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Sharp partial order of core-nilpotent endomorphisms

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Abstract. The aim of this work is to offer a definition of the sharp partial order for the set of Core-Nilpotent endomorphisms. The main properties of this partial order are studied, generalizing the notion of sharp partial order of finite square matrices. In order to do so, the structure of the set of $\{g\}$-commuting inverses of a Core-Nilpotent endomorphism is described, and in particular, the structure of the set of $\{g\}$-commuting inverses of finite square matrices with entries in an arbitrary field is determined and an algorithm for its computation is provided.

1. Introduction

As it was stated in [6] by R.E. Hartwig, in a semigroup S there are two types of relations that can be defined on the set of idempotent elements of S. These relations are the standard partial ordening relation [1], for $e, f \in S$,

$$e \leq f \Longleftrightarrow e^2 = e = ef = fe$$

and the equivalence relation of Kaplansky [7]

$$e \sim f \Longleftrightarrow e = \hat{p}p, f = p\hat{p}$$

for some $\hat{p} \in eSf, p \in fSe$ where an element a in eSf denotes $a = esf$ for some $s \in S$. In [6], a natural problem related with this elements is studied, which is how to extend these relations to a larger class of elements mantaining their respective character. The present work aims to follow this philosophy.
Developments in Matrix Orders have been growing alongside advances in Matrix Generalized Inverses, with numerous applications during the past years. See for instance, [4], with the use of the Drazin inverse to study the theory of finite Markov chains, linear programming and computational applications. There are also several approaches to this problems, using purely linnear algebra techniques, (such us [9] and [17]), or working in the functional analysis frame, (see for example [3]).
Let us consider an arbitrary $(n \times n)-$matrix A with entries in a general field k. In [9, Theorem 2.2.21], it has been proven that A can be written as the sum of two matrices A_1 and A_2 such that

- $rk(A_1) = rk(A_1^2)$ (where rk denotes the rank of the matrix);

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- A_2 is nilpotent;
- $A_1 A_2 = A_2 A_1 = 0$.

The matrices A_1 and A_2 are called the core and nilpotent parts of A, respectively, and this decomposition is unique. This ideas where generalized to arbitrary vector spaces (in general, of infinite dimension, see [12],[13] for details), leading to a new theory of Core-Nilpotent endomorphisms. The main idea in this notes is to generalize the theory presented in [9] to Core-Nilpotent endomorphisms.

The paper is organized as follows. In Section 2, basic definitions and results of the theory of finite potent endomorphisms, Core-Nilpotent endomorphisms and Generalized Inverses are summarized recalling the statements of [12] and [13]. Section 3 deals with the characterization of the set of $\{g\}-$commuting inverses of a Core-Nilpotent endomorphism on an arbitrary vector space, containing an algorithm for computing the set of $\{g\}-$commuting inverses of a square matrix and an example with the explicit application of this algorithm. Section 4 is devoted to prove that the Sharp Order defines a partial order in the set of Core-Nilpotent endomorphisms, including several other interesting properties as the construction of the group inverse of a Core-Nilpotent endomorphism using two $\{g\}-$commuting inverses and an exhaustive study of the structure of the subspaces induced by the AST decomposition (which will be introduced later) for CN-endomorphisms of the endomorphisms involved in the order relation. Also, several examples are added showing that the generalization is not trivial.

As far as the author knows, the study of Matrix Orders on infinite vector spaces (using linear algebra techniques) is not present in the literature. This new approach should lead to a new way of understanding the theory as well as bringing out new useful properties. Finally, the author wants to point out that this generalizations are compatible with the theory exposed in [9] which has been used as a model for studying the topic.

2. Preliminaries

This section is added for the sake of completeness and to fix notation.

2.1. Finite Potent Endomorphisms

Let k be an arbitrary field and let V be a k-vector space. Let us now consider an endomorphism f of V. We say that f is "finite potent" if $f^n V$ is finite dimensional for some n. This definition was introduced by J. Tate in [15] as a basic tool for his elegant definition of Abstract Residues.

In 2007, M. Argerami, F. Szechtman and R. Tifenbach showed in [2] that an endomorphism f is finite potent if and only if V admits a f-invariant decomposition $V = U_f \oplus W_f$ such that $f_{|U_f}$ is nilpotent, W_f is finite dimensional and $f_{|W_f} : W_f \xrightarrow{\sim} W_f$ is an isomorphism.

Indeed, if $k[x]$ is the algebra of polynomials in the variable x with coefficients in k, we may view V as an $k[x]$-module via f, and the explicit definition of the above f-invariant subspaces of V is:

- $U_f = \{v \in V \text{ such that } f^m(v) = 0 \text{ for some } m\}$;
- $W_f = \left\{\begin{matrix} v \in V \text{ such that } p(f)(v) = 0 \text{ for some } p(x) \in k[x] \\ \text{relatively prime to } x \end{matrix}\right\}$.

If the annihilator polynomial of f is $x^m \cdot p(x)$ with $(x, p(x)) = 1$, then $U_f = \operatorname{Ker} f^m$ and $W_f = \operatorname{Ker} p(f)$.

Hence, this decomposition is unique. We shall call this decomposition the f-invariant AST-decomposition of V.

Moreover, we shall call "index of f", $i(f)$, to the nilpotent order of $f_{|U_f}$. One has that $i(f) = 0$ if and only if V is a finite-dimensional vector space and f is an automorphism.

2.2. Jordan Basis of a nilpotent endomorphism

Let V be a vector space over an arbitrary field k and let $g \in \operatorname{End}_k(V)$ be a nilpotent endomorphism. If n is the nilpotency index of g, according to the statements of [11], setting $U_i^g = \operatorname{Ker} g^i/[\operatorname{Ker} g^{i-1} + f(\operatorname{Ker} g^{i+1})]$ with $i \in \{1, 2, \dots, n\}$, $\mu_i(V, g) = \dim_k U_i^g$ and $S_{\mu_i(V,g)}$ a set such that $\# S_{\mu_i(V,g)} = \mu_i(V, g)$ with $S_{\mu_i(V,g)} \cap S_{\mu_j(V,g)} = \emptyset$ for all $i \neq j$, one has that there exists a family of vectors $\{v_{s_i}\}$ that determines a Jordan basis of g:

$$B = \bigcup_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq n}} \{v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\}\,. \tag{2.1}$$

Moreover, if we write $H_{s_i}^g = \langle v_{s_i}, g(v_{s_i}), \dots, g^{i-1}(v_{s_i})\rangle$, the basis B induces a decomposition

$$V = \bigoplus_{\substack{s_i \in S_{\mu_i(V,g)} \\ 1 \leq i \leq n}} H_{s_i}^g\,. \tag{2.2}$$

2.3. Core-Nilpotent Endomorphisms

Let us consider a square matrix A with entries in a general field k, the index of A, $i(A) \geq 0$, is the smallest integer such that $rk(A^{i(A)}) = rk(A^{i(A)+1})$. In [9, Theorem 2.2.21], it has been proven that A can be written as the sum of two matrices A_1 and A_2 such that

- $rk(A_1) = rk(A_1^2)$ i.e. $i(A_1) \leq 1$;
- A_2 is nilpotent;
- $A_1 A_2 = A_2 A_1 = 0$.

