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Characterizations and representations of w-core inverses in rings

Qi Zhang[a], Chengcheng Wang[b], Huihui Zhu[b,*]

[a]*College of Mathematics and Computer Science, Tongling University, Tongling 244000, China.*
[b]*School of Mathematics, Hefei University of Technology, Hefei 230009, China.*

Abstract. Let R be an associate ring with involution and let $a, w \in R$. The notion of EI along an element is introduced. An element w is called EI along a if $w^{\|a}$ exists and $w^{\|a}w = ww^{\|a}$. Its several characterizations are given by w-core inverses. Several necessary and sufficient conditions such that $a_w^{\text{\textcircled{\#}}}aw$ and $wa_w^{\text{\textcircled{\#}}}a$ are projections are derived. In particular, it is shown that $a_w^{\text{\textcircled{\#}}}aw$ is a projection if and only if aw is Moore-Penrose invertible with $(aw)^\dagger = a_w^{\text{\textcircled{\#}}}$ if and only if aw is group invertible with $(aw)^\# = a_w^{\text{\textcircled{\#}}}$. Also, $wa_w^{\text{\textcircled{\#}}}a$ is a projection if and only if a is Moore-Penrose invertible with $a^\dagger = wa_w^{\text{\textcircled{\#}}}$. Then, we describe the existence of w-core inverse of a by the existence of (the unique) projection $p \in R$ and idempotent $q \in R$ satisfying $pR = aR = awR = qR$ and $Rq = Raw$.

1. Introduction

Idempotents (projections) have strongly connections with the theory of generalized inverses, which have attracted much research in many branches of mathematics, including matrices, bounded linear operators, semigroups and more general setting of rings. Several papers [5, 9–11, 15, 18] investigated the related topic. Since we prefer to take a ring theoretic point, corresponding characterizations for idempotents (or projections) generated by the recently introduced w-core inverses will be presented in a general ring.

The paper is organized as follows. In Section 2, we present the notations and basic properties of several generalized inverses. In Section 3, we define and characterize EI along an element by idempotents. As an application, the well known existence criterion for EP elements is given. In Section 4, we determine when the idempotent of the form $a_w^{\text{\textcircled{\#}}}aw$ or $wa_w^{\text{\textcircled{\#}}}a$ is a projection. In particular, it is shown that $a_w^{\text{\textcircled{\#}}}aw$ is a projection if and only if aw is Moore-Penrose invertible with $(aw)^\dagger = a_w^{\text{\textcircled{\#}}}$ if and only if aw is group invertible with $(aw)^\# = a_w^{\text{\textcircled{\#}}}$. In this case, aw is EP. Also, $wa_w^{\text{\textcircled{\#}}}a$ is a projection if and only if a is Moore-Penrose invertible with $a^\dagger = wa_w^{\text{\textcircled{\#}}}$. Then, we describe the existence of the w-core inverse by the existence of (the unique) projection $p \in R$ and idempotent $q \in R$ satisfying $pR = aR = awR = qR$ and $Rq = Raw$. Finally, for any $a, w \in R$, it is shown that a is regular (resp., $\{1,3\}$-invertible, $\{1,4\}$-invertible, Moore-Penrose invertible, group invertible and core invertible) if and only if awa is regular (resp., $\{1,3\}$-invertible, $\{1,4\}$-invertible, Moore-Penrose invertible, group invertible and core invertible), provided that $w^{\|a}$ exists. Also, their expressions are given.

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* Corresponding author: Huihui Zhu
Email addresses: zhangqi830917@163.com (Qi Zhang), ccwmath@163.com (Chengcheng Wang), hhzhu@hfut.edu.cn (Huihui Zhu)

2. Preliminaries

In this section, we mainly introduce some notations, and give some fundamental results of generalized inverses, which will be useful in the sequent sections.

Let R be an associate ring with unity 1. We say that an element $a \in R$ is (von Neumann) regular if $a \in aRa$. Any $x \in R$ satisfying $a = axa$ is called an inner inverse of a, and is denoted by a^-. By R^- and $a\{1\}$ we denote the sets of all regular elements in R and all inner inverses of a, respectively.

Definition 2.1. [4] *Let R be an associate ring with unity 1. An element $a \in R$ is called Drazin invertible if there exist an element $b \in R$ and a non-negative integer k such that*

$$\text{(i) } a^k = a^{k+1}b, \text{ (ii) } bab = b, \text{ (iii) } ab = ba.$$

Any element b satisfying the above conditions (i)−(iii) *is unique if it exists, and is denoted by a^D. The smallest non-negative integer k is called the Drazin index of a and is denoted by* $\mathrm{ind}(a)$.

