Filomat 37:10 (2023), 3191–3200
https://doi.org/10.2298/FIL2310191A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Note on weakly nil clean and π-regular rings

Khaled Alhazmy[a], Fuad Ali Ahmed Almahdi[a], El Mehdi Bouba[b], Mohammed Tamekkante[c]

[a]*Department of Mathematics, Faculty of Sciences, King Khalid University, P. O. Box. 9004, Abha, Saudi Arabia*
[b]*Mathematics Department, Pluridisciplinary faculty, Mohammed First University, B.P. 300, Selouane, Nador 62700, Morocco*
[c]*Department of Mathematics, Faculty of Science, University Moulay Ismail Meknes, Box 11201, Zitoune, Morocco*

Abstract. Let R be a commutative ring with identity $1 \neq 0$. The ring R is called weakly nil clean if every element x of R can be written as $x = n + e$ or $x = n - e$, where n is a nilpotent element of R and e is an idempotent element of R. The ring R is called weakly nil neat if every proper homomorphic image of R is weakly nil clean. Among other results, this paper gives some new characterizations of weakly nil clean (resp. weakly nil neat) rings. An element $x \in R$ is said to be von Neumann regular if $x = x^2y$ for some $y \in R$, and x is said to be π-regular if $x^n = x^{2n}y$ for some $y \in R$ and some integer $n \geq 1$. It is proved that an element $x \in R$ is π-regular if and only if it can be written as $x = n + r$, where n is a nilpotent element and r is a von Neumann regular element. In this paper, we study the uniqueness of this expression.

1. Introduction

Throughout, all rings considered are commutative with identity$\neq 0$. For a ring R, U(R), nil(R), Jac(R), and Idem(R) denote, respectively, the group of all units of R, the nil-radical of R, the Jacobson radical of R, and the set of all idempotents of R.

In [14], Nicholson defined a ring R to be clean if every element can be written as the sum of an idempotent and a unit. A list of several characterizations of clean rings is given in [12]. In the last years, clean rings have been widely studied and many variants of this notion have been proposed and studied. One of these variants is the concept of nil clean rings introduced by Diesl in [11] (over associative rings). A ring is called nil clean if every element can be written as the sum of an idempotent and a nilpotent. It is proved that a ring R is nil clean if and only if $R/\text{nil}(R)$ is boolean ([11, Corollary 3.20]). It is clear that nil clean rings are clean. However, the two concepts are different. Ahn and Anderson in [1] generalized the notion of clean ring by introducing the notion of weakly clean rings. A ring R is called weakly clean if every element x of R can be written as $x = u + e$ or $x = u - e$, where $u \in$ U(R) and $e \in$ Idem(R). Inspired by this idea, Danchev and McGovern, in [8], consider the class of weakly clean rings which lies strictly between the class of clean rings and the class of nil clean rings. A ring R is called weakly nil clean if every element x of R can be written as $x = n + e$ or $x = n - e$, where $n \in$ nil(R) and $e \in$ Idem(R).

Recall that a ring R is von Neumann regular if for every $x \in R$ there is $y \in R$ such that $x = x^2y$, and that R is π-regular if for every $x \in R$ there are $y \in R$ and an integer $n \geq 1$ such that $x^n = x^{2n}y$. It is proved

2020 *Mathematics Subject Classification*. Primary 13A15; Secondary 13A99
Keywords. Weakly nil clean rings, π-regular rings
Received: 28 May 2022; Revised: 17 July 2022; Accepted: 22 July 2022
Communicated by Dijana Mosić
Email addresses: khazmy@kku.edu.sa (Khaled Alhazmy), fuadalialmahdy@hotmail.com (Fuad Ali Ahmed Almahdi), mehdi8bouba@hotmail.fr (El Mehdi Bouba), tamekkante@yahoo.fr (Mohammed Tamekkante)

that R is π-regular (resp. von Neumann regular) if and only if R is zero-dimensional (resp. reduced and zero-dimensional). Let R be a ring and $x \in R$. Following Anderson and Badawi in [2], x is said to be von Neumann regular if $x = x^2 y$ for some $y \in R$, and x is said to be π-regular if $x^n = x^{2n} y$ for some $y \in R$ and some integer $n \geq 1$. Let $\mathrm{vnr}(R) := \{x \in R \mid x \text{ is von Neumann regular}\}$ and $\pi\mathrm{-r}(R) := \{x \in R \mid x \text{ is } \pi - \text{regular}\}$. It is clear that R is a von Neumann regular (resp. π-regular) ring if and only if $\mathrm{vnr}(R) = R$ (resp. $\pi\mathrm{-r}(R) = R$).
Section 2 of this paper begins with a new characterization of weakly nil clean rings. Namely, Theorem 2.1 states that a ring R is weakly nil clean if and only if R is zero-dimensional and $\mathrm{U}(R) = \mathrm{nil}(R) \pm 1$. This result is using to characterize when some ring extensions (namely, direct product of rings, amalgamated duplication of a ring along an ideal, and trivial ring extension of a ring by a module) are weakly nil clean. Recall that a ring R is weakly nil neat if every proper homomorphic image of R is weakly nil clean. This class of rings was introduced and studied in [9]. Theorem 2.8 and Corollary 2.9 propose some new characterizations of weakly nil neat rings.
Following [2, Theorem 4.2], an element x of a ring R is π-regular if and only if it can be written as $x = n + r$, where n is a nilpotent element and r is a von Neumann regular element. Section 3 deals the uniqueness of this expression.