The matrices A_1 and A_2 are called the core and nilpotent parts of A, respectively, and this decomposition is unique.

If k is a field, given a finite-dimensional k-vector space E we can define the index of an endomorphism $f \in \operatorname{End}_k(V)$, $i(f)$, as the smallest integer such that $\operatorname{Im} f^{i(f)} = \operatorname{Im} f^{i(f)+1}$. Bearing in mind the well known relationship between endomorphisms and matrices one can easily translate the previous theory to endomorphisms on a finite dimensional k-vector space.

Let us now consider an arbitrary k-vector space V (in general, infinite dimensional). For a finite potent endomorphism $f \in \operatorname{End}_k(V)$ the index is the smallest integer such that $dim_k \operatorname{Im} f^{i(f)} = dim_k \operatorname{Im} f^{i(f)+1} < \infty$, and this coincides with the nilpotency order of $f_{|U_f}$ where $V = W_f \oplus U_f$ is the AST-decomposition induced by f. In [13], the author showed that every finite potent endomorphism admits a unique core-nilpotent decomposition. Moreover, in [12], the general theory of core-nilpotent endomorphisms of arbitrary vector spaces was developed.

Definition 2.1. *We say that an endomorphism $f \in \operatorname{End}_k(V)$ is core-nilpotent (CN-endomorphism) when there exists two endomorphisms $f_1, f_2 \in \operatorname{End}_k(V)$ such that*

- $f = f_1 + f_2$;
- $i(f_1) \leq 1$;
- f_2 *is nilpotent;*
- $f_1 \circ f_2 = f_2 \circ f_1 = 0$.

Basic examples of CN-endomorphisms are all endomorphisms of a finite-dimensional vector space, finite potent endomorphisms, automorphisms and nilpotent endomorphisms of infinite-dimensional vector spaces. Several results were given, as the characterization of CN-endomorphisms:

Theorem 2.2. *If $f \in \operatorname{End}_k(V)$, then the following conditions are equivalent:*

1. *f is a CN-endomorphism.*
2. $\operatorname{Ker} f^n = \operatorname{Ker} f^{n+1}$ *and* $\operatorname{Im} f^n = \operatorname{Im} f^{n+1}$ *for a certain* $n \in \mathbb{N}$.
3. $V = \operatorname{Ker} f^n \oplus \operatorname{Im} f^n$ *for a certain* $n \in \mathbb{N}$.
4. *There exists a unique decomposition $V = W_f \oplus U_f$, where W_f and U_f are f-invariant k-subspaces of V, $f_{|W_f} \in Aut_k(W_f)$ and $f_{|U_f}$ is nilpotent.*

Obtaining as a corollary that the core-nilpotent decomposition of a CN-endomorphism is unique. From this characterization, several properties of Core-Nilpotent endomorphisms were studied, in particular, bearing in mind the following definition:

Definition 2.3. *An endomorphism $f \in \operatorname{End}_k(V)$ has index 0 when $f \in Aut_k(V)$. Given $n \in \mathbb{N}$, an endomorphism $f \in \operatorname{End}_k(V)$ has index n when* $\operatorname{Ker} f^n \neq \operatorname{Ker} f^{n-1}$ *or* $\operatorname{Im} f^n \neq \operatorname{Im} f^{n-1}$, $\operatorname{Ker} f^n = \operatorname{Ker} f^{n+1}$ *and* $\operatorname{Im} f^n = \operatorname{Im} f^{n+1}$. *The index of an endomorphism f is denoted $i(f)$.*

It was shown that:

Lemma 2.4. *The index of an endomorphism $f \in \operatorname{End}_k(V)$ exists if and only if f is a Core-Nilpotent endomorphism.*

2.4. Generalized Inverses.

Let k be a ground field and V an arbitrary k-vector space (in general, infinite-dimensional).

Definition 2.5. *An endomorphism $f^- \in \operatorname{End}_k(V)$ is a $\{1\}$-inverse of $f \in \operatorname{End}_k(V)$ when*

$$f \circ f^- \circ f = f\,.$$

Definition 2.6. *An endomorphism $f^- \in \operatorname{End}_k(V)$ is a $\{2\}$-inverse of $f \in \operatorname{End}_k(V)$ when f is a $\{1\}$-inverse of f^-, that is:*

$$f^- \circ f \circ f^- = f^-\,.$$

Definition 2.7. *An endomorphism $f^- \in \operatorname{End}_k(V)$ is a reflexive generalized inverse of $f \in \operatorname{End}_k(V)$ when it is a $\{1\}$-inverse and a $\{2\}$-inverse of f simultaneously.*

Definition 2.8. *An endomorphism $f \in \operatorname{End}_k(V)$ has a Drazin inverse when there exists $f^D \in \operatorname{End}_k(V)$ such that :*

1. $f^{m+1} \circ f^D = f^m$;
2. $f^D \circ f \circ f^D = f^D$;
3. $f^D \circ f = f \circ f^D$;

for a certain non-negative integer m.

Definition 2.9. *An endomorphism $f^\# \in \operatorname{End}_k(V)$ is a group inverse of f when it verifies that:*

1. $f \circ f^\# \circ f = f$;
2. $f^\# \circ f \circ f^\# = f^\#$;
3. $f \circ f^\# = f^\# \circ f$.

This is, $f^\#$ is a group inverse when it is a commuting reflexive generalized inverse of f.

In [12], the author showed that, with the previous notation, an endomorphism $f \in \operatorname{End}_k(V)$ has a Drazin inverse if and only if f is a CN-endomorphism (with $m = i(f)$). Also, that the group inverse of $f \in \operatorname{End}_k(V)$ exists if and only if f is a CN-endomorphism with $i(f) \leq 1$ and $f^\# = f^D$. Thus, showing that when the group inverse exists, then it is unique.

3. Study of the $\{g\}$-commuting inverses of Core-Nilpotent endomorphisms.

The aim of this section is to characterize the set of $\{g\}-$commuting inverses of a Core-Nilpotent endomorphism, this is, to state explicitely the conditions required for them to exist and their expression.

Let V be a k-vector space and let $f \in \operatorname{End}_k(V)$ be an arbitrary endomorphism.

Definition 3.1. *An endomorphism $g_{-} \in \operatorname{End}_k(V)$ is a $\{g\}$-conmmuting inverse of f when*

$$\begin{aligned} f \circ g_{-} \circ f &= f \\ f \circ g_{-} &= g_{-} \circ f. \end{aligned}$$

One can briefly describe this $\{g\}$-commuting inverses as $\{1\}$-inverses of f that conmmute with f.

We are going to characterize all the endomorphisms for which there exists a $\{g\}$-comuting inverse.