An element $a \in R$ is group invertible if it is Drazin invertible and $\mathrm{ind}(a) = 1$. The group inverse of a is denoted by $a^\#$. Moreover, $a \in R$ is group invertible if and only if $a \in a^2R \cap Ra^2$. In this case, $a^\# = yax = ax^2 = y^2a$, where $a = a^2x = ya^2$ for some $x, y \in R$. As usual, we denote by R^D and $R^\#$ the sets of all Drazin invertible and group invertible elements in R, respectively.

Let $*$ be an involution on R, that is the involution $*$ satisfying $(x^*)^* = x$, $(xy)^* = y^*x^*$ and $(x + y)^* = x^* + y^*$ for all $x, y \in R$. We call R a $*$-ring if there exists an involution on R. Throughout this paper, any ring R considered, unless otherwise noted, is assumed to be a $*$-ring.

Definition 2.2. [12] *Let $a \in R$. An element $a \in R$ is called Moore-Penrose invertible if there exists an $x \in R$ satisfying the following equations*

$$\text{(i) } axa = a, \quad \text{(ii) } xax = x, \quad \text{(iii) } (ax)^* = ax, \quad \text{(iv) } (xa)^* = xa.$$

Any element x satisfying the equations (i)−(iv) *is called the Moore-Penrose inverse of a. It is unique if it exists, and is denoted by $a^\dagger$.*

We say that a is $\{1, 3\}$-invertible if x satisfies the equations (i) and (iii) in Definition 2.2. Such x is called a $\{1, 3\}$-inverse of a, and is denoted by $a^{(1,3)}$. By $a\{1, 3\}$ we denote the set of all $\{1, 3\}$-inverses of a. Similarly, a is $\{1, 4\}$-invertible if x satisfies the equations (i) and (iv). Such x is called a $\{1, 4\}$-inverse of a, and is denoted by $a^{(1,4)}$. By $a\{1, 4\}$ we denote the set of all $\{1, 4\}$-inverses of a. As usual, we denote by $R^\dagger$, $R^{(1,3)}$ and $R^{(1,4)}$ the sets of all Moore-Penrose invertible, {1,3}-invertible and {1,4}-invertible elements in R, respectively. It is known from [21, Lemma 2.1] that $a \in R^{(1,3)}$ if and only if $a \in Ra^*a$. If $a = ya^*a$ for some $y \in R$, then y^* is a {1,3}-inverse of a. Dually, $a \in R^{(1,4)}$ if and only if $a \in aa^*R$. If $a = aa^*x$ for some $x \in R$, then x^* is a {1,4}-inverse of a. It is well known that $a \in R^\dagger$ if and only if $a \in aa^*R \cap Ra^*a$.

In 2011, Mary introduced the inverse along an element in a semigroup by the Green's preorder relations. We herein state the notion of the inverse along an element in an associate ring R with unity 1.

Definition 2.3. [6, Definition 4] *Let $a, d \in R$. An element a is called invertible along d if there exists some $b \in R$ such that $bad = d = dab$ and $b \in dR \cap Rd$. Such an element b is called the inverse of a along d. It is unique if it exists, and is denoted by $a^{\|d}$.*

As was stated in [6, Theorem 11], the inverse along an element recovers the group inverse, the Drazin inverse and the Moore-Penrose inverse. More precisely, given any $a \in R$, a is group invertible if and only if a is invertible along a, a is Drazin invertible if and only if a is invertible along a^n for some positive integer n, a is Moore-Penrose invertible if and only if a is invertible along a^*. In 2013, Mary [7] presented the existence criterion for the inverse along an element by the intersection of ideals, i.e, a is invertible along d if and only if $d \in dadR \cap Rdad$. In particular, if $d = dadx = ydad$, then $a^{\|d} = dx = yd$. By the symbol $R^{\|d}$ we denote the set of all elements which are invertible along d in a ring R.

Extending the (Mary's) inverse along an element, Drazin in 2012 introduced the (b, c)-inverse. For any $a, b, c \in R$, a is (b, c)-invertible [3] if there is some $y \in R$ such that $y \in bRy \cap yRc$, $yab = b$ and $cay = c$. Such an y is unique if it exists, and is denoted by $a^{(b,c)}$. It is noted that the (d, d)-inverse of a is exactly the inverse of a along d. By $R^{(b,c)}$ we denote the set of all (b, c)-invertible elements in R.

Recently, the author introduced a class of generalized inverses, called the w-core inverse [19], generalizing the core inverse, the core-EP inverse and the Moore-Penrose inverse.