2. Note on weakly nil clean rings

Following [2, Theorem 4.2], $\pi\mathrm{-r}(R) = \mathrm{vnr}(R) + \mathrm{nil}(R)$. In other words, a ring R is π-regular if and only if every element can be written as a sum of a nilpotent element and a von Neumann regular element. Hence, since $\mathrm{Idem}(R) \cup -\mathrm{Idem}(R) \subseteq \mathrm{vnr}(R)$, the notion of π-regular rings generalizes this of weakly nil clean rings. This fact was mentioned and proved in [8]. Note also that a π-regular ring needs not to be weakly nil clean as witnessed by the ring $\mathbb{Z}_3 \times \mathbb{Z}_3$. So, which sub-class of zero-dimensional rings (or equivalently π-rings) is the class of weakly nil clean rings. The answer is given by [8, Theorem 1.17] which states that a ring R is weakly nil clean if and only if R is zero-dimensional and there is at most one maximal ideal of R, say M, which satisfies $R/M = \mathbb{Z}_3$ while for all other maximal ideals N of R we have $R/N = \mathbb{Z}_2$.
Our first result gives a new characterization of weakly nil clean rings and disclose, in another way, the relation between weakly nil clean rings and zero-dimensional rings. It looks like the way that Diesl in [11] characterize nil clean rings. Namely, [11, Corollary 3.11] states that a ring R is nil clean if and only if R is zero dimensional and $\mathrm{U}(R) = \mathrm{nil}(R) + 1$.

Theorem 2.1. *Let R be a ring. Then, the following are equivalent:*

1. *R is weakly nil clean.*
2. *For each $x \in R$, $x^2 + x \in \mathrm{nil}(R)$ or $x^2 - x \in \mathrm{nil}(R)$.*
3. *R is zero-dimensional and $\mathrm{U}(R) = \mathrm{nil}(R) \pm 1$.*

Proof. (1) ⇒ (2) Let $x \in R$. Then, $x = n - e$ or $x = n + e$, where $n \in \mathrm{nil}(R)$ and $e \in \mathrm{Idem}(R)$. If $x = n - e$ then $x^2 + x = n^2 - 2ne + n \in \mathrm{nil}(R)$ and if $x = n + e$ then $x^2 - x = n^2 + 2ne - n \in \mathrm{nil}(R)$.
(2) ⇒ (3) Let $x \in R$. Over $R/\mathrm{nil}(R)$, we have $\overline{x}^2 = \pm\overline{x}$. Thus, $R/\mathrm{nil}(R)$ is von Neumann regular, and so R is zero-dimensional. Let u be a unit of R. We have $u + u^2 \in \mathrm{nil}(R)$ or $u - u^2 \in \mathrm{nil}(R)$. Then, over $R/\mathrm{nil}(R)$, $-\overline{u}$ is an idempotent or $\overline{u}$ is an idempotent. Since idempotents lift modulo any nil ideal, either $-u = e + n$ or $u = e + n$ for some idempotent element e of R and some nilpotent element n of R. In the both cases, $e \in \mathrm{nil}(R) + \mathrm{U}(R) = \mathrm{U}(R)$. So, $e = 1$. Hence, $u = -1 - n$ or $u = 1 + n$. Thus, $u \in \mathrm{nil}(R) \pm 1$. Consequently, $\mathrm{U}(R) \subseteq \mathrm{nil}(R) \pm 1$. The other inclusion is trivial.
(3) ⇒ (1) Let $x \in R$. Since R is π-regular, by [2, Theorem 4.2], $x = n + ue$ for some $n \in \mathrm{nil}(R)$, $u \in \mathrm{U}(R)$, and $e \in \mathrm{Idem}(R)$. By hypothesis, $u = m \pm 1$ for some $m \in \mathrm{nil}(R)$. Thus, $x = n + me \pm e$. Thus, x is weakly nil clean. □

Corollary 2.2. *Let R be a ring. Then, the following are equivalent:*

1. *R is a reduced weakly nil clean ring.*

2. *For each* $x \in R$, $x^2 + x = 0$ *or* $x^2 - x = 0$.

3. *R is von Neumann regular and* $\mathrm{U}(R) = \{\pm 1\}$.

Remark 2.3. *Using [1, Theorem 1.12], the equivalence* (1) ⇔ (2) *in the previous Corollary coincides with [8, Theorem 1.13].*

The class of weakly nil clean rings is not closed under finite products. For example, $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not weakly nil clean even if $\mathbb{Z}_3$ is weakly nil clean. Observing Theorem 2.1, one can easily understand that the problem comes from the fact that the condition $\mathrm{U}(R) = \mathrm{nil}(R) \pm 1$ is not compatible with direct product of rings. The next result gives necessary and sufficient conditions for a direct product of rings to be weakly nil clean.