Lemma 3.2. *Given an endomorphism $f \in \operatorname{End}_k(V)$ such that $V = \operatorname{Ker} f^n \oplus \operatorname{Im} f^n$ for a certain $n \in \mathbb{N}$, then $f_{|\operatorname{Im} f^n} \in Aut_k(\operatorname{Im} f^n)$.*

Proof. Given $v \in \operatorname{Im} f^n$ with $v = f^n(v')$ for $v' \in V$, then $f(v) = f(f^n(v')) = f^n(f(v')) \in \operatorname{Im} f^n$, so the k-subspace $\operatorname{Im} f^n$ is f invariant. Therefore, we can consider the linear map

$$f_{|\operatorname{Im} f^n} : \operatorname{Im} f^n \to \operatorname{Im} f^n.$$

Since $\operatorname{Im} f^n \cap \operatorname{Ker} f^n = \{0\}$, then $\operatorname{Im} f^n \cap \operatorname{Ker} f = \{0\}$ and $\operatorname{Ker}(f_{|\operatorname{Im} f^n}) = \{0\}$, deducing that $f_{|\operatorname{Im} f^n}$ is injective. If $\tilde{v} \in \operatorname{Im} f^n$ with $\tilde{v} = f^n(v)$, writing $v = v_1 + f^n(v_2)$ with $v_1 \in \operatorname{Ker} f^n$ and $f^n(v_2) \in \operatorname{Im} f^n$, one has that

$$\tilde{v} = f^n(v) = f^n(f^n(v_2))$$

which shows that $f_{|\operatorname{Im} f^n}$ is surjective. Hence, $f_{|\operatorname{Im} f^n} \in Aut_k(\operatorname{Im} f^n)$ as we wanted to see. □

Theorem 3.3. *Given $f \in \operatorname{End}_k(V)$. Then g_{-} exists if and only if $V = \operatorname{Ker} f \oplus \operatorname{Im} f$.*
In this case, both $\operatorname{Ker} f$ *and* $\operatorname{Im} f$ *are (g_{-})-invariant and g_{-} is characterized by*

$$g_{-}(v) = \begin{cases} (f_{|\operatorname{Im} f})^{-1}(v) & if \quad v \in \operatorname{Im} f \\ \tilde{v} & if \quad v \in \operatorname{Ker} f \end{cases}$$

with $\tilde{v} \in \operatorname{Ker} f$.

Proof. Firstly, let us assume that g_{-} exists. Given $v \in V$ then

$$f(f(g_{-}(v)) - v) = 0 \Longrightarrow f(g_{-}(v)) - v \in \operatorname{Ker} f \Longrightarrow f(g_{-}(v)) = v + \tilde{v}$$

with $\tilde{v} \in \operatorname{Ker} f$, so we have that

$$V = \operatorname{Ker} f + \operatorname{Im} f.$$

Lets now consider $v \in \operatorname{Ker} f \cap \operatorname{Im} f$. As $v \in \operatorname{Im} f$, writing $v = f(\bar{u})$ we have

$$v = f(\bar{u}) = f(g_{-}(f(\bar{u}))) = g_{-}(f(v)) = g_{-}(0) = 0.$$

Therefore, it is

$$V = \operatorname{Ker} f \oplus \operatorname{Im} f.$$

Lets check this decomposition is (g_{-})-invariant.
If $u \in \operatorname{Ker} f$ then

$$f(g_{-}(u)) = g_{-}(f(u)) = g_{-}(0) = 0.$$

If $w \in \operatorname{Im} f$ we can write $w = f(w')$ with $w' \in V$ and then

$$g_-(w) = g_-(f(w')) = f(g_-(w')) \in \operatorname{Im} f.$$

Let us now determine the explicit expression of g_-.
Given $v \in \operatorname{Im} f$, since $f^2(g_-(v)) = f(v)$ we have that

$$f(g_-(v)) - v \in \operatorname{Ker} f \Longrightarrow f(g_-(v)) = v + \tilde{v} \Longrightarrow g_-(f(v)) = v + \tilde{v}$$

with $\tilde{v} \in \operatorname{Ker} f$, but it must be $\tilde{v} = 0$ because $(g_-)_{|\operatorname{Im} f} \subseteq \operatorname{Im} f$. So, using the previous lemma for $n = 1$, it is

$$g_-(v) = (f_{|\operatorname{Im} f})^{-1}(v)$$

getting the expression we looked for.
If $v \in \operatorname{Ker} f$

$$f^2(g_-(v)) = g_-(f^2(v)) = g(0) = 0.$$

Therefore we have that $g_-(v) \in \operatorname{Ker} f^2$. Lets now see that $\operatorname{Ker} f^2 = \operatorname{Ker} f$.
The inclusion $\operatorname{Ker} f \subseteq \operatorname{Ker} f^2$ is inmediate for all linear maps. Given $v \in \operatorname{Ker} f^2$, due to Definition 3.1, we have that

$$f(v) = f(g_-(f(v)) = g_-(f^2(v)) = g_-(0) = 0.$$

Then $v \in \operatorname{Ker} f$ proving that $\operatorname{Ker} f^2 \subseteq \operatorname{Ker} f$.
Conversely, let us prove the existence of that g_- from the assumptions made. According to Definition 3.1 we shall show the existence of some $g_- \in \operatorname{End}_k(V)$ satisfying

$$f \circ g_- \circ f = f$$

$$f \circ g_- = g_- \circ f.$$

Assuming $V = \operatorname{Ker} f \oplus \operatorname{Im} f$, using the previous lemma for $n = 1$, we can define $\bar{g} \in \operatorname{End}_k(V)$ such that

$$\bar{g}(v) = \begin{cases} (f_{|\operatorname{Im} f})^{-1}(v) & if \quad v \in \operatorname{Im} f \\ \tilde{v} & if \quad v \in \operatorname{Ker} f \end{cases}$$

with $\tilde{v} \in \operatorname{Ker} f$. It is clear that with this definition, both $\operatorname{Ker} f$ and $\operatorname{Im} f$ are $\bar{g}$−invariant.
If $v \in \operatorname{Im} f$ then

- $f(\bar{g}(f(v)) = f((f_{|\operatorname{Im} f})^{-1}(f(v)) = f(v);$
- $f(\bar{g}(v)) = f((f_{|\operatorname{Im} f})^{-1}(v)) = v$ and $\bar{g}(f(v)) = (f_{|\operatorname{Im} f})^{-1}(f(v)) = v.$

If $v \in \operatorname{Ker} f$ then, bearing in mind that $\operatorname{Ker} f$ is $\bar{g}$−invariant we get $f(g_-(v)) = 0$. The rest of the possible cases are trivial.
Thus taking $\bar{g} = g_-$ the claim is proved. □

Definition 3.4. *We will say that an endomorphism $f \in \operatorname{End}_k(V)$ is admissible for a g-inverse when there exists a $\{g\}$-commuting inverse of f in the sense of Theorem 3.3. This can be stated analogously by saying that $f \in \operatorname{End}_k(V)$ is admissible for a $\{g\}$-inverse when there exists a decomposition $V = \operatorname{Ker} f \oplus \operatorname{Im} f$.*

Corollary 3.5. *An arbitrary endomorphism $f \in \operatorname{End}_k(V)$ admits a $\{g\}$ −commuting inverse if and only if f is a CN-endomorphism of $i(f) \leq 1$.*

Corollary 3.6. *If we consider the subset*

$$X^-_{fp}(V) = \{\textit{finite potent endomorphisms of index} \leq 1\} \subset \operatorname{End}_k(V)$$

then $\{g\}$-commuting inverses exist for every element of $X^-_{fp}(V)$.