Definition 2.4. [19, Definition 2.1] *Let $a, w \in R$. An element a is called w-core invertible if there exists some $x \in R$ such that $awx^2 = x$, $xawa = a$ and $(awx)^* = awx$. Such an x is called a w-core inverse of a. It is unique if it exists, and is denoted by $a^{\#}_{w}$.*

It is shown in [19] that a is w-core invertible if and only if there exists some $x \in R$ such that $awxa = a$, $xR = aR$ and $Rx = Ra^*$ if and only if w is invertible along a and a is $\{1, 3\}$-invertible. In this case, $a^{\#}_{w} = w^{\|a}a^{(1,3)}$. In particular, an element is called core invertible if it is 1-core invertible, the standard notion of the core inverse in a ring can be seen in [13]. The core inverse of a is denoted by $a^{\#}$. In [19], it is illustrated that a is core invertible if and only if it is a-core invertible if and only if it is 1-core invertible, in which case, $a^{\#} = a^{\#}aa^{(1,3)}$; a is Moore-Penrose invertible if and only if it is a^*-core invertible. More results on w-core inverses can be referred to mathematical literature [20].

3. EI along an element

It is known that the Drazin inverse has the double commutant property, that is, a^D double commutes with a, precisely, if $ax = xa$ then $a^D x = xa^D$ for any $x \in R$. However, for the case of the Moore-Penrose inverse, $aa^\dagger$ is not equal to $a^\dagger a$ in general. More generally, $w^{\|a}$ does not commute with w.

For any $a, w \in R$, w is called EI (equal idempotent) along a if $w \in R^{\|a}$ and $ww^{\|a} = w^{\|a}w$, which is the equivalent condition such that w is completely invertible along a [14, Corollary 3.5]. The standard notion of the complete inverse along an element can be found in [14]. Also, Benítez and Boasso in [1, Section 6] have shown some characterizations of $ww^{\|a} = w^{\|a}w$.

The following result determines when w and $w^{\|a}$ commute by the existence of the w-core invertibility of a. Firstly, an auxiliary lemma is given below.

Lemma 3.1. *Given any $a, w \in R$ with $w \in R^{\|a}$, then $w^{\|a} = w^{\|a}a^-a = aa^-w^{\|a}$.*

Proof. Note that $w^{\|a}wa = a = aww^{\|a}$ and $w^{\|a} \in aR \cap Ra$. Then $a \in awRa \subseteq aRa$, i.e., a is regular. Also, it follows from $w^{\|a} \in Ra$ that $w^{\|a} = ta$ for some $t \in R$, and consequently $w^{\|a}a^-a = taa^-a = ta = w^{\|a}$. A dual statement gives $w^{\|a} = aa^-w^{\|a}$. □

Proposition 3.2. *Let $a, w \in R$ and $a \in R^{\#}_{w}$. Suppose $e = wa^{\#}_{w}a$ and $f = a^{\#}_{w}aw$. Then the following conditions are equivalent*:

(i) *w is EI along a.*
(ii) *$e = f$.*
(iii) *$eR = fR$ and $ef = fe$.*
(iv) *$eR = fR$ and $efe = fef$.*
(v) *$eR = fR$ and $(ef)^2 = (fe)^2$.*
If one of the above conditions holds, then $a \in R^{\#}$. Moreover, $a^{\#} = wa^{\#}_{w}$.

Proof. (i) ⇔ (ii) Note that $a \in R^{\#}_{w}$ if and only if $w \in R^{\|a}$ and $a \in R^{(1,3)}$. Then, by Lemma 3.1, $w^{\|a} = w^{\|a}a^{(1,3)}a$. Given $ww^{\|a} = w^{\|a}w$, then $e = wa^{\#}_{w}a = ww^{\|a}a^{(1,3)}a = ww^{\|a} = w^{\|a}w = w^{\|a}a^{(1,3)}aw = a^{\#}_{w}aw = f$. Conversely, if $e = f$, then $ww^{\|a} = ww^{\|a}a^{(1,3)}a = wa^{\#}_{w}a = e = f = a^{\#}_{w}aw = w^{\|a}a^{(1,3)}aw = w^{\|a}w$.

(ii) ⇒ (iii) and (iii) ⇒ (iv) are clear.

(iv) ⇒ (v) Note that $(ef)^2 = efef = (efe)f = fef$ and $(fe)^2 = fefe = efe$. Then $(ef)^2 = (fe)^2$.

(v) ⇒ (ii) As $eR = fR$, then $e = fe$ and $f = ef$. So, $e = e^2 = (fe)^2 = (ef)^2 = f^2 = f$.