Theorem 2.4. *Let* $\{R_i\}_{i=1}^k$ *be a family of rings with* $k \geq 2$. *Then,* $R = \prod_{i=1}^k R_i$ *is weakly nil clean if and only if each* R_i *is weakly nil clean and at most one* R_i *is not nil clean.*

Proof. (⇒) The class of weakly nil clean rings is closed under homomorphic images (by [8, Proposition 1.9]). Hence, each R_i is weakly nil clean. Suppose that R_1 is not nil clean. Hence, by [8, Proposition 1.10], $2 \notin \mathrm{nil}(R_1)$. Using Theorem 2.1, $\mathrm{U}(R) = \mathrm{nil}(R) \pm 1$. Hence,

$$\prod_{i=1}^{k} \mathrm{U}(R_i) = \left(\prod_{i=1}^{k} \mathrm{nil}(R_i) \right) \pm 1_R.$$

Thus, $(-1, 1, \cdots, 1) = (n_1, n_2, \cdots, n_k) + (1, 1, \cdots, 1)$ or $(-1, 1, \cdots, 1) = (n_1, n_2, \cdots, n_k) - (1, 1, \cdots, 1)$, where $n_i \in \mathrm{nil}(R_i)$ for each $i = 1, \cdots, k$. In the first case, we obtain that $2 = -n_1 \in \mathrm{nil}(R_1)$, a contradiction. Thus, $(-1, 1, \cdots, 1) = (n_1, n_2, \cdots, n_k) - (1, 1, \cdots, 1)$, and so $2 = n_i \in \mathrm{nil}(R_i)$ for each $i = 2, \cdots, k$. Using [8, Proposition 1.10], we conclude that R_i is nil clean for each $i = 2, \cdots, k$.
(⇐) We have $\dim(R) = \sup\{\dim(R_i) \mid i = 1, \cdots, k\} = 0$. Hence, R is zero-dimensional. If all R_i are nil clean then so is R (by [11, Proposition 3.13]), and then R is weakly nil clean. Now, suppose that R_1 is not nil clean. Hence, for each $i = 2, \cdots, k$, R_i is nil clean, and so $2_{R_i} \in \mathrm{nil}(R_i)$ (by [8, Proposition 1.10]). Let $u = (u_1, u_2, \cdots, u_k)$ be a unit of R. Then, u_i is a unit of R_i for each $i = 1, \cdots, k$. By [11, Corollary 3.10], for each $i = 2, \cdots, k$, we have $u_i = n_i + 1$ for some $n_i \in \mathrm{nil}(R_i)$. By Theorem 2.1, $u_1 = n_1 + 1$ or $u_1 = n_1 - 1$ for some $n_1 \in \mathrm{nil}(R_1)$. If $u_1 = n_1 + 1$ then, $u = (n_1, n_2, \cdots, n_k) + (1, 1, \cdots, 1) \in \mathrm{nil}(R) + 1_R$. Suppose now that $u_1 = n_1 - 1$. Then, $u = (n_1 - 1, n_2 + 1, \cdots, n_k + 1) = (n_1, n_2 + 2, \cdots, n_k + 2) - (1, 1, \cdots, 1) \in \mathrm{nil}(R) - 1_R$. Consequently, , $\mathrm{U}(R) \subseteq \mathrm{nil}(R) \pm 1_R$, and so $\mathrm{U}(R) = \mathrm{nil}(R) \pm 1$. Hence, by Theorem 2.1, R is weakly nil clean. □

Corollary 2.5. *Let R be a ring. Then,* $R \times R$ *is weakly nil clean if and only if R is nil clean.*

Let R be a ring, I an ideal of R, and $\pi : R \to R/I$ the canonical surjection. The amalgamated duplication of R along I, denoted by $R \bowtie I$, is the special pullback of π and π; i.e., the subring of $R \times R$ given by

$$R \bowtie I = \Big\{(r, r + i) \mid x \in R, i \in I\Big\}.$$

In particular, $R \bowtie R = R \times R$ and $R \bowtie (0) \cong R$. The construction $R \bowtie I$ was introduced and its basic properties were studied by D'Anna and Fontana in [5, 6].
The next result extends Corollary 2.5 to the context of an amalgamated duplication of a ring along an ideal.

Theorem 2.6. *Let R be a ring and I be an ideal of R. Then,* $R \bowtie I$ *is weakly nil clean if and only if R is weakly nil clean and* $2I \subseteq \mathrm{nil}(R)$.