Corollary 3.7. *Let $f, g_- \in \mathrm{End}_k(V)$, where f is a Core Nilpotent endomorphism with $i(f) \leq 1$ and g_- is a $\{g\}$-commuting inverse of f. If $V = W_f \oplus U_f = \mathrm{Im}\, f \oplus \mathrm{Ker}\, f$ is the AST-decomposition induced by f on V then:*

1. $f^D \circ g_- = g_- \circ f^D$.
   Moreover, if g_- is a Core-Nilpotent endomorphism then $f^D \circ (g_-)^D = (g_-)^D \circ f^D$. Where f^D and $(g_-)^D$ stand for the Drazin inverse of f and g_- respectively.
2. *If $f = f_1 + f_2$ is the core-nilpotent decomposition of f then:*

$$f_1 \circ g_- = g_- \circ f_1$$

$$f_2 \circ g_- = g_- \circ f_2.$$

Proof. It is obtained directly from the fact that

$$f^D(v) = \begin{cases} (f_{|W_f})^{-1}(v) & if \quad v \in W_f \\ 0 & if \quad v \in U_f \end{cases}$$

and the statement of Theorem 3.3. □

Remark 3.8. *I wish to point out that Theorem 3.3 generalizes the notion of $\{g\}$-commuting inverse of a matrix that is presented in the works of Sujit Kumar Mitra, P. Bhimasankaram and Saroj B. Malik in [9] which is used to study the sharp order.*

3.1. Algorithm for the computation of $\{g\}$-commuting inverses of a square matrix

The goal of this section is to apply the previous results for the characterization of the set $A_{(C,-)}$ of $\{g\}$-commuting inverses of a finite square matrix A with entries in an arbitrary field k and to offer an algorithm for the explicit computation of $A_{(C,-)}$.

Let us consider a finite potent endomorphism $f \in \mathrm{End}_k(V)$, and $g_- \in \mathrm{End}_k(V)$ as a $\{g\}$-commuting inverse of f. We know that the AST-decomposition induced by f can be written as

$$V = W_f \oplus U_f = \mathrm{Im}\, f \oplus \mathrm{Ker}\, f.$$

We also know that $i(f) \leq 1$ and denoting

$$B_V = B_{W_f} \cup B_{U_f} = B_{\mathrm{Im}\, f} \cup B_{\mathrm{Ker}\, f},$$

using the notation introduced in Section 2.2.2, a basis induced from the AST-decomposition can be written as

$$B_{\mathrm{Im}\, f} = \{w_1, \dots, w_r\},$$

being a basis of $\mathrm{Im}\, f$ where $r = \dim_k \mathrm{Im}\, f$, and

$$B_{\mathrm{Ker}\, f} = \bigcup_{s_1 \in S_{\mu_1(\mathrm{Ker}\, f, f)}} \{v_1\},$$

being a basis of $\mathrm{Ker}\, f$ in our conditions.

Hence, if we denote by $X_f(g_-)$ the set of all $\{g\}$- commuting inverses of a finite potent endomorphism $f \in \mathrm{End}_k(V)$, from the previous characterization of g_- we obtain the existence of a bijection

$$X_f(g_-) \simeq \prod_{\# S_{\mu_1(\mathrm{Ker}\, f, f)}} \mathrm{Ker}\, f\,. \tag{3.1}$$

Proof. Using Theorem 3.3 we have the map

$$\begin{aligned} X_f(g_-) &\longrightarrow \prod_{\#S_{\mu_1(\operatorname{Ker} f, f)}} \operatorname{Ker} f \\ g_- &\longmapsto (g_-(v_{s_1}))_{s_1 \in S_{\mu_1}(U_f, f)} \end{aligned}$$

which is clearly a bijection. □

Let $A \in Mat_{m\times m}(k)$ be now an square matrix with entries in an arbitrary field k. Bearing in mind (3.1) and the previous discussion and denoting by $N(A)$ the nullspace of matrix A, we can determine the structure of $A_{(C,-)}$ from the following bijection :

$$A_{(C,-)} \simeq \prod_{dimN(A)} N(A)^{[dimN(A)]} \simeq k^{[dimN(A)]^2} \tag{3.2}$$

Proof. Bearing in mind the well-known relationship between finite square matrices and endomorphisms of finite-dimensional vector spaces, the statement is immediately deduced from the relation mentioned in (3.1). □

Indeed, the explicit algorithm for computing $A_- \in A_{(C,-)}$ is the following :

1. Write A in its Jordan canonical form: $A = PJP^{-1}$.
2. If $J = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$ with C invertible, compute the inverse C^{-1}.
3. Set $J' = \begin{pmatrix} C^{-1} & 0 \\ 0 & R \end{pmatrix}$, with R arbitrary.
4. Finally, it is
$$A_- = PJ'P^{-1}.$$

Notice that the expression of A_- depends on $[dimN(A)]^2$ parametres.

Example 1. We shall now characterize the set of $\{g\}$-commuting inverses of the matrix

$$A = \begin{pmatrix} 1 & -4 & 4 & -16 & -4 & 28 \\ 0 & 1 & 1 & 4 & 7 & 15 \\ 0 & 0 & 1 & 0 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in {}_{\mathrm{Mat}}6{\times}6(\mathbb{R})\,.$$

Since the Jordan canonical form of A is

$$A = \begin{pmatrix} -1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 & -3 & 5 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 4 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & -3 & 8 & -1 & -25 \\ 0 & 1 & 0 & 4 & 4 & 4 \\ 0 & 0 & -1 & 0 & -3 & -11 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

We now set

$$J' = \begin{pmatrix} 1 & -4 & -6 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & d & e & f \\ 0 & 0 & 0 & g & h & i \end{pmatrix}.$$

Applying the algorithm we obtain that A_- is a $\{g\}$-commuting inverse of A if and only if

$$A_- = \begin{pmatrix} -1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 & -3 & 5 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -4 & -6 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & d & e & f \\ 0 & 0 & 0 & g & h & i \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & -3 & 8 & -1 & -25 \\ 0 & 1 & 0 & 4 & 4 & 4 \\ 0 & 0 & -1 & 0 & -3 & -11 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 4 & -8 & 16 & -8 & -72 \\ 0 & 1 & -1 & -4a+4 & -4a+4b+1 & -4a+8b-4c-7 \\ 0 & 0 & 1 & 3d-5g & 3d-5g+5h-3e+3 & 3d+3f-5g+10h-6e+11-5i \\ 0 & 0 & 0 & a+d+g & a-b+d+g-h-e & a-2b+c+d+f+g-2h-2e+i \\ 0 & 0 & 0 & -d-2g & -d-2g+2h+e & -d-f-2g+4h+2e-2i \\ 0 & 0 & 0 & g & g-h & g-2h+i \end{pmatrix},$$

with $a, b, c, d, e, f, g, h, i \in \mathbb{R}$.