We next give the criterion and the formula of $a^{\#}$. As $ae = a = fa$, then $a = fa = ea = wa^{\#}_{w}a^2 \in Ra^2$. Also, $a = af = a^2wa^{\#}_{w}f \in a^2R$. So, $a \in R^{\#}$ and $a^{\#} = w(a^{\#}_{w})^2aw$. It follows from Lemma 3.1 that $a^{\#}_{w}aa^{(1,3)} = (w^{\|a}a^{(1,3)})aa^{(1,3)} = (w^{\|a}a^{(1,3)}a)a^{(1,3)} = w^{\|a}a^{(1,3)} = a^{\#}_{w}$. Thus, $a^{\#} = a^{\#}aa^{(1,3)} = w(a^{\#}_{w})^2awaa^{(1,3)} = wa^{\#}_{w}aa^{(1,3)} = wa^{\#}_{w}$. □

Recall that an element $b \in R$ is EP if $b \in R^{\#} \cap R^{\dagger}$ and $b^{\#} = b^{\dagger}$. A well known characterization for EP element is that b is EP if and only if $b \in R^{\dagger}$ and $bb^{\dagger} = b^{\dagger}b$. It is known [6] that $b \in R^{\dagger}$ if and only if $b^{\|b^*}$ exists. Moreover, $b^{\dagger} = b^{\|b^*}$. We herein remind the reader that the EP element is a special case of EI along an element. Indeed, if b is EP, then $bb^{\dagger} = b^{\dagger}b$, i.e., $bb^{\|b^*} = b^{\|b^*}b$, and consequently, b is EI along b^*.

Take $w = b$ and $a = b^*$, then $e = b(b^*)^{\circledast}_{b}b^* = bb^{\|b^*}(b^*)^{(1,3)}b^* = bb^{\|b^*} = bb^{\dagger}$ and $f = (b^*)^{\circledast}_{b}b^*b = b^{\dagger}b$. As a special result of Theorem 3.2, we get the following characterizations for EP elements.

Corollary 3.3. *Let $b \in R$. Suppose $e = bb^{\dagger}$ and $f = b^{\dagger}b$. Then the following conditions are equivalent:*
(i) *b is EP.*
(ii) *$e = f$.*
(iii) *$eR = fR$ and $ef = fe$.*
(iv) *$eR = fR$ and $efe = fef$.*
(v) *$eR = fR$ and $(ef)^2 = (fe)^2$.*

We close this section with the characterization for the inverse along an element.

Theorem 3.4. *Let $a, w \in R$. Then the following conditions are equivalent:*
(i) *w is invertible along a.*
(ii) *there exists $e^2 = e \in R$ such that $a \in Re$ and ew is invertible along a.*
(iii) *there exists $f^2 = f \in R$ such that $a \in fR$ and wf is invertible along a.*
In this case, $w^{\|a} = (ew)^{\|a} = (wf)^{\|a}$.

Proof. (i) ⇒ (ii) Given (i), then, by Lemma 3.1, a is regular. Write $e = a^-a$, then $e^2 = e$ and $a = ae \in Re$. Hence, $a \in aewaR \cap Raewa$ since $a \in awaR \cap Rawa$, i.e., ew is invertible along a.

(ii) ⇒ (i) Since $a \in Re$, there is some $x \in R$ such that $a = xe$ and hence $aw = xew = xeew = aew$. As ew is invertible along a, then $a \in aewaR \cap Raewa$, and whence $a \in awaR \cap Rawa$, as required. Moreover, $w^{\|a} = (ew)^{\|a}$.

(i) ⇔ (iii) Setting $f = aa^-$, it can be similarly proved as (i) ⇔ (ii). □

4. When $a^{\circledast}_{w}aw$ and $wa^{\circledast}_{w}a$ are projections

Recall that an element $p \in R$ is an idempotent if $p = p^2$, and p is a projection if $p = p^2 = p^*$.

It is known from [19] that if a is w-core invertible, then there exists some $x \in R$ such that $awxa = a$, $xawx = x$, $awx = (awx)^*$, $awx^2 = x$ and $xawa = a$. Clearly, xaw and wxa are both idempotent. However, they may not be projections.

The following two theorems characterize when xaw and wxa are projections, provided that x is the w-core inverse of a.

Lemma 4.1. [19, Theorem 2.9] *Let $a, w \in R$. Then $a \in R^{\circledast}_{w}$ if and only if $w^{\|a}$ and $a^{(1,3)}$ both exist. In this case, $a^{\circledast}_{w} = w^{\|a}a^{(1,3)}$ and $w^{\|a} = a^{\circledast}_{w}a$.*

Theorem 4.2. *Let $a, w \in R$ with $a \in R^{\circledast}_{w}$. If $x = a^{\circledast}_{w} \in R$, then the following conditions are equivalent:*
(i) *$xaw = (xaw)^*$.*
(ii) *$(aw)^*R = aR$.*
(iii) *$(aw)^*R \subseteq aR$.*
(iv) *$R(xa)^* = R(xaw)$.*
(v) *$R(xaw) \subseteq R(xa)^*$.*
(vi) *$aw \in R^{\dagger}$ with $(aw)^{\dagger} = x$.*
(vii) *$aw \in R^{\#}$ with $(aw)^{\#} = x$.*

Proof. (i) ⇒ (ii) As $xaw = (xaw)^*$, then $(aw)^* = (awxaw)^* = (xaw)^*(aw)^* = xaw(aw)^* = awx^2aw(aw)^* \in aR$. Also, $a = xawa = (xaw)^*a = (aw)^*x^*a \in (aw)^*R$. So, $(aw)^*R = aR$.

