathbb{C}^2$, we have

$$|D^\nu_z \varphi_{\mu,\lambda}(r,x)| \leq \|(r,x)\|^{|\nu|}\exp(2\|(r,x)\|\,\|\mathrm{Im} z\|), \tag{1}$$

where

$$D^\nu_z = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1}\partial z_2^{\nu_2}} \quad \text{and} \quad |\nu| = \nu_1 + \nu_2.$$

In particular, for all $\nu \in \mathbb{N}^2$, $(r,x) \in \mathbb{R}^2_+$ and $z = (\mu,\lambda) \in \mathbb{C}^2$:

$$|\varphi_{\mu,\lambda}(r,x)| \leq 1. \tag{2}$$

Next, consider the set Γ defined as

$$\Gamma = \mathbb{R}^2 \cup \Big\{(it,x) : (t,x) \in \mathbb{R}^2, |t| \leq |x|\Big\}.$$

and let Γ_+ denotes the subset:

$$\Gamma_+ = \mathbb{R}^2_+ \cup \Big\{(it,x) : (t,x) \in \mathbb{R}^2, 0 \leq t \leq |x|\Big\},$$

then for all $(\mu,\lambda) \in \Gamma$, we have

$$\sup_{(r,x)\in\mathbb{R}^2} |\varphi_{\mu,\lambda}(r,x)| = 1.$$

In the following, we denote by

- $d\nu_\alpha(r,x)$ the measure defined on $\mathbb{R}^2_+$ by

$$d\nu_\alpha(r,x) = k_\alpha r^{2\alpha+1}dr \otimes dx,$$

with

$$k_\alpha = \frac{1}{2^\alpha\Gamma(\alpha+1)(2\pi)^{1/2}}.$$

- For $p \in [1,\infty]$, p' denotes as in all that follows, the conjugate exponent of p.
- $L^p(d\nu_\alpha), 1 \le p \le \infty$, the space of measurable functions on $\mathbb{R}^2_+$, satisfying

$$\begin{aligned} \|f\|_{L^p(d\nu_\alpha)} &= \left(\int_{\mathbb{R}^2_+} |f(r,x)|^p d\nu_\alpha(r,x)\right)^{1/p} < \infty, \quad 1 \le p < \infty, \\ \|f\|_{L^\infty(d\nu_\alpha)} &= \underset{(r,x)\in\mathbb{R}^2_+}{ess\ sup}\ |f(r,x)| < \infty, \quad p = \infty. \end{aligned}$$

- $\mathcal{B}_{\Gamma_+}$ the σ-algebra defined on Γ_+ by

$$\mathcal{B}_{\Gamma_+} = \left\{\theta^{-1}(B) : \quad B \in \mathcal{B}_{Bor}(\mathbb{R}^2_+)\right\},$$

where θ defined on the set Γ_+ by

$$\theta(\mu,\lambda) = (\sqrt{\mu^2+\lambda^2}, \lambda). \tag{3}$$

- $d\gamma_\alpha$ the measure defined on $\mathcal{B}_{\Gamma_+}$ by

$$\forall A \subset \mathcal{B}_{\Gamma_+}, \quad \gamma_\alpha(A) = \nu_\alpha(\theta(A)).$$

- $L^p(d\gamma_\alpha), 1 \le p \le \infty$, the space of measurable functions on Γ_+, satisfying

$$\begin{aligned} \|f\|_{L^p(d\gamma_\alpha)} &= \left(\int_{\Gamma_+} |f(\mu,\lambda)|^p d\gamma_\alpha(\mu,\lambda)\right)^{1/p} < \infty, \quad 1 \le p < \infty, \\ \|f\|_{L^\infty(d\gamma_\alpha)} &= \underset{(\mu,\lambda)\in\Gamma_+}{ess\ sup}\ |f(\mu,\lambda)| < \infty, \quad p = \infty. \end{aligned}$$

We have the following properties.

Proposition 2.1. *i) For every non-negative measurable function g on Γ_+, we have*

$$\int_{\Gamma_+} f(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = k_\alpha\Big[\int_{\mathbb{R}^2_+} f(\mu,\lambda)(\mu^2+\lambda^2)^\alpha \mu d\mu_\alpha d\lambda + \int_{\mathbb{R}}\int_0^{|\lambda|} f(i\mu,\lambda)(\lambda^2-\mu^2)^\alpha \mu d\mu_\alpha d\lambda\Big].$$

ii) For every non-negative measurable function f on $\mathbb{R}^2_+$ (resp. integrable on $\mathbb{R}^2_+$ with respect to the measure $d\nu_\alpha$), $f \circ \theta$ is a measurable non-negative function on Γ_+, (resp. integrable on Γ_+ with respect to the measure $d\gamma_\alpha$) and we have

$$\int_{\Gamma_+} f \circ \theta(\mu,\lambda)d\gamma_\alpha(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x)d\nu_\alpha(r,x). \tag{4}$$

Remark 2.2. *The eigenfunction $\varphi_{\mu,\lambda}$, satisfies the following product formula*

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi \varphi_{\mu,\lambda}\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta d\theta.$$

Following is the definition of the translation operator $\tau_{(r,x)}$ associated with the Riemann-Liouville operator.

Definition 2.2. *Let f be in $L^p(d\nu_\alpha)$, $p \in [1,\infty]$, for all $(r,x) \in \mathbb{R}^2_+$, we define the translation operator $\tau_{(r,x)}$ associated with the Riemann-Liouville operator by*

$$\tau_{(r,x)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta d\theta, \tag{5}$$

for all $(s,y) \in \mathbb{R}^2_+$.

Proposition 2.2. *For every $f \in L^p(d\nu_\alpha)$, $1 \leqslant p \leqslant \infty$ and $(r,x) \in \mathbb{R}^2_+$, the function $\tau_{(r,x)}(f)$ belongs to $L^p(d\nu_\alpha)$ and we have*

$$\left\|\tau_{(r,x)}(f)\right\|_{L^p(d\nu_\alpha)} \leqslant \left\|f\right\|_{L^p(d\nu_\alpha)}. \tag{6}$$

Definition 2.3. *The convolution product of $f, g \in L^1(d\nu_\alpha)$ is defined by*

$$f *_\alpha g(r,x) = \int_{\mathbb{R}^2_+} \tau_{(r,x)}(\check{f})(s,y)g(s,y)d\nu_\alpha(s,y), \quad \text{for all} \quad (r,x) \in \mathbb{R}^2_+, \tag{7}$$

with $\check{f}(s,y) = f(s,-y)$.

Proposition 2.3. *Let $1 \leq p,q,r \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If f is a function in $L^p(d\nu_\alpha)$ and g an element of $L^q(d\nu_\alpha)$, then $f *_\alpha g$ belongs to $L^r(d\nu_\alpha)$ and we have*

$$\left\|f *_\alpha g\right\|_{L^r(d\nu_\alpha)} \leq \left\|f\right\|_{L^p(d\nu_\alpha)} \left\|g\right\|_{L^q(d\nu_\alpha)}. \tag{8}$$

Next, we have the notion of generalized Fourier transform $\mathcal{F}_\alpha$ associated with the Riemann-Liouville operator $\mathcal{R}_\alpha$.

Definition 2.4. *The Fourier transform associated with the Riemann-Liouville operator is defined on $L^1(d\nu_\alpha)$ by*

$$\mathcal{F}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu_\alpha(r,x), \quad \forall\ (\mu,\lambda) \in \Gamma. \tag{9}$$

Below, we recall some fundamental properties of the generalized Fourier transform $\mathcal{F}_\alpha$.

(i) For all $f \in L^1(d\nu_\alpha)$,

$$\|\mathcal{F}_\alpha(f)\|_{L^\infty(d\gamma_\alpha)} \leq \|f\|_{L^1(d\nu_\alpha)}. \tag{10}$$

(ii) For every $f \in L^1(d\nu_\alpha)$, we have

$$\mathcal{F}_\alpha(f)(\mu,\lambda) = \widetilde{\mathcal{F}}_\alpha(f) \circ \theta(\mu,\lambda), \quad (\mu,\lambda) \in \Gamma,$$

where for every $(\mu,\lambda) \in \mathbb{R}^2$,

$$\widetilde{\mathcal{F}}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x)j_\alpha(r\mu)e^{-i\lambda x}d\nu_\alpha(r,x)$$

and θ is the function defined by the relation (3).

(iii) For $f \in L^1(d\nu_\alpha)$ such that $\mathcal{F}_\alpha(f) \in L^1(d\gamma_\alpha)$, we have the inversion formula for $\mathcal{F}_\alpha$: for almost every $(r,x) \in \mathbb{R}^2_+$,

$$f(r,x) = \int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma_\alpha(\mu,\lambda). \tag{11}$$

Theorem 2.1. i*) (Plancherel's formula for $\mathcal{F}_\alpha$). For every f in $\mathcal{S}_*(\mathbb{R}^2)$, we have*

$$\int_{\Gamma_+} |\mathcal{F}_\alpha(f)(\mu,\lambda)|^2 d\gamma_\alpha(\mu,\lambda) = \int_{\mathbb{R}^2_+} |f(r,x)|^2 d\nu_\alpha(r,x). \tag{12}$$

In particular, the generalized Fourier transform $\mathcal{F}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto $L^2(d\gamma_\alpha)$.

ii*) (Parseval's formula for $\mathcal{F}_\alpha$). For all f, g in $L^2(d\nu_\alpha)$ we have*

$$\int_{\Gamma_+} \mathcal{F}_\alpha(f)(\mu,\lambda)\overline{\mathcal{F}_\alpha(g)(\mu,\lambda)}d\gamma_\alpha(\mu,\lambda) = \int_{\mathbb{R}^2_+} f(r,x)\overline{g(r,x)}d\nu_\alpha(r,x). \tag{13}$$

2.2. Basic Riemann-Liouville wavelet theory

In this subsection, we shall recall some fundamental results on the Riemann-Liouville wavelet transforms due to Rachdi and Herch [20].