Bearing in mind that dim $N(A) = 3$, it is known from (3.2) that $A_{(C,-)} \simeq \mathbb{R}^9$, which is coherent with the computations made in this example.

4. Sharp partial order of Core-Nilpotent endomorphisms.

The main goal of this section is to generalize the theory presented in [9, Chapter 4], about the sharp order studied for matrices, to Core-Nilpotent endomorphisms.
Let us begin by introducing the sharp order for matrices.
Recall from [9, Chapter 2] that a matrix G is said to be a conmmuting g-inverse of a matrix A if

- $AGA = A$
- $AG = GA$

Moreover, such a commuting g-inverse for A exists if and only if A is of index ≤ 1. Let us denote $\mathcal{I}_{1,n}$ the set of all $(n \times n)$ matrices of index ≤ 1.

Definition 4.1. *Let $A, B \in \mathcal{I}_{1,n}$. A is said to be below B under the sharp order if there exist commuting g-inverses G_1 and G_2 of A such that $AG_1 = BG_1$ and $G_2A = G_2B$.*
When A is below B under the sharp order, we write $A <^{\#} B$.

Let V be an arbitrary k-vector space.

Definition 4.2. *Let us consider two Core-Nilpotent endomorphisms $f, h \in \mathrm{End}_k(V)$ with $i(f) = i(h) \leq 1$, both admissible for g-commuting inverses. We say that f is under h for the sharp order if and only if there exist Core-Nilpotent $\{g\}$−commuting inverses g_1 and g_2 of f such that $f \circ g_1 = h \circ g_1$ and $g_2 \circ f = g_2 \circ h$.*
When f is below h under the sharp order, we write $f <^{\#} h$.

Lemma 4.3. *If g_1, g_2 are g-commuting inverses of f, then*

$$g_1 \circ f \circ g_2 = f^{\#}$$

where $f^{\#}$ denotes the group inverse of f (Definition 2.9).

Proof. By the uniqueness of the group inverse, we show that $g_1 \circ f \circ g_2$ is a reflexive commuting $\{g\}$−inverse of f in the sense of Definition 2.7. This is obvious since

$$f \circ (g_1 \circ f \circ g_2) \circ f = (f \circ g_1 \circ f) \circ g_2 \circ f = f \circ g_2 \circ f = f;$$

also

$$\begin{aligned}(g_1 \circ f \circ g_2) \circ f \circ (g_1 \circ f \circ g_2) &= g_1 \circ (f \circ g_2 \circ f) \circ g_1 \circ f \circ g_2 \\ &= g_1 \circ (f \circ g_1 \circ f) \circ g_2 = g_1 \circ f \circ g_2;\end{aligned}$$

and finally, the conmmuting part is deduced from the fact that both g_1, g_2 are $\{g\}$ −commuting inverses of f so

$$(g_1 \circ f \circ g_2) \circ f = f \circ (g_1 \circ f \circ g_2)$$

and we have that $g_1 \circ f \circ g_2$ is a generalized conmmuting inverse of f as we wanted to see.
□

Corollary 4.4. *In particular, one has that $g_1 \circ f \circ g_2$ is a Core-Nilpotent endomorphism.*

Proposition 4.5. *Let $f, g \in \mathrm{End}_k(V)$ be two Core-Nilpotent endomorphisms . We have that $f <^{\#} h$ if and only if $f \circ f^{\#} = h \circ f^{\#}$ and $f^{\#} \circ f = f^{\#} \circ h$, with $f^{\#}$ being the group inverse of f.*

Proof. Let g_1 and g_2 be commuting g-inverses of f such that $f \circ g_1 = h \circ g_1$ and $g_2 \circ f = g_2 \circ h$. From the previous lemma we know that $g_1 \circ f \circ g_2 = f^{\#}$. Therefore, we have

$$f \circ f^{\#} = f \circ g_1 \circ f \circ g_2 = h \circ g_1 \circ f \circ g_2 = h \circ f^{\#}$$

and

$$f^{\#} \circ f = g_1 \circ f \circ g_2 \circ f = g_1 \circ f \circ g_2 \circ h = f^{\#} \circ h.$$

Conversely, as $f^{\#}$ is a commuting g-inverse of f it is trivial. □

Corollary 4.6. *Let us consider $f, h \in \mathrm{End}_k(V)$ two Core-Nilpotent endomorphisms with $i(f) = i(h) \leq 1$, then $f <^{\#} h$ if and only if $f^2 = f \circ h = h \circ f$.*

Proof. Let us suppose that $f <^{\#} h$. Using the previous theorem we have that $f \circ f^{\#} = h \circ f^{\#}$ and $f^{\#} \circ f = f^{\#} \circ h$. In fact, it is

$$f^2 \circ f^{\#} = f = f^{\#} \circ f^2$$

and then

$$f^2 = f \circ f^{\#} \circ f^2 = h \circ f^{\#} \circ f^2 = h \circ f.$$

Analogoulsy, one can prove that $f^2 = f \circ h$.
Conversely, it follows directly from

$$(f^{\#})^2 \circ f = f^{\#} = f \circ (f^{\#})^2,$$

that

$$f \circ f^{\#} = f^2 \circ (f^{\#})^2 = h \circ f \circ (f^{\#})^2 = h \circ f^{\#}$$

and

$$f^{\#} \circ f = (f^{\#})^2 \circ f^2 = (f^{\#})^2 \circ f \circ h = f^{\#} \circ h$$

and hence, the statement is proved. □

Corollary 4.7. *Note that, if $f <^{\#} h$, then $\mathrm{Im}(h - f) \in \mathrm{Ker}\, f$ and $\mathrm{Im}\, f \in \mathrm{Ker}(h - f)$.*

The objective now is to give the explicit relations in the k-subspaces of V of the decomposition induced by f and h when $f <^{\#} h$.