(ii) ⇒ (iii) is trivial.

(iii) $\Rightarrow$ (i) Given $(aw)^*R \subseteq aR$, then there exists suitable $t \in R$ such that $(aw)^* = at = xawat = xaw(aw)^*$. Post-multiplying the equation above by x^* gives $(xaw)^* = xaw(xaw)^* = xaw$, as required.

(i) $\Rightarrow$ (iv) Once given $xaw = (xaw)^*$, then $Rxaw = R(xaw)^* = Rw^*(xa)^* \subseteq R(xa)^*$. Again, we have $(xa)^* = (xawxa)^* = (xa)^*(xaw)^* = (xa)^*xaw$, which implies $R(xa)^* \subseteq Rxaw$. Thus, $R(xa)^* = Rxaw$.

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i) Since $Rxaw \subseteq R(xa)^*$, it follows that $xaw = t(xa)^* = t(xawxa)^* = t(xa)^*(xaw)^* = xaw(xaw)^* = (xaw)^*$ for some $t \in R$.

(i) $\Rightarrow$ (vi) Note that $x \in R$ satisfies $awxaw = aw$, $xawx = x$ and $awx = (awx)^*$. Again, as $xaw = (xaw)^*$, then $aw \in R^\dagger$ and $(aw)^\dagger = x$.

(vi) $\Rightarrow$ (i) is obvious since $x \in (aw)\{1, 4\}$.

(i) $\Rightarrow$ (vii) Since $xaw = (xaw)^*$, we have $xaw = awx^2aw = (awx)^*(xaw)^* = (xawawx)^* = (awx)^* = awx$. Consequently, by $awxaw = aw$ and $xawx = x$, one gets $aw \in R^\#$ and $(aw)^\# = x$.

(vii) $\Rightarrow$ (i) follows from $awx = xaw$ and $awx = (awx)^*$. □

From Theorem 4.2 above, for any $a \in R_w^{\textcircled{\#}}$, we get that any one condition of (i)-(vii) is equivalent to the fact that (viii) aw is EI along $(aw)^*$ (i.e., aw is EP). Indeed, given (vi), and hence (vii), then $aw \in R^\# \cap R^\dagger$ and $(aw)^\# = (aw)^\dagger$, hence aw is EI along $(aw)^*$. For the converse statement, it is clear that we have the implication (viii) $\Rightarrow$ (vi).

Theorem 4.3. *Let $a, w \in R$ with $a \in R_w^{\textcircled{\#}}$. If $x = a_w^{\textcircled{\#}}$, then the following conditions are equivalent:*

(i) $wxa = (wxa)^*$.

(ii) $(xa)^*R = (wxa)R$.

(iii) $(xa)^*R \subseteq (wxa)R$.

(iv) $a^*R = (wa)R$.

(v) $a^*R \subseteq (wa)R$.

(vi) $a \in R^\dagger$ *with* $a^\dagger = wx$.

Proof. To begin with, (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v) are trivial.

(i) $\Rightarrow$ (ii) Given $wxa = (wxa)^*$, then $(wxa)R = (wxa)^*R = (xa)^*w^*R \subseteq (xa)^*R$. Also, $(xa)^*R = (xawxa)^*R = (wxa)^*(xa)^*R \subseteq (wxa)^*R = (wxa)R$.

(iii) $\Rightarrow$ (i) It follows from $(xa)^*R \subseteq (wxa)R$ that $(xa)^* = wxat = wx(awxa)t = wxa(wxat) = wxa(xa)^*$ for some $t \in R$. We have $(wxa)^* = (xa)^*w^* = wxa(xa)^*w^* = wxa(wxa)^* = wxa$.

(i) $\Rightarrow$ (iv) Since $wxa = (wxa)^*$, we have $a^* = (awxa)^* = (wxa)^*a^* = wxaa^* = w(awx^2)aa^*$, which gives $a^*R \subseteq waR$. Also, $wa = w(xawa) = (wxa)^*wa = a^*(wx)^*wa \in a^*R$. Therefore, $a^*R = (wa)R$.

(v) $\Rightarrow$ (i) From $a^*R \subseteq (wa)R$, there exists some $t \in R$ such that $a^* = wat = w(xawa)t = wxa(wat) = wxaa^*$. Post-multiplying the equality $a^* = wxaa^*$ by $(wx)^*$ yields $(wxa)^* = wxa(wxa)^* = wxa$.