Proof. (⇒) Since $R \cong R \bowtie I/0 \times I$, it is clear that R is weakly nil clean. If $2 \in \mathrm{nil}(R)$ then, $2I \subseteq \mathrm{nil}(R)$. Hence, we may assume that $2 \notin \mathrm{nil}(R)$ and let $i \in I$. Note that $\mathrm{nil}(R \bowtie I) = \{(r, r+i) \mid r \in \mathrm{nil}(R) \text{ and } i \in \mathrm{nil}(R) \cap I\}$. Clearly, $(1, 1-i)^2 + (1, 1-i) = (2, 2-2i+i^2-i) \notin \mathrm{nil}(R \bowtie I)$. Thus, by Theorem 2.1, $(0, i^2-i) = (1, 1-i)^2 - (1, 1-i) \in \mathrm{nil}(R \bowtie I)$. Hence, $i^2 - i \in \mathrm{nil}(R)$ for each $i \in I$. In particular, $4i^2 - 2i = (2i)^2 - 2i \in \mathrm{nil}(R)$. But, $4i^2 - 4i \in \mathrm{nil}(R)$. Thus, $2i \in \mathrm{nil}(R)$. So, $2I \subseteq \mathrm{nil}(R)$.

(⇐) Let $i \in I$. By Theorem 2.1, we have $i^2 - i \in \text{nil}(R)$ or $i^2 + i \in \text{nil}(R)$. Since $2i \in \text{nil}(R)$, we have both $i^2 - i \in \text{nil}(R)$ and $i^2 + i \in \text{nil}(R)$.
Let $x \in R$. Then, $x^2 - x \in \text{nil}(R)$ or $x^2 + x \in \text{nil}(R)$. Suppose that $x^2 - x \in \text{nil}(R)$. Then, $(x, x+i)^2 - (x, x+i) = (x^2 - x, x^2 - x + 2xi + i^2 - i)$. Since $2xi + i^2 - i \in I \cap \text{nil}(R)$, we get that $(x, x+i)^2 - (x, x+i) \in \text{nil}(R \bowtie I)$. Similarly, if $x^2 + x \in \text{nil}(R)$, we have $(x, x+i)^2 + (x, x+i) = (x^2 + x, x^2 + x + 2xi + i^2 + i)$. Since $2xi + i^2 + i \in I \cap \text{nil}(R)$, we get that $(x, x+i)^2 + (x, x+i) \in \text{nil}(R \bowtie I)$. Consequently, $R \bowtie I$ is weakly nil clean. □

Let R be a ring and M an R-module. The trivial ring extension of R by M is the ring $R := R \propto M$, where the underlying group is $R \times M$ and the multiplication is defined by $(a, m)(b, m') = (ab, am' + bm)$. It is also called the (Nagata) idealization of M over R and is denoted by $R(+)M$. This construction was first introduced, in 1962, by Nagata [13] with the objective to emphasize the interaction between rings and their modules and, more importantly, to provide numerous families of examples of rings with zero-divisors.
Next, we characterize weakly nil clean trivial rings extension.

Theorem 2.7. *Let R be a ring and M an R-module. Then, $R \propto M$ is weakly nil clean if and only if R is weakly nil clean.*

Proof. (⇒) Note that $R \cong R \propto M/0 \propto M$ is a homomorphic image of $R \propto M$. Hence R is weakly nil clean.
(⇐) The ring $R \propto M$ is zero-dimensional since R is so. A unit element of $R \propto M$ has the form (u, m) with u is a unit of R and $m \in M$. By Theorem 2.1, $u = n \pm 1$ for some $n \in \text{nil}(R)$. Thus, $(u, m) = (n, m) \pm (1, 0)$. Note that $(n, m) \in \text{nil}(R \propto M)$ and $1_{R \propto M} = (1, 0)$. Thus, $\text{U}(R \propto M) = \text{nil}(R \propto M) \pm 1$. So, by Theorem 2.1, $R \propto M$ is weakly nil clean. □

In [8], Danchev and McGovern asked about the class of rings for which every proper homomorphic image is weakly nil clean and they left the question as an open problem. This class of rings was considered recently in [9] under the name of weakly nil neat rings. The main result of this last paper ([9, Theorem 2.8]) gives a complete characterization of weakly nil neat rings. Our next two results propose some new descriptions for such rings.

Theorem 2.8. *Let R be a ring. Then, R is weakly nil neat if and only if either*

(a) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, *or*

(b) there is at most one nonzero prime ideal of R, say P, which satisfies $R/P \cong \mathbb{Z}_3$ while for all other nonzero prime ideals P' of R we have $R/P' \cong \mathbb{Z}_2$

Proof. (⇒) Let P be a nonzero prime ideal of R. Then, R/P is a weakly nil clean domain, and so $R/P \cong \mathbb{Z}_2$ or $R/P \cong \mathbb{Z}_3$ (by [8, Proposition 1.9]). Suppose that there are two different nonzero prime ideals of R, say P_1 and P_2, such that $R/P_1 \cong R/P_2 \cong \mathbb{Z}_3$. Clearly, P_1 and P_2 are maximal and so, by the Chinese Reminder Theorem, we get $R/P_1 \cap P_2 \cong R/P_1 \times R/P_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $\mathbb{Z}_3 \times \mathbb{Z}_3$ is not weakly nil clean, we should have $P_1 \cap P_2 = (0)$. Hence, $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.
(⇐) If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ then proper homomorphic image of R are isomorphic to $\mathbb{Z}_3$ which is weakly nil clean. Hence, R is weakly nil neat.
Suppose that (b) holds. Let I be a proper ideal of R. Every prime ideal $P \supseteq I$ of R is maximal since $I \neq (0)$. Thus, R/I is zero-dimensional. Let M/I be a maximal ideal of R/I. Then, $\frac{R/I}{M/I} \cong R/M$. Comparing [8, Proposition 1.17] with the condition (b), we conclude that R/I is weakly nil clean. Hence, R is weakly nil neat. □