Proposition 4.8. *Let us consider $f, h \in \mathrm{End}_k(V)$ two Core-Nilpotent endomorphisms with $i(f) = i(h) \leq 1$ such that $f <^{\#} h$, then*

$$f|_{\mathrm{Im}\, f} = h|_{\mathrm{Im}\, f}.$$

Proof. As $f <^{\#} h$ then we have $f \circ g_- = h \circ g_-$ for some Core-Nilpotent $\{g\}$-commuting inverse of f. Given $v \in \mathrm{Im}\, f$ using Theorem 3.3 it is $h(g_-(v)) = f(g_-(v)) = v$ and therefore $h((f_{|\mathrm{Im}\, f})^{-1}(v)) = v$ and $h = f$ when restricting to $\mathrm{Im}\, f$ as we wanted to see. □

Lemma 4.9. *Let us consider $f, h \in \mathrm{End}_k(V)$ two Core-Nilpotent endomorphisms with $i(f) = i(h) \leq 1$ such that $f <^{\#} h$, then $\mathrm{Ker}\, f$ and $\mathrm{Im}\, f$ are both invariant under h.*

Proof. Let us consider $v \in \mathrm{Ker}\, f$, as $f <^{\#} h$ then we have $g_-(h(v)) = g_-(f(v)) = 0$, for some Core-Nilpotent $\{g\}$-commuting inverse of f. This is $g_-(h(v)) = 0$ and therefore $h(v) \in \mathrm{Ker}\, g_- \subsetneq \mathrm{Ker}\, f$ and the first statement is proved.
Let us now consider $v \in \mathrm{Im}\, f$, this is $v = f(v')$ for some $v' \in V$. As $f <^{\#} h$, using Corollary 4.6 we have that

$$h(v) = h(f(v')) = f(h(v')) \in \mathrm{Im}\, f$$

as we wanted to see. □

Lemma 4.10. *Let us consider $f, h \in \mathrm{End}_k(V)$ two Core-Nilpotent endomorphisms with $i(f) = i(h) \leq 1$ such that $f <^{\#} h$, then*

$$\mathrm{Ker}\, h \subseteq \mathrm{Ker}\, f.$$

Proof. Let us consider $v \in \mathrm{Ker}\, h$. Then, by Corollary 4.6 it is

$$f^2(v) = h(f(v)) = f(h(v)) = f(0) = 0.$$

This shows that $\mathrm{Ker}\, h \subseteq \mathrm{Ker}\, f^2 = \mathrm{Ker}\, f$, as f is a Core-Nilpotent endomorphism of $i(f) \leq 1$. □

Lemma 4.11. *Let us consider $f, h \in \mathrm{End}_k(V)$ two Core-Nilpotent endomorphisms with $i(f) = i(h) \leq 1$ such that $f <^{\#} h$, then*

$$\mathrm{Im}\, f \subseteq \mathrm{Im}\, h.$$

Proof. Let us consider $u \in \mathrm{Im}\, f$ with $u = f(u')$ for some $u' \in V$. We can write $u' = u'_1 + u'_2$ with $u'_1 \in \mathrm{Ker}\, h$ and $u'_2 \in \mathrm{Im}\, h$ according to the decomposition of V induced by h. Hence, using Corollary 4.6 and Lemma 4.10 it is

$$u = f(u') = f(u'_1) + f(u'_2) = f(h(\bar{u})) = h(f(\bar{u})) \in \mathrm{Im}\, h$$

with $u'_2 = h(\bar{u}) \in \mathrm{Im}\, h$, as we wanted to see. □

Lemma 4.12. *Let us consider $f, h \in \mathrm{End}_k(V)$ two Core-Nilpotent endomorphisms with $i(f) = i(h) \leq 1$ such that $f <^{\#} h$, then*

$$\mathrm{Im}\, h \cap \mathrm{Ker}\, f$$

is invariant under h.

Proof. For the $\mathrm{Im}\, h$ there is nothing to say and $\mathrm{Ker}\, f$ is also invariant under h according to Lemma 4.9. □

Proposition 4.13. *Let us consider two Core-Nilpotent endomorphisms $f, h \in \mathrm{End}_k(V)$ with $i(f) = i(h) \leq 1$ such that $f <^{\#} h$, then*

$$\mathrm{Ker}\, f = \mathrm{Ker}\, h \oplus (\mathrm{Im}\, h \cap \mathrm{Ker}\, f).$$

Proof. We have to prove that $\mathrm{Ker}\, f = \mathrm{Ker}\, h + (\mathrm{Im}\, h \cap \mathrm{Ker}\, f)$ and that $\{0\} = \mathrm{Ker}\, h \cap (\mathrm{Im}\, h \cap \mathrm{Ker}\, f)$. Let us consider $v \in \mathrm{Ker}\, f$ and let us express this v as $v = v_1 + v_2$ with $v_1 \in \mathrm{Ker}\, h$ and $v_2 \in \mathrm{Im}\, h$ using the decomposition of V induced by h. Then, we can write $v_2 = v - v_1 \in \mathrm{Ker}\, f$ using Lemma 4.10 ($v_2 \in \mathrm{Ker}\, h \subseteq \mathrm{Ker}\, f$), which shows that $v_2 \in \mathrm{Ker}\, f \cap \mathrm{Im}\, h$. As $v_1 \in \mathrm{Ker}\, h$ and $v_2 \in \mathrm{Ker}\, f \cap \mathrm{Im}\, h$ we conclude that $\mathrm{Ker}\, f = \mathrm{Ker}\, h + (\mathrm{Im}\, h \cap \mathrm{Ker}\, f)$. The other claim is trivial as we already know that $\mathrm{Ker}\, h \cap \mathrm{Im}\, h = \{0\}$, so $\mathrm{Ker}\, f = \mathrm{Ker}\, h \oplus (\mathrm{Im}\, h \cap \mathrm{Ker}\, f)$ as we wanted to prove. □

Proposition 4.14. *Let us consider two Core-Nilpotent endomorphisms $f, h \in \mathrm{End}_k(V)$ with $i(f) = i(h) \leq 1$ such that $f <^{\#} h$, then*

$$\mathrm{Im}\, h = \mathrm{Im}\, f \oplus (\mathrm{Im}\, h \cap \mathrm{Ker}\, f).$$

Proof. Again, we shall prove that $\mathrm{Im}\, h = \mathrm{Im}\, f + (\mathrm{Im}\, h \cap \mathrm{Ker}\, f)$ and that $\{0\} = \mathrm{Im}\, f \cap (\mathrm{Im}\, h \cap \mathrm{Ker}\, f)$. Let us consider $v \in \mathrm{Im}\, h$ and let us express this v as $v = v_1 + v_2$ with $v_1 \in \mathrm{Ker}\, f$ and $v_2 \in \mathrm{Im}\, f$, using the decomposition of V induced by f. Therefore, it is $v_1 = v - v_2 \in \mathrm{Im}\, h$ according to Lemma 4.11 ($v_2 \in \mathrm{Im}\, f \subseteq \mathrm{Im}\, h$). As $v_2 \in \mathrm{Im}\, f$ and $v_1 \in \mathrm{Ker}\, f \cap \mathrm{Im}\, h$, we have $\mathrm{Im}\, h = \mathrm{Im}\, f + (\mathrm{Im}\, h \cap \mathrm{Ker}\, f)$. Again, the second claim is trivial as we already know that $\mathrm{Ker}\, f \cap \mathrm{Im}\, f = \{0\}$. Accordingly, we have that

$$\mathrm{Im}\, f \cap (\mathrm{Ker}\, f \cap \mathrm{Im}\, h) \subseteq \mathrm{Ker}\, f \cap \mathrm{Im}\, f = \{0\}.$$

□

Lemma 4.15. *Let us consider $f, h \in \mathrm{End}_k(V)$ two Core-Nilpotent endomorphisms with $i(f) = i(h) \leq 1$ such that $f <^{\#} h$, then*

$$h : \mathrm{Im}\, h \cap \mathrm{Ker}\, f \longrightarrow \mathrm{Im}\, h \cap \mathrm{Ker}\, f$$

is an isomorphism of k-vector subspaces.