For $(a,b) \in (0,\infty)\times\mathbb{R}^*$, the dilation operator $D_{(a,b)}$ of any measurable function h on $\mathbb{R}^2_+$ is defined by

$$D_{(a,b)}(h)(r,x) := a^{\alpha+1}|b|^{\frac{1}{2}}h(ar,bx), \quad \forall\,(r,x)\in\mathbb{R}^2_+. \tag{14}$$

In the following proposition, we assemble some fundamental properties of the dilation operators.

Proposition 2.4. *(i) For all $(a,b),(c,d)\in(0,\infty)\times\mathbb{R}^*$, we have*

$$D_{(a,b)}\circ D_{(c,d)} = D_{ac,bd}. \tag{15}$$

(ii) Let $(a,b)\in(0,\infty)\times\mathbb{R}^*$. For all $h\in L^p(d\nu_\alpha)$, $p\in[1,\infty]$. The function $D_{(a,b)}h$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\|D_{(a,b)}h\|_{L^p(d\nu_\alpha)} = a^{(2\alpha+2)(\frac{1}{2}-\frac{1}{p})}|b|^{\frac{p-2}{2p}}\|h\|_{L^p(d\nu_\alpha)}. \tag{16}$$

In particular, $D_{(a,b)}$ is an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself whose the inverse operator is $D_{(\frac{1}{a},\frac{1}{b})}$. Moreover we have

$$\forall\,(\mu,\lambda)\in\mathbb{R}^2_+,\quad \widetilde{\mathcal{F}}_\alpha(D_{(a,b)}(h))(\mu,\lambda) = \frac{1}{a^{\alpha+1}|b|^{\frac{1}{2}}}\widetilde{\mathcal{F}}_\alpha(h)(\frac{\mu}{a},\frac{\lambda}{b}). \tag{17}$$

(iii) Let $(a,b)\in(0,\infty)\times\mathbb{R}^*$. For all h,g in $L^2(d\nu_\alpha)$, we have

$$\langle D_{(a,b)}(h),g\rangle_{L^2(d\nu_\alpha)} = \langle h, D_{(\frac{1}{a},\frac{1}{b})}(g)\rangle_{L^2(d\nu_\alpha)}. \tag{18}$$

(iv) Let $(a,b)\in(0,\infty)\times\mathbb{R}^*$ and $(r,x)\in\mathbb{R}^2_+$. We have

$$D_{(a,b)}\tau_{(r,x)} = \tau_{(\frac{r}{a},\frac{x}{b})}\,D_{(a,b)}. \tag{19}$$

Definition 2.5. *A generalized wavelet on $\mathbb{R}^2_+$ is a measurable function h on $\mathbb{R}^2_+$ satisfying for almost all (μ,λ) belongs to $(0,\infty)\times\mathbb{R}^*$, the condition*

$$0 < C_h := c_\alpha\int_0^\infty\int_{\mathbb{R}}\left|\widetilde{\mathcal{F}}_\alpha(h)(\frac{\mu}{a},\frac{\lambda}{b})\right|^2\frac{da}{a}\frac{db}{|b|} < \infty, \tag{20}$$

where $c_\alpha = \frac{1}{2^\alpha\Gamma(\alpha+1)(2\pi)^{\frac{1}{2}}}$.

For $(a,b)\in(0,\infty)\times\mathbb{R}^*$ and $h\in L^p(d\nu_\alpha)$, $p\in[1,\infty]$, consider the family $h_{a,b,r,x}$, $(r,x)\in\mathbb{R}^2_+$, of generalized wavelets on $\mathbb{R}^2_+$ in $L^p(d\nu_\alpha)$ defined by

$$h_{a,b,r,x}(s,y) := \tau_{(r,-x)}(D_{(a,b)}h)(s,y), \quad (s,y)\in\mathbb{R}^2_+, \tag{21}$$

where $\tau_{(r,-x)}$, $(r,x)\in\mathbb{R}^2_+$, are the generalized translation operators given by (5).

Remark 2.3. *Let h be in $L^2(d\nu_\alpha)$. We have*

$$\forall\,(a,b)\in(0,\infty)\times\mathbb{R}^*,\ \forall\,(r,x)\in\mathbb{R}^2_+,\quad \|h_{a,b,r,x}\|_{L^2(d\nu_\alpha)} \le \|h\|_{L^2(d\nu_\alpha)}. \tag{22}$$

Notation. We denote by
$L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $p \in [1,\infty]$, the space of measurable functions f on $\mathbb{R}^2_+ \times \mathbb{R}^2_+$ such that

$$\begin{aligned} \|f\|_{L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)} &:= \left(\int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} |f(a,b,r,x)|^p d\mu_\alpha(a,b,r,x)\right)^{1/p} < \infty, \quad 1 \le p < \infty, \\ \|f\|_{L^\infty_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)} &:= \operatorname*{ess\,sup}_{(a,b,r,x)\in \mathbb{R}^2_+ \times \mathbb{R}^2_+} |f(a,b,r,x)| < \infty, \end{aligned}$$

where the measure μ_α is defined by

$$d\mu_\alpha(a,b,r,x) = d\nu_\alpha(a,b) d\nu_\alpha(r,x), \quad \forall\,(a,b,r,x) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+.$$

Definition 2.6. *Let h be a generalized wavelet on $\mathbb{R}^2_+$ in $L^2(d\nu_\alpha)$. The generalized continuous wavelet transform Φ^α_h on $\mathbb{R}^2_+$ is defined for regular functions f on $\mathbb{R}^2_+$ by*

$$\Phi^\alpha_h(f)(a,b,r,x) = \int_{\mathbb{R}^2_+} f(s,y)\overline{h_{a,b,r,x}(s,y)} d\nu_\alpha(s,y), \quad \forall\,(a,b) \in (0,\infty) \times \mathbb{R}^*,\ (r,x) \in \mathbb{R}^2_+. \tag{23}$$

Definition 2.6 can be recast as

$$\Phi^\alpha_h(f)(a,b,r,x) = f *_\alpha \overline{D_{(a,b)}h}(r,x), \tag{24}$$

where $*_\alpha$ is the generalized convolution product given by (7).

We note that the adjoint of Φ^α_h is $(\Phi^\alpha_h)^* : L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) \to L^2(d\nu_\alpha)$ and is defined as

$$(\Phi^\alpha_h)^*(F)(s,y) = \frac{1}{C_h} \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} F(a,b;r,x) h_{a,b;r,x}(s,y) d\mu_\alpha(a,b;r,x),\ (s,y) \in \mathbb{R}^2_+. \tag{25}$$

Theorem 2.2. *(Plancherel's formula for Φ^α_h). Let h be a generalized wavelet on $\mathbb{R}^2_+$ in $L^2(d\nu_\alpha)$. For all f in $L^2(d\nu_\alpha)$ we have*

$$\int_{\mathbb{R}^2_+} |f(r,x)|^2 d\nu_\alpha(r,x) = \frac{1}{C_h} \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} |\Phi^\alpha_h(f)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x). \tag{26}$$

Corollary 2.1. *(Parseval's formula for Φ^α_h). Let h be a generalized wavelet on $\mathbb{R}^2_+$ in $L^2(d\nu_\alpha)$ and f_1, f_2 in $L^2(d\nu_\alpha)$. Then, we have*

$$\int_{\mathbb{R}^2_+} f_1(r,x)\overline{f_2(r,x)} d\nu_\alpha(r,x) = \frac{1}{C_h} \int_{\mathbb{R}^2_+ \times \mathbb{R}^2_+} \Phi^\alpha_h(f_1)(a,b,r,x)\overline{\Phi^\alpha_h(f_2)(a,b,r,x)} d\mu_\alpha(a,b,r,x). \tag{27}$$

Remark 2.4. *Let h be a generalized wavelet in $L^2(d\nu_\alpha)$. Then from the relations (23) and (22), for all f in $L^2(d\nu_\alpha)$ we have*

$$\|\Phi^\alpha_h(f)\|_{L^\infty_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)} \le \|f\|_{L^2(d\nu_\alpha)} \|h\|_{L^2(d\nu_\alpha)}. \tag{28}$$

2.3. Schatten-von Neumann classes

In this sub-section, we recall the notion of Schatten-von Neumann classes. Prior to that, we set the following notation:

- $l^p(\mathbb{N})$, $1 \le p \le \infty$, the set of all infinite sequences of real (or complex) numbers $x := (x_j)_{j\in\mathbb{N}}$, such that

$$\|x\|_p = \Big(\sum_{j=1}^{\infty} |x_j|^p\Big)^{1/p} < \infty, \quad \text{if} \quad 1 \le p < \infty,$$

$$\|x\|_\infty = \sup_{j\in\mathbb{N}} |x_j| < \infty.$$

For $p = 2$, we provide this space $l^2(\mathbb{N})$ with the scalar product

$$\langle x, y\rangle_2 := \sum_{j=1}^{\infty} x_j \overline{y_j}.$$

- $B(L^p(d\nu_\alpha))$, $1 \leq p \leq \infty$, the space of bounded operators from $L^p(d\nu_\alpha)$ into itself.