Corollary 2.9. *Let R be a ring. Then, R is weakly nil neat if and only if either*

(a) R is a field, or

(b) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, *or*

(c) R is weakly nil clean, or

(d) *R is a 1-dimensional domain and $R/\mathrm{Jac}(R)$ is weakly nil clean.*

Proof. ($\Rightarrow$) If R is not a domain then following Theorem 2.8 together with [8, Theorem 1.17], $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or R is weakly nil clean since (0) is not prime.
Next suppose that R is a domain but not a field. Following Theorem 2.8, $\dim(R) = 1$ since (0) is prime and every nonzero prime ideal is maximal. Suppose that $\mathrm{Jac}(R) = (0)$. By Theorem 2.8, there is at most one nonzero prime ideal of R, say P, which satisfies $R/P \cong \mathbb{Z}_3$. Let $x \in R$. We have $x^2 + x \in P$ or $x^2 - x \in P$. While for all other nonzero prime ideals P' of R we have $R/P' \cong \mathbb{Z}_2$. Hence, $x^2 + x \in P'$ and $x^2 - x \in P'$. If $x^2 + x \in P$ (resp. $x^2 - x \in P$) then $x^2 + x \in \mathrm{Jac}(R) = (0)$ (resp. $x^2 - x \in \mathrm{Jac}(R) = (0)$). Since R is a domain, we get that $x = 0$ or $x = 1$ or $x = -1$. Thus, $R \cong \mathbb{Z}_2$ or $R \cong \mathbb{Z}_3$, a contradiction since $\dim(R) = 1$. Thus, $\mathrm{Jac}(R) \neq (0)$. Hence, $R/\mathrm{Jac}(R)$ is weakly nil clean.
($\Leftarrow$) If R is a field or $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or R is weakly nil clean then R is weakly nil neat. Next, suppose that R is a 1-dimensional domain and $R/\mathrm{Jac}(R)$ is weakly nil clean. Let I be a nonzero ideal of R. The radical ideal $\sqrt{I}$ is the intersection of all prime (and so maximal) ideals containing I. Hence, $\mathrm{Jac}(R) \subseteq \sqrt{I}$. Hence, $R/\sqrt{I} \cong \frac{R/\mathrm{Jac}(R)}{\sqrt{I}/\mathrm{Jac}(R)}$ is weakly nil clean. Since $R/\sqrt{I} \cong \frac{R/I}{\sqrt{I}/I}$ and $\sqrt{I}/I = \mathrm{nil}(R/I)$, we conclude by [8, Proposition 1.9] that R/I is weakly nil clean. Hence, R is weakly nil neat. □

The following example gives a 1-dimensional weakly nil neat ring.

Example 2.10. *Let $\mathbb{Z}_{(2)}$ be the ring of integers localized at the prime ideal (2). That is*

$$\mathbb{Z}_{(2)} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z} \text{ and } b \in 2\mathbb{Z} + 1 \right\}.$$

It is clear that $\mathbb{Z}_{(2)}$ is a 1-dimensional local domain with maximal ideal $M = \mathrm{Jac}(R) = 2\mathbb{Z}_{(2)}$. Moreover, $\mathbb{Z}_{(2)}/2\mathbb{Z}_{(2)} \cong \mathbb{Z}/2\mathbb{Z}$ is (weakly) nil clean. Hence, following Corollary 2.9, $\mathbb{Z}_{(2)}$ is weakly nil neat.

Corollary 2.11. *Let $\{R_i\}_{i=1}^k$ be a family of rings with $k \geq 2$. Then, $R = \prod_{i=1}^k R_i$ is weakly nil neat if and only if either*

(a) *$k = 2$ and $R_1 \cong R_2 \cong \mathbb{Z}_3$ or*

(b) *each R_i is weakly nil clean and at most one R_i is not nil clean.*

Proof. Since R cannot be a domain, by Corollary 2.9, R is weakly nil neat if and only if either $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or R is weakly nil clean. Moreover, it is easy to see that $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ if and only if $k = 2$ and $R_1 \cong R_2 \cong \mathbb{Z}_3$. Hence, our result follows from Theorem 2.4. □

Corollary 2.12. *Let R be a ring and $I \neq (0)$ an ideal of R. Then, $R \bowtie I$ is weakly nil neat if and only if $R \cong \mathbb{Z}_3$ or R is weakly nil clean with $2I \subseteq \mathrm{nil}(R)$.*