Proof. Let us consider $v \in (\operatorname{Im} h \cap \operatorname{Ker} f) \cap \operatorname{Ker} h \subseteq \operatorname{Im} h \cap \operatorname{Ker} h = \{0\}$ then $v = 0$ and injectivity is proved. Now, let us consider $\tilde{v} \in \operatorname{Im} h \cap \operatorname{Ker} f$. As $h_{|\operatorname{Im} h} \in Aut_k(\operatorname{Im} h)$, we know that there exists a unique $\tilde{v}' \in \operatorname{Im} h$ so that $h(\tilde{v}') = \tilde{v}$ and $f(\tilde{v}) = 0$. Hence, it is $f(h(\tilde{v}')) = 0$, this is $h(f(\tilde{v}')) = 0$ so, using Lemma 4.11, we get that $f(\tilde{v}') \in \operatorname{Ker} h \cap \operatorname{Im} h = \{0\}$ and the claim is deduced. □

After this discussion, considering $f <^{\#} h$ with $f, h \in \operatorname{End}_k(V)$ two Core-Nilpotent endomorphisms with $i(f) = i(h) \leq 1$, we are able to explicitely characterize the $\{g\}$−commuting inverses of h in terms of the decomposition of V induced by f. This will enable us to prove one of the main results of the paper, which is to give the relation between the $\{g\}$−commuting inverses of the endomorphisms that appear in the order. Let us consider $f <^{\#} h$ with $f, h \in \operatorname{End}_k(V)$ two Core-Nilpotent endomorphisms with $i(f) = i(h) \leq 1$, and let us denote $S = \operatorname{Im} h \cap \operatorname{Ker} f$. Firstly, according to Theorem 3.3 :

$$g_{h-}(v) = \begin{cases} (h_{|\operatorname{Im} h})^{-1}(v) & \text{if} \quad v \in \operatorname{Im} h \\ \tilde{v} & \text{if} \quad v \in \operatorname{Ker} h \end{cases}$$

with $\tilde{v} \in \operatorname{Ker} h$. Then, if $v \in \operatorname{Im} f \subseteq \operatorname{Im} h$ (Lemma 4.11), we know that $f^{-1}(v) = h^{-1}(v) \in \operatorname{Im} f$ by Lemma 4.8 and using that $f_{|\operatorname{Im} f} \in Aut_k(\operatorname{Im} f)$. Finally, using the explicit decomposition of $\operatorname{Im} h$ (Proposition 4.14) and the statement of Lema 4.15 we get :

$$g_{h-}(v) = \begin{cases} f^{-1}(v) & \text{if} \quad v \in \operatorname{Im} f \\ \tilde{v}' & \text{if} \quad v \in S \in \operatorname{Ker} f \\ \tilde{v} & \text{if} \quad v \in \operatorname{Ker} h \subseteq \operatorname{Ker} f \end{cases}$$

with $\tilde{v}' \in S \subseteq \operatorname{Ker} f$ and $\tilde{v} \in \operatorname{Ker} h \subseteq \operatorname{Ker} f$.
Remembering Theorem 3.3, one has inmediately the following

Corollary 4.16. *If $f <^{\#} h$, then $X_h(g_-) \subseteq X_f(g_-)$.*

Corollary 4.17. *If $f <^{\#} h$, then $h^{\#} \in X_f(g_-)$.*

Theorem 4.18. *The relation "$<^{\#}$" (Definition 4.2) defines a partial order in the set of Core-Nilpotent endomorphisms of index ≤ 1.*

Proof. Reflexivity holds trivially. In order to prove anti-symmetry let us consider two Core-Nilpotent endomorphisms $f, h \in \operatorname{End}_k(V)$ such that $f <^{\#} h$ and $h <^{\#} f$. Then, according to Corollary 4.16, it is $X_h(g_-) = X_f(g_-)$ and there exists $g_- \in X_h(g_-) = X_f(g_-)$ satisfying that

$$f = f \circ g_- \circ f = h \circ g_- \circ f = h \circ g_- \circ h = h$$

as we wanted to see.
Let us now fix three Core-Nilpotent endomorphisms $f, h, t \in \operatorname{End}_k(V)$ such that $f <^{\#} h$ and $h <^{\#} t$. Therefore, we can write $f \circ g_f = h \circ g_f$ and $h \circ g_h = t \circ g_h$. We have to construct $\{g\}$−commuting inverses of f such that $f \circ g = t \circ g$ and $g' \circ f = g' \circ t$. According to Lemma 4.3, $g_h \circ f \circ g_f$ is, in particular, a Core-Nilpotent endomorphism and a $\{g\}$− commuting inverse of f, bearing in mind that $X_h(g_-) \subseteq X_f(g_-)$ as $f <^{\#} h$. Using that $h <^{\#} t$, the definition of $\{g\}$−commuting inverse and Corollary 4.6 we get the chain of equalities:

$$t \circ g_h \circ f \circ g_f = h \circ g_h \circ f \circ g_f = g_h \circ h \circ f \circ g_f = g_h \circ f \circ h \circ g_f = f \circ g_h \circ h \circ g_f .$$

Finally, as $f <^{\#} h$ it is

$$f \circ g_h \circ h \circ g_f = f \circ g_h \circ f \circ g_f .$$

So, taking $g = g_h \circ f \circ g_f$ we have as desired. For the named g' an analogous reasoning shows that $g' = g_f \circ f \circ g_h$ works and we conclude. □

We give now a series of equivalences that come from the previous study of the Sharp Order, similar to the ones in [9, Theorem 4.2.12.].

Definition 4.19. *A Core-Nilpotent endomorphism $f : V \to V$ is a projector when $f^2 = f$.*

Theorem 4.20. *Let us consider $f, h \in \mathrm{End}_k(V)$ two Core-Nilpotent endomorphisms such that $f <^{\#} h$, then the following statements are equivalent :*

1. $f <^{\#} h$.
2. $f^2 = f \circ h = h \circ f$ *or equivalently* $f \circ (h - f) = (h - f) \circ f = 0$.
3. *There exists a projector P such that $f = P \circ h = h \circ P$.*
4. $f^{\#} \circ f = f^{\#} \circ h$ *and* $(h - f) \circ f = 0$.
5. $f \circ f^{\#} = h \circ f^{\#}$ *and* $f \circ (h - f) = 0$.