Definition 2.7. *(i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2(d\nu_\alpha))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.*

(ii) For $1 \leq p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$\|A\|_{S_p} := \Big(\sum_{n=1}^{\infty} (s_n(A))^p \Big)^{\frac{1}{p}}. \tag{29}$$

Remark 2.5. *We note that the space S_2 is the space of Hilbert-Schmidt operators, and S_1 is the space of trace class operators.*

Definition 2.8. *The trace of an operator A in S_1 is defined by*

$$tr(A) = \sum_{n=1}^{\infty} \langle A v_n, v_n\rangle_{L^2(d\nu_\alpha)} \tag{30}$$

where $(v_n)_n$ is any orthonormal basis of $L^2(d\nu_\alpha)$.

Remark 2.6. *If A is positive, then*

$$tr(A) = \|A\|_{S_1}. \tag{31}$$

Moreover, a compact operator A on the Hilbert space $L^2(d\nu_\alpha)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$\|A\|^2_{HS} := \|A\|^2_{S_2} = \|A^*A\|_{S_1} = tr(A^*A) = \sum_{n=1}^{\infty} \|A v_n\|^2_{L^2(d\nu_\alpha)} \tag{32}$$

for any orthonormal basis $(v_n)_n$ of $L^2(d\nu_\alpha)$.

Definition 2.9. *We define $S_\infty := B(L^2(d\nu_\alpha))$, equipped with the norm,*

$$\|A\|_{S_\infty} := \sup_{v\in L^2(d\nu_\alpha):\|v\|_{L^2(d\nu_\alpha)}=1} \|Av\|_{L^2(d\nu_\alpha)}. \tag{33}$$

Remark 2.7. *It is obvious that $S_p \subset S_q$, $1 \leq p \leq q \leq \infty$.*

2.4. Localization operators for the generalized continuous wavelet transform.

In this subsection, we shall recall some fundamental results associated with the Riemann-Liouville wavelet localization operators [15].

Definition 2.10. *Let h be measurable function on $\mathbb{R}^2_+$ and σ be measurable function on the set $\mathbb{R}^2_+ \times \mathbb{R}^2_+$, we define the localization operator for the generalized continuous wavelet transform, denoted by $\mathcal{L}_h(\sigma)$, on $L^p(d\nu_\alpha)$, $1 \le p \le \infty$, by $\forall\,(s,y) \in \mathbb{R}^2_+$,*

$$\mathcal{L}_h(\sigma)(f)(s,y) = \frac{1}{C_h}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \sigma(a,b,r,x)\Phi^\alpha_h(f)(a,b,r,x)\, h_{a,b,r,x}(s,y)d\mu_\alpha(a,b,r,x). \tag{34}$$

Often it is more convenient to interpret the definition of $\mathcal{L}_h(\sigma)$ in a weak sense, that is, for f in $L^p(d\nu_\alpha)$, $1 \le p \le \infty$, and g in $L^{p'}(d\nu_\alpha)$

$$\langle \mathcal{L}_h(\sigma)(f), g\rangle_{L^2(d\nu_\alpha)} = \frac{1}{C_h}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \sigma(a,b,r,x)\Phi^\alpha_h(f)(a,b,r,x)\overline{\Phi^\alpha_h(g)(a,b,r,x)}d\mu_\alpha(a,b,r,x). \tag{35}$$

For the sake of simplicity, we will call the above defined operator $\mathcal{L}_h(\sigma)$ as the localization operator.

Proposition 2.5. *Let $p \in [1,\infty)$. The adjoint of the localization operator*

$$\mathcal{L}_h(\sigma) : L^p(d\nu_\alpha) \to L^p(d\nu_\alpha)$$

is $\mathcal{L}_{k,h}(\overline{\sigma}) : L^{p'}(d\nu_\alpha) \to L^{p'}(d\nu_\alpha)$.

Theorem 2.3. *Let σ be in $L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $1 \le p \le \infty$. Then there exists a unique bounded linear operator $\mathcal{L}_h(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$, such that*

$$\|\mathcal{L}_h(\sigma)\|_{S_\infty} \leqslant (\frac{1}{C_h})^{\frac{1}{p}}\|\sigma\|_{L^p_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}. \tag{36}$$

Proposition 2.6. *Let σ be in $L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, then the localization operator*

$$\mathcal{L}_h(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$$

is in S_2 and we have

$$\|\mathcal{L}_h(\sigma)\|_{S_2} \leqslant \frac{1}{C_h}\|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}.$$

Proposition 2.7. *Let σ be in $L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $1 \leqslant p < \infty$. Then, the localization operator $\mathcal{L}_h(\sigma)$ is compact.*

Theorem 2.4. *Let σ be in $L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$. Then,*

$$\frac{1}{C_h}\|\widetilde{\sigma}\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \leqslant \|\mathcal{L}_h(\sigma)\|_{S_1} \leqslant \frac{1}{C_h}\|\sigma\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}, \tag{37}$$

where $\widetilde{\sigma}$ is given by

$$\widetilde{\sigma}(a,b,r,x) = \langle \mathcal{L}_h(\sigma)(h_{a,b,r,x}), h_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)}, \quad (a,b,r,x) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+.$$

Corollary 2.2. *For σ in $L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, we have the following trace formula*

$$tr(\mathcal{L}_h(\sigma)) = \frac{1}{C_h}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \sigma(a,b,r,x)\|h_{a,b,r,x}\|_{L^2(d\nu_\alpha)}d\mu_\alpha(a,b,r,x). \tag{38}$$

Corollary 2.3. *Let σ be in $L^p_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$, $1 \leqslant p \leqslant \infty$. Then, the localization operator*

$$\mathcal{L}_h(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is in S_p and we have

$$\|\mathcal{L}_h(\sigma)\|_{S_p} \leqslant \left(\frac{1}{C_h}\right)^{\frac{1}{p}}\|\sigma\|_{L^p_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}.$$

3. Mean dispersion theorem for the wavelet transform

In this section, we shall present some useful results regarding the concentration of $\Phi_h^\alpha(f)$ on small sets.

Proposition 3.1. *Suppose that $U \subset \mathbb{R}_+^2 \times \mathbb{R}_+^2$ satisfies*

$$\mu_\alpha(U) < \frac{C_h}{\|h\|^2_{L^2(d\nu_\alpha)}}, \tag{39}$$

then, for all f in $L^2(d\nu_\alpha)$, we have

$$\|\chi_{U^c}\Phi_h^\alpha(f)\|_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \geq \sqrt{C_h}\sqrt{1-\frac{\|h\|^2_{L^2(d\nu_\alpha)}}{C_h}\mu_\alpha(U)}\|f\|_{L^2(d\nu_\alpha)}, \tag{40}$$

where χ_{U^c} denotes the characteristic function of the complementary U^c of U.

Proof. From Plancherel's Theorem 2.2, we have

$$C_h\|f\|^2_{L^2(d\nu_\alpha)} = \|\Phi_h^\alpha(f)\|^2_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} = \|\Phi_h^\alpha(f)\|^2_{L^2_{\mu_\alpha}(U)} + \|\Phi_h^\alpha(f)\|^2_{L^2_{\mu_\alpha}(U^c)}. \tag{41}$$

On the other hand from the relation (28), we have

$$\begin{aligned} \int_U |\Phi_h^\alpha(f)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x) &\leq \|\Phi_h^\alpha(f)\|^2_{L^\infty_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}\mu_\alpha(U) \\ &\leq \mu_\alpha(U)\|f\|^2_{L^2(d\nu_\alpha)}\|h\|^2_{L^2(d\nu_\alpha)}. \end{aligned} \tag{42}$$

Thus, the result follows immediately from the relations (41) and (42). □

Remark 3.1. *Let U be a subset of $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ satisfying the relation (39). If $\Phi_h^\alpha(f)$ is supported in U, then $f = 0$.*

Proposition 3.2. *Let h be a generalized wavelet such that $\|h\|_{L^2(d\nu_\alpha)} = 1$. Let $s > 0$. Then the following uncertainty inequality hold.*
There exists a constant $C(s) > 0$ such that, for all f in $L^2(d\nu_\alpha)$, we have

$$\Big\| \|(a,b,r,x)\|^s \Phi_h^\alpha(f)\Big\|_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \geq C(s)\|f\|_{L^2(d\nu_\alpha)}. \tag{43}$$

Proof. Let $\delta > 0$. We consider the subset V_δ of $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ defined by

$$V_\delta = \Big\{(a,b,r,x) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : \|(a,b,r,x)\| < \delta\Big\},$$

and satisfying $0 < \mu_\alpha(V_\delta) < C_h$. By applying the relation (40) with $U = V_\delta$ we obtain

$$\begin{aligned} \|f\|^2_{L^2(d\nu_\alpha)} &\leq \tfrac{1}{C_h - \mu_\alpha(V_\delta)} \int_{V_\delta^c} |\Phi_h^\alpha(f)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x) \\ &\leq \tfrac{1}{\delta^{2s}(C_h - \mu_\alpha(V_\delta))} \int_{\|(a,b,r,x)\|\geq\delta} \|(a,b,r,x)\|^{2s}|\Phi_h^\alpha(f)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x) \\ &\leq \tfrac{1}{\delta^{2s}(C_h - \mu_\alpha(V_\delta))}\Big\| \|(a,b,r,x)\|^s\Phi_h^\alpha(f)\Big\|^2_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)}. \end{aligned}$$