(i) $\Rightarrow$ (vi) As $awxa = a$, $awx = (awx)^*$ and $xawx = x$, then $awxa = a$, $awx = (awx)^*$ and $wxawx = wx$, which combining with $wxa = (wxa)^*$ imply $a \in R^\dagger$ and $a^\dagger = wx$.

(vi) $\Rightarrow$ (i) As $a^\dagger = wx$, then $a^\dagger a = (a^\dagger a)^*$ gives $wxa = (wxa)^*$. □

Remark 4.4. For any $a \in R_w^{\textcircled{\#}}$, if a and x satisfy any one condition of (i)-(vi) in Theorem 4.3, then a may not be group invertible. See the following counterexample.

Example 4.5. Let $R = M_2(\mathbb{C})$ be the ring of all 2 by 2 complex matrices and let the involution be the conjugate transpose. Take $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $w = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in R$, by a direct calculation, any such form matrix $\begin{bmatrix} * & * \\ 1 & 0 \end{bmatrix} \in a\{1, 3\}$, $w^{\parallel a} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, and consequently $a_w^{\textcircled{\#}} = w^{\parallel a}a^{(1,3)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Clearly, $wa_w^{\textcircled{\#}}a = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = (wa_w^{\textcircled{\#}}a)^*$. However, $a \notin R^\#$, i.e., $a \notin a^2R \cap Ra^2$ since $a^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

For any given $a, w \in R$, the authors in [19] showed that a is w-core invertible if and only if aw is invertible along aa^*, provided that $a \in R^\dagger$. Another characterization for the existence of the w-core inverse is given below.

Theorem 4.6. *Let $a, w \in R$ and let $a \in R^{\dagger}$. Then $a \in R_w^{\textcircled{\#}}$ if and only if $(a^{\dagger})^* w$ is invertible along aa^*. In this case, $a_w^{\textcircled{\#}} = ((a^{\dagger})^* w)^{\| aa^*}(a^{\dagger})^* a^{\dagger}$.*

Proof. For the "only if" part, suppose $a \in R_w^{\textcircled{\#}}$. Then $w^{\| a}$ and $a^{(1,3)}$ exist, and hence $a \in awaR \cap Rawa \cap Ra^*a$. So, $aa^* \in Rawaa^* = Raa^{\dagger}awaa^* = Raa^*(a^{\dagger})^* waa^*$. Also, $a \in awaR = awaa^*(a^{\dagger})^* R \subseteq awaa^*R = aa^{\dagger}awaa^*R = aa^*(a^{\dagger})^* waa^*R$. Therefore, $(a^{\dagger})^* w$ is invertible along aa^*.

For the "if" part, we have $((a^{\dagger})^* w)^{\| aa^*}(a^{\dagger})^* waa^* = aa^* = aw((a^{\dagger})^* w)^{\| aa^*}$ and $((a^{\dagger})^* w)^{\| aa^*} \in aa^*R \cap Raa^*$. Suppose $x = ((a^{\dagger})^* w)^{\| aa^*}(a^{\dagger})^* a^{\dagger}$. Then we can prove that x is the w-core inverse of a by the following three steps.

(1) $awx = aw((a^{\dagger})^* w)^{\| aa^*}(a^{\dagger})^* a^{\dagger} = aa^*(a^{\dagger})^* a^{\dagger} = aa^{\dagger} = (awx)^*$.

(2) Since $x \in aa^*R$, there exists some $t \in R$ such that $x = aa^*t$ and hence $awx^2 = aa^{\dagger}x = aa^{\dagger}aa^*t = aa^*t = x$.

(3) We have

$$\begin{aligned} xawa &= ((a^{\dagger})^* w)^{\| aa^*}(a^{\dagger})^* a^{\dagger}awa = ((a^{\dagger})^* w)^{\| aa^*}(a^{\dagger})^*(a^{\dagger}a)^* wa \\ &= ((a^{\dagger})^* w)^{\| aa^*}(a^{\dagger})^* wa = ((a^{\dagger})^* w)^{\| aa^*}(a^{\dagger})^* waa^*(a^{\dagger})^* \\ &= aa^*(a^{\dagger})^* \\ &= a. \end{aligned}$$

The proof is completed. □

Recall that an element $a \in R$ is called a partial isometry if $aa^*a = a$. Various characterizations of partial isometries in a ring with involution can be referred to [8]. It follows that if $a \in R$ is a partial isometry, then $a \in R^{\dagger}$ and $a^{\dagger} = a^*$. Moreover, $(a^{\dagger})^* w = aw$. As a consequence of Theorem 4.6, we have the following result.