Proof. ($\Rightarrow$) Since $I \neq (0)$, $R \bowtie I$ cannot be a domain since $(0, i)(i, 0) = (0, 0)$ for each $i \in I$. Thus, by Corollary 2.9, $R \bowtie I \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or $R \bowtie I$ is weakly nil clean. Suppose that $R \bowtie I \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Proper homomorphic image of $\mathbb{Z}_3 \times \mathbb{Z}_3$ are isomorphic to $\mathbb{Z}_3$. Since $R \cong R \bowtie I/(0) \times I$ is a proper homomorphic image of $R \bowtie I$, we get that $R \cong \mathbb{Z}_3$. Next, if $R \bowtie I$ is weakly nil clean then, by Theorem 2.6, R is weakly nil clean with $2I \subseteq \mathrm{nil}(R)$.
($\Leftarrow$) If $R \cong \mathbb{Z}_3$ then $I = R$, and so $R \bowtie I = R \times R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Thus, $R \bowtie I$ is weakly nil neat. Now, if R is weakly nil clean with $2I \subseteq \mathrm{nil}(R)$ then, by Theorem 2.6, $R \bowtie I$ is weakly nil clean, and so weakly nil neat. □

Corollary 2.13. *Let R be a ring and $M \neq 0$ an R-module. Then, $R \propto M$ is weakly nil neat clean if and only if R is weakly nil clean.*

Proof. Follows from Theorem 2.8 and Corollary 2.9 since $R \propto M$ cannot be reduced. □

If R is a reduced weakly nil clean ring, then $\mathrm{U}(R)$ is a group of at most two elements. However, a weakly nil clean ring with $\mathrm{U}(R) = \{\pm 1\}$ needs not to be reduced. Take for example the ring $\mathbb{Z}_4$. Next, we characterize weakly nil clean rings R with $\mathrm{U}(R) = \{\pm 1\}$.

Theorem 2.14. *Let R be a ring. The following are equivalent:*

1. *R is weakly nil clean and* $\mathrm{U}(R) = \{\pm 1\}$.
2. *R is zero-dimensional and* $\mathrm{vnr}(R) = \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$.

Proof. (1) ⇒ (2) We have only to prove that $\mathrm{vnr}(R) = \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$ since weakly nil clean rings are zero-dimensional. Using [2, Theorem 2.2], we get $\mathrm{vnr}(R) = \{ue \mid u \in \mathrm{U}(R) \text{ and } e \in \mathrm{Idem}(R)\} = \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$ since $\mathrm{U}(R) = \{\pm 1\}$.
(2) ⇒ (1) Let u be a unit of R. Then, since $\mathrm{U}(R) \subseteq \mathrm{vnr}(R)$, we have $u = e$ or $u = -e$ for some $e \in \mathrm{Idem}(R)$. Thus, $u = 1$ or $u = -1$. Hence, $\mathrm{U}(R) = \{\pm 1\} \subseteq \mathrm{nil}(R) \pm 1 \subseteq \mathrm{U}(R)$. Thus, $\mathrm{U}(R) = \{\pm 1\} = \mathrm{nil}(R) \pm 1$. Using Theorem 2.1, we conclude that R is weakly nil clean. □

Corollary 2.15. *Let R be a non reduced ring.The following are equivalent:*

1. *R is weakly nil clean and* $\mathrm{U}(R) = \{\pm 1\}$.
2. *R is nil clean and* $\mathrm{U}(R) = \{\pm 1\}$.
3. *For each* $x \in R$, $x^2 = x$ *or* $x^2 = x + 2$.

Proof. (1) ⇒ (2) Let $n \in \mathrm{nil}(R)$. Then, $1 - n \in \mathrm{U}(R) = \{\pm 1\}$. Hence, $\mathrm{nil}(R) = \{0, 2\}$ since R is not reduced. Thus, by [8, Proposition 1.10], R is nil clean.
(2) ⇒ (1) Clear.
(1) ⇒ (3) As above, we get that $\mathrm{nil}(R) = \{0, 2\}$. By Theorem 2.1, we get $x^2 - x \in \{0, 2\}$ or $x^2 + x \in \{0, 2\}$ for each $x \in R$. But $2x \in \mathrm{nil}(R)$. Hence, $x^2 - x \in \{0, 2\}$ if and only if $x^2 + x \in \{0, 2\}$. Hence, for each $x \in R$, $x^2 - x \in \{0, 2\}$, as desired.
(3) ⇒ (1) Let $n \in \mathrm{nil}(R)$. If $n^2 = n$ then $n = 0$. If $n^2 = n + 2$ then $(n+1)(n-1) = n^2 - 1 = n + 1$. Hence, $n - 1 = 1$ since $n + 1$ is a unit. Thus, $n = 2$. So, $\mathrm{nil}(R) = \{0, 2\}$. Consequently, for each $x \in R$, $x^2 - x \in \mathrm{nil}(R)$. Thus, by Theorem 2.1, R is weakly nil clean. Let $u \in \mathrm{U}(R)$. If $u^2 = u$ then $u = 1$. Suppose now that $u^2 = u + 2$. Then, $u - 1 = 2u^{-1} \in \mathrm{nil}(R) = \{0, 2\}$. Thus, $u = 1$ or $u = 3 = -1$ (since 4=0). Consequently, $\mathrm{U}(R) = \{\pm 1\}$. □

Corollary 2.16. *Let R be a ring such that* $\mathrm{U}(R) = \{\pm 1\}$. *Then, R is weakly nil clean if and only if R is von Neumann regular or R is nil clean.*