Thus, we obtain the relation (43) with $C(s) := \delta^s\sqrt{C_h - \mu_\alpha(V_\delta)}$. □

Proposition 3.3. *([19]). Let h be a generalized wavelet on $\mathbb{R}^2_+$ in $L^2(d\nu_\alpha)$. Then, $\Phi^\alpha_h(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space with kernel function*

$$\mathcal{K}_h(a',b',r',x';a,b,r,x) := \frac{1}{C_h}\int_{\mathbb{R}^2_+} h_{a',b',r',x'}(s,y)\overline{h_{a,b,r,x}(s,y)}d\nu(s,y). \tag{44}$$

The kernel satisfies:

$$\forall\,(a',b',r',x'),\,(a,b,r,x)\in\mathbb{R}^2_+\times\mathbb{R}^2_+,\quad |\mathcal{K}_h(a',r',x';a,b,r,x)|\leq \frac{\|h\|^2_{L^2(d\nu_\alpha)}}{C_h}. \tag{45}$$

Notation. We shall adopt the following notations:
(i) $P_h : L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)\to L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ denotes the orthogonal projection from $L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ onto $\Phi^\alpha_h(L^2(d\nu_\alpha))$.
(ii) $P_U : L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)\to L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ denotes the orthogonal projection from $L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ onto the subspace of functions of $L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)$ supported in a subset $U\subset\mathbb{R}^2_+\times\mathbb{R}^2_+$ satisfying

$$0<\mu_\alpha(U) := \int_U d\mu_\alpha(a,b,r,x)<\infty. \tag{46}$$

Next, we recall that

$$\begin{aligned}\|P_UP_h\|_{HS} &:= \Big(\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+\times\mathbb{R}^2_+\times\mathbb{R}^2_+}|\chi_U(a,b,r,x)|^2|\mathcal{K}_h(a',b',r',x';a,b,r,x)|^2d\mu_\alpha(a',b',r',x')d\mu_\alpha(a,b,r,x)\Big)^{\frac{1}{2}}\\ &\leq \frac{\|h\|_{L^2(d\nu_\alpha)}}{\sqrt{C_h}}\sqrt{\mu_\alpha(U)}<\infty.\end{aligned} \tag{47}$$

That is, P_UP_h is a Hilbert-Schmidt operator and, therefore it is a compact operator.

Remark 3.2. *i) The operator $P_h=\Phi^\alpha_h(\Phi^\alpha_h)^*$ can be explicitly expressed as an integral operator*

$$P_hF(z)=\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}F(a,b,r,x)\mathcal{K}_h(z;a,b,r,x)d\mu_\alpha(a,b,r,x),\ z=(a',b',r',x')\in\mathbb{R}^2_+\times\mathbb{R}^2_+,$$

with integral kernel $\mathcal{K}_h$.
ii) As $\mathcal{K}_h$ is the integral kernel of an orthogonal projection, it satisfies

$$\mathcal{K}_h(z;z')=\overline{\mathcal{K}_h(z';z)},\quad \text{for all } z,z'\in\mathbb{R}^2_+\times\mathbb{R}^2_+, \tag{48}$$

and

$$\mathcal{K}_h(z;z')=\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\mathcal{K}_h(z;z'')\mathcal{K}_h(z'';z')d\mu_\alpha(z''),\ z,z'\in\mathbb{R}^2_+\times\mathbb{R}^2_+. \tag{49}$$

iii) If $\{v_n : n\in\mathbb{N}\}$ is an orthonormal basis of $\Phi^\alpha_h(L^2(d\nu_\alpha))$, $\mathcal{K}_h$ can be expanded as

$$\mathcal{K}_h(z;z')=\sum_{n=1}^{\infty}v_n(z)\overline{v_n(z')},\ z,z'\in\mathbb{R}^2_+\times\mathbb{R}^2_+. \tag{50}$$

Definition 3.1. *Let $0<\varepsilon<1$ and let $f\in L^2(d\nu_\alpha)$ be a non-zero function.*
We say that Φ^α_h is ε-time-concentrated on U, if

$$\left\|\Phi^\alpha_h(f)\right\|_{L^2_{\mu_\alpha}(U^c)}\leq\varepsilon\|f\|_{L^2(d\nu_\alpha)}\|h\|_{L^2(d\nu_\alpha)}.$$

Proposition 3.4. *Let h be a generalized wavelet and $\left(u_\beta\right)_{\beta\in\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$ and U be a measurable subset of $\mathbb{R}^2_+\times\mathbb{R}^2_+$. If $\mu_\alpha(U)<\infty$, then for every non-empty finite subset $\mathcal{K}\subset\mathbb{N}^2$, we have*

$$\sum_{\beta\in\mathcal{K}}\left(1-\|\chi_{U^c}\Phi_h^\alpha(u_\beta)\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\right)\leqslant\frac{\|h\|^2_{L^2(d\nu_\alpha)}}{C_h}\mu_\alpha(U).$$

Proof. As P_UP_h is an Hilbert-Schmidt operator then by (32)

$$\begin{aligned}\sum_{\beta\in\mathcal{K}}\langle P_U\Phi_h^\alpha(u_\beta),\Phi_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}&=\sum_{\beta\in\mathcal{K}}\langle P_hP_UP_h\Phi_h^\alpha(u_\beta),\Phi_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\\&\le tr(P_hP_UP_h)\\&=\|P_UP_h\|^2_{HS}.\end{aligned}$$

Then by (47) we get

$$\sum_{\beta\in\mathcal{K}}\langle P_U\Phi_h^\alpha(u_\beta),\Phi_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\leqslant\frac{\|h\|^2_{L^2(d\nu_\alpha)}}{C_h}\mu_\alpha(U).\tag{51}$$

Now by the Cauchy-Schwartz inequality we have for every $\beta\in\mathcal{K}$,

$$\begin{aligned}\langle P_U\Phi_h^\alpha(u_\beta),\Phi_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}&=1-\langle P_{U^c}\Phi_h^\alpha(u_\beta),\Phi_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\\&\geqslant1-\|\chi_{U^c}\Phi_h^\alpha(u_\beta)\|^2_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\end{aligned}$$

in particular, using relation (51), we obtain

$$\sum_{\beta\in\mathcal{K}}\left(1-\|\chi_{U^c}\Phi_h^\alpha(u_\beta)\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\right)\leqslant\sum_{\beta\in\mathcal{K}}\langle P_U\Phi_h^\alpha(u_\beta),\Phi_h^\alpha(u_\beta)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\leqslant\frac{\|h\|^2_{L^2(d\nu_\alpha)}}{C_h}\mu_\alpha(U).$$

□

As a consequence of the proposition 3.4, we shall demonstrate that, if the generalized continuous wavelet transform of an othornormal sequence are ε time-frequency concentrated in a given centred ball of $\mathbb{R}^2_+\times\mathbb{R}^2_+$, then such a sequence is necessarily finite.

Proposition 3.5. *Let ε and δ be positive real numbers such that $0<\varepsilon<1$, and h be a generalized wavelet. Let $\mathcal{K}\subset\mathbb{N}^2$ be a non-empty subset and $\left(u_\beta\right)_{\beta\in\mathcal{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. If $\Phi_h^\alpha(u_\beta)$ is ε-time-frequency concentrated in the set*

$$B_\delta:=\left\{(a,b,r,x)\in\mathbb{R}^2_+\times\mathbb{R}^2_+:\|(a,b,r,x)\|\le\delta\right\}$$

for every $\beta\in\mathcal{K}$, then $\mathcal{K}$ is finite and

$$\mathrm{Card}(\mathcal{K})\leqslant\frac{\delta^{4\alpha+6}}{(1-\varepsilon)}M(\alpha,h).\tag{52}$$

where $M(\alpha,h)=\frac{\|h\|^2_{L^2(d\nu_\alpha)}}{C_h}\mu_\alpha(B_1)$.

Proof. Let $\mathcal{M}\subset\mathcal{K}$ be a non-empty finite subset, then by Proposition 3.4, we deduce that

$$\sum_{\beta\in\mathcal{M}}\left(1-\|\chi_{B^c_\delta}\Phi_h^\alpha(u_\beta)\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\right)\leqslant\frac{\|h\|^2_{L^2(d\nu_\alpha)}}{C_h}\mu_\alpha(B_\delta),\tag{53}$$

however for every $\beta \in \mathcal{M}$, $\|\chi_{B_\delta^c}\Phi_h^\alpha(u_\beta)\|_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} \leqslant \varepsilon$, and

$$\mu_\alpha(B_\delta) = \mu_\alpha(B_1)\delta^{4\alpha+6}, \tag{54}$$

hence by combining relations (53) and (54), we deduce that

$$\mathrm{Card}(\mathcal{M}) \leqslant \frac{\mu_\alpha(B_1)\|h\|^2_{L^2(d\nu_\alpha)}}{(1-\varepsilon)C_h}\delta^{4\alpha+6},$$

which means that $\mathcal{K}$ is finite and satisfies relation (52). □

Let p be a positive real number, h be a generalized wavelet and $f \in L^2(d\nu_\alpha)$, we define the generalized p^{th} time-frequency dispersion of $\Phi_h^\alpha(f)$ by

$$\rho_p(\Phi_h^\alpha(f)) = \left(\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} \|(a,b,r,x)\|^p \left|\Phi_h^\alpha(f)(a,b;r,x)\right|^2 d\mu_\alpha(a,b,r,x)\right)^{\frac{1}{p}}.$$

Corollary 3.1. *Let A, p be positive real numbers and $h \in L^2(d\nu_\alpha)$ be a generalized wavelet. Let $\mathcal{K} \subset \mathbb{N}^2$ be a non-empty subset and $\left(u_\beta\right)_{\beta\in\mathcal{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Assume that for every $\beta \in \mathcal{K}$,*

$$\rho_p(\Phi_h^\alpha(u_\beta)) \leqslant A,$$

then $\mathcal{K}$ is finite and

$$\mathrm{Card}(\mathcal{K}) \leqslant A^{4\alpha+6}M'(\alpha,p,h),$$

where $M'(\alpha,p,h) = 2^{1+\frac{8\alpha+12}{p}}M(\alpha,h)$.