Corollary 4.7. *Let $a, w \in R$ and $a = aa^*a$. Then $a \in R_w^{\textcircled{\#}}$ if and only if aw is invertible along aa^*. In this case, $a_w^{\textcircled{\#}} = (aw)^{\| aa^*}$.*

Given any $a \in R$, the set of all right annihilators of a is denoted by $a^0 = \{x \in R : ax = 0\}$ and the set of all left annihilators of a is denoted by ${}^0a = \{x \in R : xa = 0\}$. For any $a, b \in R$, if $aR = bR$ then ${}^0a = {}^0b$, and if $Ra = Rb$ then $a^0 = b^0$ (see e.g., [13, Lemmas 2.5 and 2.6]).

We now come to our main characterization theorem for w-core inverses, which are given by ideals generated by idempotents and projections.

Theorem 4.8. *Let $a, w \in R$. Then the following conditions are equivalent:*

(i) *a is w-core invertible.*

(ii) *there exist a unique projector $p \in R$ and an idempotent $q \in R$ such that $pR = aR = awR = qR$ and $Rq = Raw$.*

(iii) *there exist a projector $p \in R$ and an idempotent $q \in R$ such that $pR = aR = awR = qR$ and $Rq = Raw$.*

(iv) *$aw \in R^-$ and there exist a projector $p \in R$ and an idempotent $q \in R$ such that ${}^0p = {}^0a = {}^0(aw) = {}^0q$ and $q^0 = (aw)^0$.*

(v) *$aw \in R^-$ and there exist a unique projector $p \in R$ and an idempotent $q \in R$ such that ${}^0p = {}^0a = {}^0(aw) = {}^0q$ and $q^0 = (aw)^0$.*

In this case, $a_w^{\textcircled{\#}} = q(aw)^- p$ for any $(aw)^- \in (aw)\{1\}$.

Proof. (i) ⇒ (ii) Let $x \in R$ be the w-core inverse of a and let $p = awx$ and $q = xaw$. Then $p^2 = p = p^*$ and $q^2 = q$. Also, $pR = awxR \subseteq awR \subseteq aR = awxaR \subseteq pR$ gives $pR = aR = awR$. From $xawx = x$, we have $qR = xR = aR$ and $Rq = Raw = Rawxaw \subseteq Rq$, which imply $Rq = Raw$.

We next show that such a projection is unique. Let p_1, p_2 satisfy (ii). Then $p_1R = aR = awR = p_2R$. There exists some $s \in R$ such that $p_1 = p_2s = p_2p_2s = p_2p_1$, and similarly $p_2 = p_1p_2$. So, $p_1 = (p_2p_1)^* = p_1p_2 = p_2$.

(ii) ⇒ (iii) and (iii) ⇒ (iv) are clear.

(iv) ⇒ (v) It suffices to show the uniqueness of p. Suppose ${}^0p_1 = {}^0p_2$. Then $(1-p_1)p_2 = 0$ and $(1-p_2)p_1 = 0$. Hence, $p_1 = p_2p_1 = p_1p_2 = p_2$.

(v) ⇒ (i) As ${}^0p = {}^0(aw)$ and $(1 - aw(aw)^-)aw = 0$, then $(1 - aw(aw)^-)p = 0$, i.e., $aw(aw)^-p = p$. It follows from $q^0 = (aw)^0$ that $aw = awq$. Set $x = q(aw)^-p$. Then x is the w-core inverse of a. Indeed, we have

(1) $awx = awq(aw)^-p = aw(aw)^-p = p = p^* = (awx)^*$.

(2) ${}^0p = {}^0a$ yields $pa = a$, and hence $awxa = pa = a$. As $(1 - awx)a = 0$, then $(1 - awx)q = 0$ since ${}^0q = {}^0a$. Thus, $awxq = q$. Post-multiplying the equality $awxq = q$ by $(aw)^- p$ gives $awx^2 = x$.

(3) As $aw(1 - (aw)^- aw) = 0$ and $q^0 = (aw)^0$, then $q = q(aw)^- aw$. Once given ${}^0q = {}^0a$, then $a = qa$ and hence $xawa = q(aw)^- pawa = q(aw)^- awa = qa = a$. □

Suppose that $a, w \in R$ and $w^{\|a}$ exists. Then, by [7, Theorem 2.1], $a \in aRa$, i.e., a is regular. As $a \in awaR \cap Rawa$, then $a \in awaRa$ and $awa \in awaRawa$, i.e., awa is regular. Conversely, the regularity of awa also guarantees the regularity of a. Indeed, since $awa \in awaRawa$, by $a \in awaR \cap Rawa$, we have $a \in awaRa \subseteq aRa$, consequently, a is regular.

It is natural to ask whether a and awa have the same generalized invertibility, provided that w is invertible along a. The following result illustrates the assumptions.