Proof. 1) $\Longleftrightarrow$ 2) It is Corollary 4.6.
2) $\Longrightarrow$ 3) We have $f = f \circ f^{\#} \circ f = f^{\#} \circ f^2 = f^{\#} \circ f \circ h$. Let us define $P = f^{\#} \circ f$. Then, P is a projector since $f^{\#} \circ f \circ f^{\#} \circ f = f^{\#} \circ f$ and we get $f = P \circ h$. Now, $h \circ P = h \circ f^{\#} \circ f = h \circ f \circ f^{\#} = f^2 \circ f^{\#} = f$ so we conclude.
3) $\Longrightarrow$ 4) Notice that $f \circ P = h \circ P^2 = h \circ P = f$ and that $P \circ f = P^2 \circ h = P \circ h = f$, using the definition of projector and the relation given by (3). Since $\mathrm{Im}\, f^{\#} = \mathrm{Im}\, f$, we have that $f^{\#} = f^{\#} \circ P = P \circ f^{\#}$. Now, $f^{\#} \circ f = f^{\#} \circ P \circ h = f^{\#} \circ h$. As $h \circ P = f \circ P$, then $(h - f) \circ P = 0$. Hence, $(h - f) \circ f = (h - f) \circ P \circ h = 0$ and the statement is proved.
4) $\Longrightarrow$ 2) Since $(h - f) \circ f = 0$ we get that $h \circ f = f^2$. As we have $f \circ f^{\#} = f^{\#} \circ h$, multiplying by f^2 it is $f^2 = f^2 \circ f^{\#} \circ f = f^2 \circ f^{\#} \circ h$. Now, from $f^2 = f^2 \circ f^{\#} \circ h$ we deduce that $f^2 = f \circ h$ as we wanted.
3) $\Longrightarrow$ 5) $\Longrightarrow$ 2) It is analogous to 3) $\Longrightarrow$ 4) $\Longrightarrow$ 2). □

Lemma 4.21. *Let us consider $f, h \in \mathrm{End}_k(V)$ two Core-Nilpotent endomorphism with $i(f) = i(h) \leq 1$. If f is a projector, then $f <^{\#} h$ if and only if $h \in X_f(g_\_)$.*

Proof. If $f <^{\#} h$, using Corollary 4.6 we get inmediately that $f^2 = f \circ h = h \circ f$, in particular $f \circ h = h \circ f$. As f is a projector, then $f \circ h \circ f = f^2 = f$ and the first claim is proved. Conversely, if $h \in X_f(g_\_)$, again by Corollary 4.6 we know that it is sufficient to prove that $f^2 = f \circ h = h \circ f$. The fact that $f \circ h = h \circ f$ is obvious since $h \in X_f(g_\_)$. Finally, $f^2 = f \circ h \circ f \circ f = h \circ f \circ f \circ f = h \circ f \circ f = h \circ f$ using that f is a projector and concluding the proof. □

Remark 4.22. *There are several parts along this chapter were, in order to prove a certain statement, it is used that the set of $\{g\}-$commuting inverses of a CN-endomorphism of index ≤ 1 is not empty. This is easily deduced from the following results.*
Firstly, recall that from [12, Proposition 4.2], the Drazin inverse of a CN-endomorphism on an infinite-dimensional k-vector space exists and it is unique. Also , according to [14, Theorem 3.5], we know that if $f \in \mathrm{End}_k(V)$ is a CN-endomorphism with $i(f) \leq 1$, which is our case in this section, then $f^D = f^{\#}$ is the unique group inverse of f, where f^D is its Drazin inverse. This enables us to know that, at least, the set $X_f(g_\_)$ of $\{g\}-$commuting inverses of a CN-endomorphism f, is not empty. In fact we can reformulate the theory for the sharp order as follows $f <^{\#} h$ if and only if $f \circ f^D = h \circ f^D$ and $f^D \circ f = f^D \circ h$. This is, $f <^{\#} h$ if and only if the Drazin inverse of f, f^D, conmutes with h.

Finally, let us include several examples that show that the generalization of the sharp order for matrices to CN-endomorphisms is not trivial.
Firstly, let us consider a $\mathbb{R}$-vector space of countable dimension

$$V = \oplus_{i \in \mathbb{N}} < v_i > .$$

Then, the endomorphisms $f, h : V \to V$ defined as:

$$f(v_i) = \begin{cases} v_1 + 2v_2 & \text{if} \quad i = 1 \\ 3v_1 + 4v_2 & \text{if} \quad i = 2 \\ 0 & \text{if} \quad i > 2 \end{cases}, h(v_i) = \begin{cases} v_1 + 2v_2 & \text{if} \quad i = 1 \\ 3v_1 + 4v_2 & \text{if} \quad i = 2 \\ \lambda v_3 & \text{if} \quad i = 3 \\ 0 & \text{if} \quad i = 2j \\ \lambda v_i & \text{if} \quad i = 2j+1 \end{cases}$$

for all $i \in \mathbb{N}$, $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ and $j > 1$, are CN-endomorphisms which verify that $f <^{\#} h$. This fact can be easily seen using, for example, Corollary 4.6, because

$$f^2(v_i) = f(h(v_i)) = h(f(v_i)) = \begin{cases} 7v_1 + 10v_2 & \text{if} \quad i = 1 \\ 15v_1 + 22v_2 & \text{if} \quad i = 2 \\ 0 & \text{if} \quad i > 2 \end{cases}$$

for all $i \in \mathbb{N}$.
Let us now fix a Core-Nilpotent endomorphism $f \in \mathrm{End}_k(V)$. We shall compute explicitely all the Core-Nilpotent endomorphisms $h \in \mathrm{End}_k(V)$ such that $f <^{\#} h$.
Considering an $\mathbb{R}-$vector space of countable dimension

$$V = \oplus_{i \in \mathbb{N}} < v_i >,$$

we define $f : V \to V$ as:

$$f(v_i) = \begin{cases} v_i & \text{if} \quad \text{i is odd} \\ 0 & \text{if} \quad \text{i is even} \end{cases}.$$

It is easy to see that f is a Core-Nilpotent endomorphism of $i(f) \leq 1$. If we write V as

$$V = < v_{2i+1} >_{i \in \mathbb{N} \cup \{0\}} \oplus < v_{2i} >_{i \in \mathbb{N}} .$$

Clearly, $\mathrm{Im}\, f = < v_{2i+1} >_{i \in \mathbb{N} \cup \{0\}}$ and $\mathrm{Ker}\, f = < v_{2i} >_{i \in \mathbb{N}}$, and therefore

$$V = \mathrm{Ker}\, f \oplus \mathrm{Im}\, f,$$

so by Theorem 2.2 we deduce the previous statement. Notice that

$$f^2(v_i) = f(v_i)$$

for all $i \in \mathbb{N}$, so f is a projector. Using Lema 4.21 and the characterization of the $\{g\}-$commuting inverses of h in terms of the decomposition of V induced by f we conclude our task and the expression of all the h´s above f is:

$$h(v_i) = \begin{cases} (f_{|\mathrm{Im}\, f})^{-1}(v_i) = v_i & \text{if} \quad \text{i is odd} \\ \sum_j \lambda_{j_i} v_{2_{j_i}} & \text{if} \quad \text{i is even} \end{cases},$$

with $\sum_j \lambda_{j_i} v_{2_{j_i}}$ being finite sums with $\lambda_{j_i} \in \mathbb{R}$ and $j \in \mathbb{N}$.

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