Proof. Assume that $\rho_p(\Phi_h^\alpha(u_\beta)) \leqslant A$ for every $\beta \in \mathcal{K}$, then we have

$$\int_{B^c_{A2^{\frac{2}{p}}}} |\Phi_h^\alpha(u_\beta)(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x) \leqslant \frac{1}{\left(A2^{\frac{2}{p}}\right)^p}\rho_p^p(\Phi_h^\alpha(f)) \leqslant \frac{1}{4}. \tag{55}$$

Relation (55) means that for every $\beta \in \mathcal{K}$, u_β is $\dfrac{1}{2}$-concentrated in the set $B_{A2^{\frac{2}{p}}}$, hence according to Proposition 3.5, we deduce that $\mathcal{K}$ is finite and

$$\mathrm{Card}(\mathcal{K}) \leqslant A^{4\alpha+6}M'(\alpha,p,h).$$

□

Lemma 3.1. *Let h be a generalized wavelet and p be a positive real number. If $\left(u_\beta\right)_{\beta\in\mathbb{N}^2}$ is an orthonormal sequence in $L^2(d\nu_\alpha)$, then there exists $j_0 \in \mathbb{Z}$ such that*

$$\forall\beta \in \mathbb{N}^2,\ \rho_p(\Phi_h^\alpha(u_\beta)) \geqslant 2^{j_0}.$$

Proof. The proof is an immediate consequence of Heisenberg-type inequality (43). □

Theorem 3.1 (Shapiro's Dispersion Theorem). *Let h be a generalized wavelet and $\left(u_\beta\right)_{\beta\in\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, then for every positive real number p and for every non-empty finite subset $\mathcal{K} \subset \mathbb{N}^2$, we have*

$$\sum_{\beta\in\mathcal{K}}\left(\rho_p(\Phi_h^\alpha(u_\beta))\right)^p \geq \frac{1}{2}\left(\frac{3}{M'(\alpha,p,h)2^{8\alpha+13}}\right)^{\frac{p}{4\alpha+6}}(\mathrm{Card}(\mathcal{K}))^{1+\frac{p}{4\alpha+6}}. \tag{56}$$

Proof. For every $j \in \mathbb{Z}$, let

$$P_j = \left\{\beta \in \mathbb{N}^2 \ : \ \rho_p(\Phi_h^\alpha(u_\beta)) \in [2^{j-1}, 2^j)\right\},$$

then for every $\beta \in P_j$

$$\int_{\mathbb{R}_+^2 \times \mathbb{R}_+^2} \|(a,b,r,x)\|^p \left|\Phi_h^\alpha(u_\beta)(a,b,r,x)\right|^2 d\mu_\alpha(a,b,r,x) \leqslant 2^{pj},$$

thus, using the relation (55) yields

$$\int_{B^c_{2^{j+\frac{2}{p}}}} \left|\Phi_h^\alpha(u_\beta)(a,b,r,x)\right|^2 d\mu_\alpha(a,b,r,x) \leqslant \frac{1}{4}\frac{\rho_p(u_\beta)^p}{2^{jp}} \leqslant \frac{1}{4}. \tag{57}$$

Therefore, as a consequence of the relation (57), we deduce that every $\beta \in P_j$, u_β is $\frac{1}{2}$-concentrated in the ball $B_{2^{j+\frac{2}{p}}}$, In other words, the sequence $(u_\beta)_{\beta \in P_j}$ satisfies the conditions of proposition 3.5, which shows that P_j is finite and

$$\mathrm{Card}(P_j) \leqslant 2^{j(4\alpha+6)} M'(\alpha,p,h). \tag{58}$$

For $m \in \mathbb{Z}$, $m \geqslant j_0$, we denote by $Q_m = \bigcup_{j=j_0}^{m} P_j$ then according to relation (58), we have

$$\mathrm{Card}\,(Q_m) = \sum_{j=j_0}^{m} \mathrm{Card}(P_j) \leqslant \frac{M'(\alpha,p,h)}{3} 2^{(m+1)(4\alpha+6)}.$$

Now, if $\mathrm{Card}(\mathcal{K}) > \dfrac{2M'(\alpha,p,h)}{3} 2^{(j_0+1)(4\alpha+6)}$, then we can choose an integer $n > j_0$ such that

$$\frac{2M'(\alpha,p,h)}{3} 2^{n(4\alpha+6)} < \mathrm{Card}(\mathcal{K}) \leqslant \frac{2M'(\alpha,p,h)}{3} 2^{(n+1)(4\alpha+6)}. \tag{59}$$

Thus, by relation (59) we get

$$\sum_{\beta \in \mathcal{K}} \left(\rho_p(\Phi_h^\alpha(v_\beta))\right)^p \geqslant \frac{\mathrm{Card}(\mathcal{K})}{2} 2^{(n-1)p} \geqslant \frac{1}{2} \left(\mathrm{Card}(\mathcal{K})\right)^{1+\frac{p}{4\alpha+6}} \left(\frac{3}{2^{8\alpha+13} M'(\alpha,p,h)}\right)^{\frac{p}{4\alpha+6}}.$$

Finally, if $\mathrm{Card}(\mathcal{K}) \leqslant \dfrac{2M'(\alpha,p,h)}{3} 2^{(j_0+1)(4\alpha+6)}$, then

$$\sum_{\beta \in \mathcal{K}} \left(\rho_p(\Phi_h^\alpha(v_\beta))\right)^p \geqslant \mathrm{Card}(\mathcal{K}) 2^{(j_0-1)p} \geqslant \mathrm{Card}(\mathcal{K})^{1+\frac{p}{4\alpha+6}} \left(\frac{3}{M'(\alpha,p,h) 2^{8\alpha+13}}\right)^{\frac{p}{4\alpha+6}}.$$

□

Remark 3.3. *By taking* $\mathrm{Card}(\mathcal{K}) = 1$*, relation (56) appears as a general version of Heisenberg-Pauli-Weyl inequality for the generalized continuous wavelet transform including the* p^{th} *dispersion with* $0 < p < 2$.

Corollary 3.2. *Let* $p > 0$*,* h *be a generalized wavelet and let* $\left(u_\beta\right)_{\beta \in \mathbb{N}^2}$ *be an orthonormal sequence in* $L^2(d\nu_\alpha)$*. Then for every* $\mathcal{K} \subset \mathbb{N}^2$

$$\sum_{\beta \in \mathcal{K}} \left(\left\| \|(a,b)\|^p \Phi_h^\alpha(u_\beta)(a,b,r,x) \right\|_{L^2_{\mu_\alpha}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)} + \left\| \|(r,x)\|^p \Phi_h^\alpha(u_\beta)(a,b,r,x) \right\|_{L^2_{\mu_\alpha}(\mathbb{R}_+^2 \times \mathbb{R}_+^2)} \right)$$
$$\geq \frac{1}{2} \left(\frac{3}{M'(\alpha,p,h) 2^{12\alpha+19}}\right)^{\frac{p}{4\alpha+6}} \mathrm{Card}(\mathcal{K})^{1+\frac{p}{4\alpha+6}}.$$

Proof. The result is an immediate consequence of the previous theorem and the fact that

$$\|(a,b,r,x)\|^p \leq 2^p(\|(a,b)\|^p + \|(r,x)\|^p).$$

□

As a consequence of the last dispersion inequality, we infer that, there does not exist an infinite sequence $\left(u_\beta\right)_{\beta\in\mathcal{K}}$ in $L^2(d\nu_\alpha)$ such that the two sequences

$$\left\| \|(a,b)\|^p \Phi_h^\alpha(u_\beta)(a,b,r,x) \right\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}$$

and

$$\left\| \|(r,x)\|^p \Phi_h^\alpha(u_\beta)(a,b,r,x) \right\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}$$

are bounded.

Corollary 3.3. *Let $p > 0$, h be a generalized wavelet and let $\left(u_\beta\right)_{\beta\in\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Then for every $\mathcal{K} \subset \mathbb{N}^2$*

$$\begin{aligned}&\sup_{\beta\in\mathcal{K}}\left(\left\| \|(a,b)\|^p \Phi_h^\alpha(u_\beta)(a,b,r,x) \right\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}, \left\| \|(r,x)\|^p \Phi_h^\alpha(u_\beta)(a,b,r,x) \right\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\right)\\ &\geq \frac{1}{4}\left(\frac{3}{M'(\alpha,p,h)2^{12\alpha+19}}\right)^{\frac{p}{4\alpha+6}} \mathrm{Card}(\mathcal{K})^{\frac{p}{4\alpha+6}}.\end{aligned}$$

In particular

$$\sup_{\beta\in\mathbb{N}^2}\left(\left\| \|(a,b)\|^p \Phi_h^\alpha(u_\beta)(a,b,r,x) \right\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} + \left\| \|(r,x)\|^p \Phi_h^\alpha(u_\beta)(a,b,r,x) \right\|_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)}\right) = \infty.$$

Theorem 3.2 (Shapiro's Umbrella Theorem). *Let h be a generalized wavelet and $\mathcal{K} \subset \mathbb{N}^2$ be a non-empty subset and $\left(u_\beta\right)_{\beta\in\mathcal{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, if there is a function $g \in L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ such that*

$$|\Phi_h^\alpha(u_\beta)(a,b,r,x)| \leqslant g(a,b,r,x),$$

for every $\beta \in \mathcal{K}$ and for almost every $(a,b,r,x) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$, then $\mathcal{K}$ is finite.