Theorem 4.9. *Let $a, w \in R$ and let $w^{\|a}$ exist. Then*

(i) *$a \in R^-$ if and only if $awa \in R^-$. In this case, $a^- w^{\|a} a^- \in (awa)\{1\}$ and $(awa)^- awawa(awa)^- \in a\{1\}$.*

(ii) *$a \in R^{(1,3)}$ if and only if $awa \in R^{(1,3)}$. In this case, $a^{(1,3)} w^{\|a} a^{(1,3)} \in (awa)\{1,3\}$ and $(awa)^{(1,3)} awawa(awa)^{(1,3)} \in a\{1,3\}$.*

(iii) *$a \in R^{(1,4)}$ if and only if $awa \in R^{(1,4)}$. In this case, $a^{(1,4)} w^{\|a} a^{(1,4)} \in (awa)\{1,4\}$ and $(awa)^{(1,4)} awawa(awa)^{(1,4)} \in a\{1,4\}$.*

(iv) *$a \in R^\dagger$ if and only if $awa \in R^\dagger$. In this case, $(awa)^\dagger = a^\dagger w^{\|a} a^\dagger$ and $a^\dagger = (awa)^\dagger awawa(awa)^\dagger$.*

(v) *$a \in R^\#$ if and only if $awa \in R^\#$. In this case, $(awa)^\# = a^\# w^{\|a} a^\#$ and $a^\# = (awa)^\# awawa(awa)^\#$.*

Proof. (i) has been proved above.

(ii) Suppose that $a \in R^{(1,3)}$. Then $a \in Ra^*a$ and hence $awa \in Ra^*awa$. Note that $w \in R^{\|a}$ implies that $a \in awaR$. Then $a = awat$ for some $t \in R$ and hence $awa \in Ra^*awa = R(awat)^*awa \subseteq R(awa)^*awa$, i.e., $awa \in R^{(1,3)}$.

Conversely, given $awa \in R^{(1,3)}$, i.e., $awa \in R(awa)^*awa$, then $a = awat \in R(awa)^*awat = R(awa)^*a \subseteq Ra^*a$. So, $a \in R^{(1,3)}$.

We next give the expressions of $a^{(1,3)}$ and $(awa)^{(1,3)}$. It is noted that $w^{\|a}$ exists if and only if $a \in awaR \cap Rawa$. Hence, there are some $s, t \in R$ such that $a = sawa = awat$. Suppose that $awa \in R^{(1,3)}$. Then $a = sawa(awa)^{(1,3)}awa = awa(awa)^{(1,3)}awat = a(awa)^{(1,3)}awa = awa(awa)^{(1,3)}a$.

Suppose that $x = (awa)^{(1,3)}awawa(awa)^{(1,3)}$. Then $x \in a\{1,3\}$. Indeed, we have $ax = a(awa)^{(1,3)}awawa(awa)^{(1,3)} = awa(awa)^{(1,3)} = (ax)^*$ and $axa = awa(awa)^{(1,3)}a = a$.

Conversely, as $a \in R^{(1,3)}$, then one can verify that $a^{(1,3)} w^{\|a} a^{(1,3)} \in (awa)\{1,3\}$.

(iii) can be proved by a similar way of (ii).

(iv) follows from (ii) and (iii).

(v) It is known that $a \in R^\#$ if and only if $a \in a^2R \cap Ra^2$. Suppose that $a \in R^\#$. Then $a \in a^2R \cap Ra^2$ and hence $awa \in awa^2R \cap Ra^2wa$. Again, from the implication $w \in R^{\|a} \Rightarrow a \in awaR \cap Rawa$, it follows that $awa \in awa^2R \cap Ra^2wa \subseteq (awa)^2R \cap R(awa)^2$.

Conversely, as $awa \in (awa)^2R \cap R(awa)^2$, then $a \in a^2R \cap Ra^2$ since $w \in R^{\|a} \Leftrightarrow a \in awaR \cap Rawa$.

By a direct calculation, $(awa)^\# = a^\# w^{\|a} a^\#$ and $a^\# = (awa)^\# awawa(awa)^\#$. □

It is known that $a \in R^{\text{#}\!\!\!\bigcirc}$ if and only if $a \in R^\# \cap R^{(1,3)}$. As a special result of Theorem 4.9, we have the following result.

Theorem 4.10. *Let $a, w \in R$ and let $w^{\|a}$ exist. Then $a \in R^{\text{#}\!\!\!\bigcirc}$ if and only if $awa \in R^{\text{#}\!\!\!\bigcirc}$. In this case, $(awa)^{\text{#}\!\!\!\bigcirc} = a^{\text{#}\!\!\!\bigcirc} w^{\|a} a^{\text{#}\!\!\!\bigcirc}$ and $a^{\text{#}\!\!\!\bigcirc} = (awa)^{\text{#}\!\!\!\bigcirc} awawa(awa)^{\text{#}\!\!\!\bigcirc}$.*

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