Proof. (⇒) Since weakly nil clean rings are zero-dimensional, if R is reduced then R is von Neumann regular. Assume now that R is not reduced. Then, by Corollary 2.15, R is nil clean.
(⇐) Follows from Corollary 2.2 and the fact that nil clean rings are weakly nil clean. □

Recall from [4], that a ring R is said to be an *UN*-ring if every nonunit element a of R is a product of a unit and a nilpotent elements. Following [16, Proposition 2.25], a ring R is *UN* if and only if every element of R is either nilpotent or unit if and only if $\mathrm{nil}(R)$ is a maximal ideal of R. A simple example of *UN*-rings is $\mathbb{Z}/9\mathbb{Z}$.

Corollary 2.17. *Let R be a ring. Then, R is zero-dimensional and* $\mathrm{vnr}(R) = \mathrm{U}(R) \cup \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$ *if and only if either*

(a) *R is a UN-ring, or*

(b) $R \cong A \times B$ *with* $A, B \in \{\mathbb{Z}_3, \mathbb{Z}_4\}$, *or*

(c) *R is weakly nil clean and* $\mathrm{U}(R) = \{\pm 1\}$.

Proof. (⇒) If R is indecomposable, then $\mathrm{Idem}(R) = \{0, 1\}$. Thus, $\mathrm{vnr}(R) = \mathrm{U}(R) \cup \{0\}$. Since R is zero-dimensional, by [2, Theorem 4.2], $R = \pi{-}\mathrm{r}(R) = \mathrm{vnr}(R) + \mathrm{nil}(R) = \mathrm{U}(R) \cup \{0\} + \mathrm{nil}(R) = \mathrm{U}(R) \cup \mathrm{nil}(R)$. Consequently, R is a *UN*-ring.

Next, suppose that R is decomposable and write, without loss of generality, $R = S \times T$. Let $s \in \mathrm{vnr}(S)$. Then, $(s, 0) \in \mathrm{vnr}(R)\backslash\mathrm{U}(R)$. Hence, $(s, 0) \in \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$. Then, $s \in \mathrm{Idem}(S) \cup -\mathrm{Idem}(S)$. Therefore, $\mathrm{vnr}(S) = \mathrm{Idem}(S) \cup -\mathrm{Idem}(S)$. Moreover, $0 = \dim(R) = \sup\{\dim(S), \dim(T)\}$. Hence, $\dim(S) = 0$. By Theorem 2.14, S is weakly nil clean and $\mathrm{U}(S) = \{\pm 1\}$. Similarly, T is weakly nil clean and $\mathrm{U}(T) = \{\pm 1\}$.

Case 1. Suppose that $2_S = 0$. In this case S becomes nil clean and so, by Theorem 2.4, R is weakly nil clean. Moreover, $\mathrm{U}(R) = \mathrm{U}(S) \times \mathrm{U}(T) = \{1\} \times \{-1, 1\} = \{(-1, -1), (1, 1)\} = \{-1_R, 1_R\}$.

Case 2. Suppose that $2_T = 0$. Similarly to Case 1, we obtain that R is weakly nil clean and $\mathrm{U}(R) = \{\pm 1\}$.

Case 3. Suppose that $2_S \neq 0$ and $2_T \neq 0$. Assume that S is decomposable and let e be a non trivial idempotent of S. Then, $(e, -1)$ is a non unit von Neumann regular element of R, and so $(e, -1) \in \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$. Thus, $(e, -1)^2 = (e, -1)$ or $(e, -1)^2 = -(e, -1)$. That is $(e, 1) = (e, -1)$ or $(e, 1) = -(e, -1)$. The first case implies that $2_T = 0$ which is impossible. Thus, $2e = 0$. Similarly, $2(1 - e) = 0$, and so $2_S = 0$ which is also impossible. Consequently, S is indecomposable. Let $n \in \mathrm{nil}(S)$. Then, $1 - n \in \mathrm{U}(S) = \{-1, 1\}$. Hence, $\mathrm{nil}(S) \subseteq \{0, 2\}$. If S is reduced then, by [8, Proposition 1.9], $S = \mathbb{Z}_3$. Next, suppose that S is not reduced. Then, S is nil clean since $2_S \in \mathrm{nil}(S)$. Let $x \in S$. Then, $x = n + e$ where n is a nilpotent element of S and e is an idempotent element of S. Thus, $S = \{0, 1, 2, 3\}$ since $\mathrm{nil}(S) = \{0, 2\}$ and $\mathrm{Idem}(S) = \{0, 1\}$. Moreover, $2_S \neq 0$ and then $4_S = 0$. Hence, S contains exactly four elements. Since $2_S \neq 0$, $S \cong \mathbb{Z}_4$. Consequently, $S \in \{\mathbb{Z}_3, \mathbb{Z}_4\}$. Similarly, we get $T \in \{\mathbb{Z}_3, \mathbb{Z}_4\}$.