Proof. Following the idea of Malinnikova [14], for every positive real number $0 < \varepsilon < 1$, there is a subset $\Delta_{g,\varepsilon} \subset \mathbb{R}^2_+ \times \mathbb{R}^2_+$, such that

$$\mu_\alpha(\Delta_{g,\varepsilon}) = \inf\left\{\mu_\alpha(U) \ : \ \int\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+\setminus U} \left|g(a,b,r,x)\right|^2 d\mu_\alpha(a,b,r,x) \leqslant \varepsilon^2\right\},$$

and

$$\int\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+\setminus\Delta_{g,\varepsilon}} \left|g(a,b,r,x)\right|^2 d\mu_\alpha(a,b,r,x) = \varepsilon^2.$$

Hence, according to the hypothesis, for every $\alpha \in \mathcal{K}$, we have

$$\int\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+\setminus\Delta_{g,\varepsilon}} \left|\Phi_h^\alpha\left(u_\beta\right)(a,b,r,x)\right|^2 d\mu_\alpha(a,b,r,x) \leqslant \varepsilon^2,$$

and by the Theorem 3.4, we get $\mathrm{Card}(\mathcal{K})(1-\varepsilon) \leqslant \mu_\alpha(\Delta_{g,\varepsilon})$. □

4. Riemann-Liouville wavelet Scalograms

The aim of this section is to study the scalograms associated with the Riemann-Liouville wavelet transform. The obtained results generalize the results proved by Ghobber in [10], in the context of Riemann-Liouville wavelet transform.

4.1. Calderón-Toeplitz operator

Definition 4.1. *Let h be a generalized wavelet on $\mathbb{R}_+^2$ in $L^2(d\nu_\alpha)$. We define the Riemann-Liouville wavelet scalogram of f as*

$$\mathbf{S}_h^\alpha(f)(a,b,r,x) = C_h^{-1}|\Phi_h^\alpha f(a,b,r,x)|^2,\ (a,b,r,x) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2. \tag{60}$$

Remark 4.1. *From the Plancherel formula associated with Φ_h^α, we have*

$$\int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} \mathbf{S}_h^\alpha(f)(a,b,r,x)d\mu_\alpha(a,b,r,x) = \|f\|^2_{L^2(d\nu_\alpha)}. \tag{61}$$

It justifies the interpretation of a scalogram as a time-frequency energy density. Also, note that (35)

$$\langle \mathcal{L}_h(\sigma)f, f\rangle_{L^2(d\nu_\alpha)} = \int_{\mathbb{R}_+^2\times\mathbb{R}_+^2} \sigma(a,b,r,x)\mathbf{S}_h^\alpha(f)(a,b,r,x)d\mu_\alpha(a,b,r,x). \tag{62}$$

In this section we shall keep our focus on localization operators $\mathcal{L}_h(\sigma)$ with symbol $\sigma = \chi_U$, and h is a generalized wavelet on $\mathbb{R}_+^2$ in $L^2(d\nu_\alpha)$, and U is subset of $\mathbb{R}_+^2 \times \mathbb{R}_+^2$ with finite measure $\mu_\alpha(U) < \infty$. For the sake of simplicity, such an operator will be denoted as $\mathcal{L}_h(U)$.

Definition 4.2. *We define the Calderón-Toeplitz operator*

$$T_{h,U} : \Phi_h^\alpha(L^2(d\nu_\alpha)) \to \Phi_h^\alpha(L^2(d\nu_\alpha))$$

by

$$T_{h,U}\, F = P_h P_U F. \tag{63}$$

Proposition 4.1. *The operator $T_{h,U} : \Phi_h^\alpha(L^2(d\nu_\alpha)) \to \Phi_h^\alpha(L^2(d\nu_\alpha))$ is trace-class and satisfies*

$$0 \le T_{h,U} \le P_U \le I, \tag{64}$$

and

$$T_{h,U} = \Phi_h^\alpha \mathcal{L}_h(U)(\Phi_h^\alpha)^*. \tag{65}$$

Proof. For all $F \in \Phi_h^\alpha(L^2(d\nu_\alpha))$,

$$\langle T_{h,U}\, F, F\rangle_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} = \langle P_h(P_U F), F\rangle_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} = \langle P_U F, F\rangle_{L^2_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} = \int_U |F(a,b,r,x)|^2 d\mu_\alpha(a,b,r,x). \tag{66}$$

Thus we deduce (64), and $T_{h,U}$ is bounded and positive.
Now, we want to prove (65). Indeed, using Φ_h^α and $(\Phi_h^\alpha)^*$, the time-frequency localization operator

$$\mathcal{L}_h(U) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$$

can be expressed as

$$\mathcal{L}_h(U)(f) = (\Phi_h^\alpha)^*(P_U \Phi_h^\alpha f), \quad f \in L^2(d\nu_\alpha).$$

Therefore,

$$(\Phi_h^\alpha \mathcal{L}_h(U)(\Phi_h^\alpha)^*)F = P_h P_U F = T_{h,U}\, F, \quad F \in \Phi_h^\alpha(L^2(d\nu_\alpha)). \tag{67}$$

Therefore, the time-frequency operator $\mathcal{L}_h(U)$ and the Calderón-Toeplitz operator $T_{h,U}$ are related by

$$T_{h,U} = \Phi_h^\alpha \mathcal{L}_h(U)(\Phi_h^\alpha)^*.$$

□

Remark 4.2. *From the above proposition, we deduce that $T_{h,U}$ and $\mathcal{L}_h(U)$ enjoy the same spectral properties, in particular, we have the following proposition.*

Proposition 4.2. *The Calderón-Toeplitz operator $T_{h,U}$ is compact and even trace class with*

$$tr(T_{h,U}) = tr(\mathcal{L}_h(U)) = M_\alpha(h, U), \tag{68}$$

where

$$M_\alpha(h, U) := \frac{1}{C_h} \int_U \|h_{a,b,r,x}\|^2_{L^2(d\nu_\alpha)} d\mu_\alpha(a, b, r, x). \tag{69}$$

Proof. Note that the operator $T_{h,U} : \Phi_h^\alpha(L^2(d\nu_\alpha)) \to \Phi_h^\alpha(L^2(d\nu_\alpha))$ is bounded and positive. Now, let $\{e_n\}_{n=1}^\infty$ be an arbitrary orthonormal basis for $\Phi_h^\alpha(L^2(d\nu_\alpha))$. Then, if we denote by $v_n = \sqrt{C_h}(\Phi_h^\alpha)^*(e_n)$, then $\{v_n\}_{n=1}^\infty$ is an orthonormal basis for $L^2(d\nu_\alpha)$.
Thus, by (35) and Fubini's theorem, we get

$$\begin{aligned}
\sum_{n=1}^\infty \langle T_{h,U}(e_n), e_n\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &= \sum_{n=1}^\infty \left\langle \mathcal{L}_h(U)(\Phi_h^\alpha)^*(e_n), (\Phi_h^\alpha)^*(e_n)\right\rangle_{L^2(d\nu_\alpha)} \\
&= \frac{1}{C_h}\sum_{n=1}^\infty \int_U |\Phi_h^\alpha(v_n)(a, b, r, x)|^2 d\mu_\alpha(a, b, r, x) \\
&= \frac{1}{C_h}\int_U \sum_{n=1}^\infty |\Phi_h^\alpha(v_n)(a, b, r, x)|^2 d\mu_\alpha(a, b, r, x) \\
&= \frac{1}{C_h}\int_U \sum_{n=1}^\infty |\langle v_n, h_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)}|^2 d\mu_\alpha(a, b, r, x) \\
&= \frac{1}{C_h}\int_U \|h_{a,b,r,x}\|^2_{L^2(d\nu_\alpha)} d\mu_\alpha(a, b, r, x) \\
&= M_\alpha(h, U).
\end{aligned}$$

Therefore, by Definition 2.8 and Remark 2.6, the operator $T_{h,U}$ is trace class with

$$\|T_{h,U}\|_{S_1} = tr(T_{h,U}) = M_\alpha(h, U).$$

□

Let $\mathbf{V}_{h,U} : L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) \to L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ the operator defined by $\mathbf{V}_{h,U} = P_h P_U P_h$. The advantage of $\mathbf{V}_{h,U}$ compared to $T_{h,U}$ is that it is defined on $L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ and consequently its spectral properties can be easily related to its integral kernel. Since $T_{h,U}$ is positive and trace-class, then using the decomposition

$$L^2_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+) = \Phi_h^\alpha(L^2(d\nu_\alpha)) \oplus \left(\Phi_h^\alpha(L^2(d\nu_\alpha))\right)^\perp,$$

we deduce that $\mathbf{V}_{h,U}$ is also positive and trace-class with

$$tr(\mathbf{V}_{h,U}) = tr(T_U^h) = M_\alpha(h, U). \tag{70}$$

In addition, we have the following result.

Proposition 4.3. *The trace of $T^2_{h,U}$ is given by*

$$tr(T^2_{h,U}) = \int_U \int_U |\mathcal{K}_h(a, b, r, x; a', b', r', x')|^2 d\mu_\alpha(a, b, r, x) d\mu_\alpha(a', b', r', x'). \tag{71}$$

Proof. Since, $\mathbf{V}_{h,U}$ is positive, then

$$tr(T^2_{h,U}) = tr(\mathbf{V}^2_{h,U}). \tag{72}$$

On the other hand using the fact that the space $\Phi^\alpha_h(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space with kernel $\mathcal{K}_h$, we get that for $F \in L^2(d\nu_\alpha)$

$$\begin{aligned}\mathbf{V}_{h,U}F(a,b,r,x) &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} F(a',b',r',x') \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \chi_U(c,d,t,y)\mathcal{K}_h(a,b,r,x;c,d,t,y) \times \\ &\quad \mathcal{K}_h(c,d,t,y;a',b',r',x')d\mu_\alpha(c,d,t,y)d\mu_\alpha(a',b',r',x'). \end{aligned}\tag{73}$$

That is, $\mathbf{V}_{h,U}$ has integral kernel

$$\mathbf{N}_{h,U}(a,b,r,x;a',b',r',x') = \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \chi_U(c,d,t,y)\mathcal{K}_h(a,b,r,x;c,d,t,y)\mathcal{K}_h(c,d,t,y;a',b',r',x')d\mu_\alpha(c,d,t,y). \tag{74}$$

Therefore,

$$\begin{aligned} tr(\mathbf{V}^2_{h,U}) &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} |\mathbf{N}_{h,U}(a,b,r,x;a',b',r',x')|^2 d\mu_\alpha(a,b,r,x)d\mu_\alpha(a',b',r',x') \\ &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \chi_U(z_1)\chi_U(z_2)\mathbf{K}_h(z_1;z_2)d\mu_\alpha(z_1)d\mu_\alpha(z_2) \end{aligned}$$

where by using the properties of the kernel of the reproducing kernel Hilbert space

$$\begin{aligned} \mathbf{K}_h(z_1;z_2) &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+}\int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \mathcal{K}_h(z_2;a,b,r,x)\mathcal{K}_h(a,b,r,x;z_1)\mathcal{K}_h(z_1;a',b',r',x')\times \\ &\quad \mathcal{K}_h(a',b',r',x';z_2)d\mu_\alpha(a,b,r,x)d\mu_\alpha(a',b',r',x') \\ &= \mathcal{K}_h(z_2;z_1)\mathcal{K}_h(z_1;z_2). \end{aligned}$$

Using (48), we get

$$\mathbf{K}_h(z_1;z_2) = |\mathcal{K}_h(z_1;z_2)|^2. \tag{75}$$

This follows us to conclude. □

4.2. Eigenvalues and eigenfunctions

Since the localization operator $\mathcal{L}_h(U) = (\Phi^\alpha_h)^*\chi_U\Phi^\alpha_h$ that we consider is a compact and self-adjoint operator, the spectral theorem gives the following spectral representation

$$\mathcal{L}_h(U)(f) = \sum_{n=1}^{\infty} s_n(U)\left\langle f, v^U_n\right\rangle_{L^2(d\nu_\alpha)} v^U_n, \quad f \in L^2(d\nu_\alpha), \tag{76}$$

where $\{s_n(U)\}_{n=1}^\infty$ are the positive eigenvalues arranged in a non increasing manner and $\{v^U_n\}_{n=1}^\infty$ is the corresponding orthonormal set of eigenfunctions. Note that $s_n(U) \searrow 0$ and by (36), we have for all $n \geq 1$,

$$s_n(U) \leq s_1(U) \leq 1. \tag{77}$$

This, together with (65), we can deduce that the Calderón-Toeplitz operator

$$T_{h,U} : \Phi^\alpha_h(L^2(d\nu_\alpha)) \to \Phi^\alpha_h(L^2(d\nu_\alpha))$$

can be diagonalized as

$$T_{h,U}F = \sum_{n=1}^{\infty} s_n(U)\left\langle F, e^U_n\right\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} e^U_n, \quad F \in \Phi^\alpha_h(L^2(d\nu_\alpha)), \tag{78}$$

where $e^U_n = \Phi^\alpha_h(v^U_n)$.

Lemma 4.1. *For all $z = (a, b, r, x) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$, we have*

$$\Theta(z) := \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \chi_U(\omega)|\mathcal{K}_h(\omega; z)|^2 d\mu_\alpha(\omega) = \sum_{n=1}^{\infty} s_n(U)\mathbf{S}^\alpha_h(v^U_n)(z). \tag{79}$$

Proof. From (44), we have for all $z = (a, b, r, x) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+$, the function $\mathcal{K}_h(.; z)$ is in $\Phi^\alpha_h(L^2(d\nu_\alpha))$. Therefore using the properties of the kernel of the reproducing kernel Hilbert space, we get

$$\begin{aligned} \langle T_{h,U}\,\mathcal{K}_h(.; z), \mathcal{K}_h(.; z)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &= \langle P_U\mathcal{K}_h(.; z), \mathcal{K}_h(.; z)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \\ &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \chi_U(\omega)\mathcal{K}_h(\omega; z)\overline{\mathcal{K}_h(\omega; z)}d\mu_\alpha(\omega) \\ &= \int_{\mathbb{R}^2_+\times\mathbb{R}^2_+} \chi_U(\omega)|\mathcal{K}_h(\omega; z)|^2 d\mu_\alpha(\omega). \end{aligned}$$

Let $\{w^U_n\}^\infty_{n=1} \subset \Phi^\alpha_h(L^2(d\nu_\alpha))$ be an orthonormal basis of $Ker(T_{h,U})$ (eventually empty).
Hence, $\{e^U_n\}^\infty_{n=1} \cup \{w^U_n\}^\infty_{n=1}$ is an orthonormal basis of $\Phi^\alpha_h(L^2(d\nu_\alpha))$ and therefore the reproducing kernel $\mathcal{K}_h$ can be written as

$$\mathcal{K}_h(a, b, r, x; a', b', r', x') = \overline{\mathcal{K}_h(a', b', r', x'; z)} = \sum_{n=1}^{\infty} e^U_n(z)\overline{e^U_n(a', b', r', x')} + \sum_{n=1}^{\infty} w^U_n(z)\overline{w^U_n(a', b', r', x')}. \tag{80}$$

Using this, we compute again

$$\begin{aligned} \langle T_{h,U}\,\mathcal{K}_h(.; z), \mathcal{K}_h(.; z)\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &= \left\langle T_{h,U} \sum_{n=1}^{\infty} \overline{e^U_n(z)}e^U_n, \sum_{k=1}^{\infty} \overline{\phi^U_\alpha(z)}\phi^U_\alpha \right\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \\ &= \sum_{n,k} \overline{e^U_n(z)}\phi^U_\alpha(z) \left\langle T_{h,U}\, e^U_n, \phi^U_\alpha \right\rangle_{L^2_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} \\ &= \sum_{n=1}^{\infty} s_n(U)|e^U_n(z)|^2, \end{aligned}$$

and the conclusion follows. □

Let $\varepsilon \in (0, 1)$ and define the quantity

$$n(\varepsilon, U) := card\Big\{n : s_n(U) \geq 1 - \varepsilon\Big\}.$$

Then an easy adaptation of the proof of Lemma 3.3 in [1], we obtain the following estimate for the eigenvalue distribution.

Proposition 4.4. *Let $\varepsilon \in (0, 1)$. We have*

$$\begin{aligned} |n(\varepsilon, U) - M_\alpha(h, U)| \leq\ & max\{\tfrac{1}{\varepsilon}, \tfrac{1}{1-\varepsilon}\}\times \\ & \Big|\frac{1}{C_h}\int_U\int_U |\mathcal{K}_h(a', b', r', x'; a, b, r, x)|^2 d\mu_\alpha(a, b, r, x)d\mu_\alpha(a', b', r', x') - M_\alpha(h, U)\Big|. \end{aligned}$$

4.3. Scalogram of a subspace

Given an $N-$dimentional subspace V of $L^2(d\nu_\alpha)$, P_V the orthogonal projection onto V with projection kernel k_V, is defined as

$$P_V f(.) = \int_{\mathbb{R}^2_+} k_V(.; t, s)f(t, s)d\nu_\alpha(t, s). \tag{81}$$

Recall that if $\{v_n\}_{n=1}^{N}$ is an orthonormal basis of V, then

$$k_V(r,x;,t,s) = \sum_{n=1}^{N} v_n(r,x)\overline{v_n(t,s)}. \tag{82}$$

The kernel k_V is independent of the choice of orthonormal basis for V.

Definition 4.3. *The scalogram of the space V with generalized wavelet h is defined*

$$\mathbf{SCAL}_h^\alpha V(a,b,r,x) := \int_{\mathbb{R}_+^2}\int_{\mathbb{R}_+^2} k_V(t,s;b,y)\overline{h_{a,b,r,x}(t,s)}h_{a,b,r,x}(b,y)d\nu_\alpha(t,s)d\nu_\alpha(b,y). \tag{83}$$

Then, we have the following result.

Lemma 4.2. *The scalogram* $\mathbf{SCAL}_h^\alpha V$ *is given by*

$$\mathbf{SCAL}_h^\alpha V = C_h \sum_{n=1}^{N} \mathbf{S}_h^\alpha(v_n). \tag{84}$$

Proof. We have

$$\begin{aligned}
\mathbf{SCAL}_h^\alpha V(a,b,r,x) &= \int_{\mathbb{R}_+^2}\int_{\mathbb{R}_+^2}\sum_{n=1}^{N} v_n(t,s)\overline{v_n(b,y)h_{a,b,r,x}(t,s)}h_{a,b,r,x}(b,y)d\nu_\alpha(t,s)d\nu_\alpha(b,y)\\
&= \sum_{n=1}^{N}\langle v_n, h_{a,b,r,x}\rangle_{L^2(d\nu_\alpha)}\overline{\langle v_n, h_{a,b,r,x}\rangle}_{L^2(d\nu_\alpha)}\\
&= \sum_{n=1}^{N}\Phi_h^\alpha(v_n)(a,b,r,x)\overline{\Phi_h^\alpha(v_n)(a,b,r,x)}\\
&= \sum_{n=1}^{N}|\Phi_h^\alpha(v_n)(a,b,r,x)|^2.
\end{aligned}$$

This completes the proof. □

Definition 4.4. *We define the time-frequency concentration of a subspace V in U as:*

$$\xi_{U,h}(V) := \frac{1}{N}\int_U \mathbf{SCAL}_h^\alpha V(a,b,r,x)d\mu_\alpha(a,b,r,x). \tag{85}$$

Then, using Lemma 4.2, we get the desired result:

$$\xi_{U,h}(V) := \frac{C_h}{N}\sum_{n=1}^{N}\int_U \mathbf{S}_h^\alpha(v_n)(a,b,r,x)d\mu_\alpha(a,b,r,x). \tag{86}$$

Theorem 4.1. *The N−dimentional signal space* $V_N = span\{v_n^U\}_{n=1}^N$ *consisting of the first N eigenfunctions of* $\mathcal{L}_h(U)$ *corresponding to the N largest eigenvalues* $\{s_n(U)\}_{n=1}^N$ *maximize the regional concentration* $\xi_{U,h}(V)$ *and*

$$\xi_{U,h}(V_N) := \frac{C_h}{N}\sum_{n=1}^{N} s_n(U). \tag{87}$$

Proof. We have

$$\xi_{U,h}(V_N) := \frac{C_h}{N}\sum_{n=1}^{N}\int_U \mathbf{S}_h^{\alpha}(v_n^U)(a,b,r,x)d\mu_\alpha(a,b,r,x). \tag{88}$$

Moreover, the min-max lemma for self-adjoint operators states that (see e. g. Sec.95 in [21])

$$s_n(U) = \int_U \mathbf{S}_h^{\alpha}(v_n^U)(a,b,r,x)d\mu_\alpha(a,b,r,x) = \max\Big\{\langle \mathcal{L}_h(U)(f), f\rangle_{L^2(d\nu_\alpha)} : \|f\|_{L^2(d\nu_\alpha)} = 1, f \perp v_1^U, ..., v_{n-1}^U\Big\}.$$

So, the eigenvalues of $\mathcal{L}_h(U)$ determine the number of orthogonal functions that have a well-concentrated scalogram in U. Thus,

$$\xi_{U,h}(V_N) = \frac{C_h}{N}\sum_{n=1}^{N} s_n(U). \tag{89}$$

The min-max characterization of the eigenvalues of compact operators implies that the first N eigenfunctions of the time-frequency operator $\mathcal{L}_h(U)$ have optimal cumulative time-frequency concentration inside U, in the sense,

$$\sum_{n=1}^{N}\Big\langle \mathcal{L}_h(U)(v_n^U), v_n^U\Big\rangle_{L^2(d\nu_\alpha)} = \max\Big\{\sum_{n=1}^{N}\langle \mathcal{L}_h(U)v_n, v_n\rangle_{L^2(d\nu_\alpha)} : \{v_n\}_{n=1}^N \text{ orthonormal}\Big\}. \tag{90}$$

Therefore any $N-$dimensional subset V of $L^2(d\nu_\alpha)$ cannot to be better concentrated in U than V_N, i.e

$$\xi_{U,h}(V) \leq \xi_{U,h}(V_N). \tag{91}$$

The proof is complete. □

Remark 4.3. *The time-frequency concentration of a subspace V_N in U satisfies,*

$$s_N(U) \leq \frac{1}{C_h}\xi_{U,h}(V_N) \leq s_1(U) \leq 1. \tag{92}$$

4.4. Accumulated scalogram

Let $\rho_{(h,U)} := \mathbf{SCAL}_h^{\alpha} V_{N_\alpha(h,U)}$, the $\rho_{(h,U)}$ is called the accumulated scalogram, provided that $N_\alpha(h,U) = [M_\alpha(h,U)]$ is the smallest integer greater than or equal to $M_\alpha(h,U)$ and

$$V_{N_\alpha(h,U)} = span\{v_n^U\}_{n=1}^{N_\alpha(h,U)}.$$

Observe that,

$$\rho_{(h,U)}(a,b,r,x) = \sum_{n=1}^{N_\alpha(h,U)} |\Phi_h^{\alpha}(v_n^U)(a,b,r,x)|^2 = \sum_{n=1}^{N_\alpha(h,U)} |e_n^U(a,b,r,x)|^2. \tag{93}$$

Also,

$$\|\rho_{(h,U)}\|_{L^1_{\mu_\alpha}(\mathbb{R}_+^2\times\mathbb{R}_+^2)} = C_h N_\alpha(h,U) = C_h M_\alpha(h,U) + O(1).$$

Moreover, since

$$\sum_{n=1}^{N_\alpha(h,U)} s_n(U) \leq tr(\mathcal{L}_h(U)) = M_\alpha(h,U)$$

then we can define the quantity

$$E(h,U) := 1 - \frac{\sum_{n=1}^{N_\alpha(h,U)} s_n(U)}{M_\alpha(h,U)}. \tag{94}$$

which satisfies,

$$0 \leq E(h,U) \leq 1. \tag{95}$$

More precisely, we have the following result.

Lemma 4.3. *Let* $\varepsilon \in (0,1)$. *We have*

$$0 \leq E(h,U) \leq 1 - (1-\varepsilon)\min(1, \frac{n(\varepsilon,U)}{M_\alpha(h,U)}). \tag{96}$$

Proof. Let $\varepsilon \in (0,1)$ and define $l_\alpha(\varepsilon,U) = \min(N_\alpha(h,U), n(\varepsilon,U))$. It follows that

$$s_n(U) \geq 1-\varepsilon, \quad 1 \leq n \leq l_\alpha(\varepsilon,U). \tag{97}$$

As $N_\alpha(h,U) \geq l_\alpha(h,U)$, we get

$$\sum_{n=1}^{N_\alpha(h,U)} s_n(U) \geq \sum_{n=1}^{l_\alpha(\varepsilon,U)} s_n(U) \geq (1-\varepsilon) l_\alpha(\varepsilon,U). \tag{98}$$

Therefore

$$0 \leq E(h,U) \leq 1 - (1-\varepsilon)\frac{l_\alpha(\varepsilon,U)}{M_\alpha(h,U)}. \tag{99}$$

As $N_\alpha(\varepsilon,U) \geq M_\alpha(h,U)$, we obtain the desired result. □

Consequently when the eigenvalues $\{s_n(U)\}_{n=0}^{n(\varepsilon,U)}$ are close to 1, then $E(h,U) \to 0$. Moreover, we have the following result bounding the error between $\rho_{(h,U)}$ and Θ.

Proposition 4.5. *We have*

$$\frac{1}{M_\alpha(h,U)} \|\rho_{(h,U)} - C_h\Theta\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)} \leq \frac{C_h}{M_\alpha(h,U)} + 2C_h E(h,U). \tag{100}$$

Proof. From Lemma 4.1, we have, for all $z = (a,b,r,x) \in U$

$$\rho_{(h,U)}(z) - C_h\Theta(z) = \sum_{n=1}^{\infty} (t_n - s_n(U))|e_n^U(z)|^2, \tag{101}$$

where $t_n = 1$ if $n \leq N_\alpha(h,U)$ and 0 otherwise. Now since

$$\||e_n^U|^2\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)} = C_h$$

and

$$\sum_{n=1}^{\infty} s_n(U) = M_\alpha(h,U),$$

we obtain

$$\begin{aligned}
\|\rho_{(h,U)} - C_h\Theta\|_{L^1_{\mu_\alpha}(\mathbb{R}^2_+\times\mathbb{R}^2_+)} &\le C_h\sum_{n=1}^{\infty}|t_n - s_n(U)| \\
&= C_h\sum_{n=1}^{N_\alpha(h,U)}(1-s_n(U)) + C_h\sum_{n>N_\alpha(h,U)}s_n(U) \\
&= C_hN_\alpha(h,U) + C_h\sum_{n=1}^{\infty}s_n(U) - 2C_h\sum_{n=1}^{N_\alpha(h,U)}s_n(U) \\
&= C_hN_\alpha(h,U) + C_hM_\alpha(h,U) - 2C_h\sum_{n=1}^{N_\alpha(h,U)}s_n(U) \\
&= C_h\Big(N_\alpha(h,U) - M_\alpha(h,U)\Big) + 2C_h\Big(M_\alpha(h,U) - \sum_{n=1}^{N_\alpha(h,U)}s_n(U)\Big) \\
&\le C_h + 2C_h\Big(M_\alpha(h,U) - \sum_{n=1}^{N_\alpha(h,U)}s_n(U)\Big),
\end{aligned}$$

and the estimate (100) follows. □

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