(⇐) If R is a *UN*-ring then R is an indecomposable zero-dimensional ring. Hence, $\mathrm{vnr}(R) = \{ue \mid u \in \mathrm{U}(R) \text{ and } e \in \mathrm{Idem}(R)\} = \mathrm{U}(R) \cup \{0\} = \mathrm{U}(R) \cup \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$.

Next, if $R \cong A \times B$ with $A, B \in \{\mathbb{Z}_3, \mathbb{Z}_4\}$ then clearly R is zero-dimensional. Also, it is easily checked that $\mathrm{vnr}(A \times B) = \mathrm{vnr}(A) \times \mathrm{vnr}(B) = \{0, 1_A, -1_A\} \times \{0, 1_B, -1_B\} = \mathrm{U}(A \times B) \cup \mathrm{Idem}(A \times B) \cup -\mathrm{Idem}(A \times B)$.

Finally, if R is weakly nil clean and $\mathrm{U}(R) = \{\pm 1\}$ then, by Theorem 2.14, R is zero-dimensional and $\mathrm{vnr}(R) = \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$. Hence, $\mathrm{vnr}(R) = \mathrm{U}(R) \cup \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$. □

Corollary 2.18 ([1, Corollary 1.13]). *Let R be a ring. Then, $R = \mathrm{U}(R) \cup \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$ if and only if either*

(a) R is a field, or

(b) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, or

(c) R is Boolean, or

(d) R is isomorphic to $B \times \mathbb{Z}_3$ for some Boolean ring B.

Proof. (⇒) Since $\mathrm{U}(R) \cup \mathrm{Idem}(R) \cup -\mathrm{Idem}(R) \subseteq \mathrm{vnr}(R)$. We obtain that $R = \mathrm{vnr}(R) = \mathrm{U}(R) \cup \mathrm{Idem}(R) \cup -\mathrm{Idem}(R)$. Hence, R is von Neumann regular. By Corollary 2.17, R is a *UN*-ring, or $R \cong A \times B$ with $A, B \in \{\mathbb{Z}_3, \mathbb{Z}_4\}$, or R is weakly nil clean with $\mathrm{U}(R) = \{\pm 1\}$.

If R is a *UN*-ring, then R is a field since it is reduced. For the same reason, the second case implies that $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Now, if R is weakly nil clean then R is either Boolean, or isomorphic to $\mathbb{Z}_3$ (and so a field) or R is isomorphic to $B \times \mathbb{Z}_3$ for some Boolean ring B (by [8, Proposition 1.13]).

(⇐) Clear. □

3. Uniquely π-regular elements

Recall that, following [2, Theorems 2.2 and 4.2],

$$\begin{aligned} \mathrm{vnr}(R) &= \{ue \mid u \in \mathrm{U}(R) \text{ and } e \in \mathrm{Idem}(R)\}, \text{ and} \\ \pi{-}\mathrm{r}(R) &= \{n + ue \mid u \in \mathrm{U}(R),\ e \in \mathrm{Idem}(R), \text{ and } n \in \mathrm{nil}(R)\}. \end{aligned}$$

For a π-regular element $x \in R$, the expression $x = n + r$ where n is a nilpotent element and r is a von Neumann regular element will be called a π-regular decomposition of x.

Definition 3.1. *An element $x \in R$ is called uniquely π-regular if there exist a unique von Neumann regular element r such that $x - r$ is nilpotent.*

Proposition 3.2. *Let R be a ring, $u \in \mathrm{U}(R)$, $e \in \mathrm{Idem}(R)$ and $n \in \mathrm{nil}(R)$. Then,*

1. *$n + ue$ is uniquely π-regular if and only if $e\mathrm{nil}(R) = \{0\}$.*
2. *ue is uniquely π-regular if and only if $e\mathrm{nil}(R) = \{0\}$.*
3. *u is uniquely π-regular if and only if R is reduced.*
4. *Every nilpotent element is uniquely π-regular. In particular, 0 is uniquely π-regular.*

Proof. (1) As mentioned above, $n + ue$ is a π-regular element and ue is a von Neumann regular element. Let $m \in \mathrm{nil}(R)$. We have $n + ue = n + uem + ue(1 - m)$. It is clear that $ue(1 - m)$ is von Neumann regular since $1 - m$ is unit. Thus, $ue = ue(1 - m)$. Then, $0 = em$ and so $e\mathrm{nil}(R) = \{0\}$.
Conversely, suppose that $e\mathrm{nil}(R) = \{0\}$. Let $n + ue = m + vf$ where $m \in \mathrm{nil}(R)$, $v \in \mathrm{U}(R)$, and $f \in \mathrm{Idem}(R)$ be a π-regular decomposition of $n + ue$. Then, $vf(1-e) = (n - m + ue)(1 - e) = (n - m)(1 - e) \in \mathrm{nil}(R)$. Therefore, $f(1 - e) \in \mathrm{nil}(R)$ since v is unit. On the other hand, $f(1 - e)$ is an idempotent. Thus, $f(1 - e) = 0$. So, $f = ef$. Similarly, $e = ef$. Hence, $e = f$. Since $e\mathrm{nil}(R) = \{0\}$, we get, $ue = (n + ue)e = (m + ve)e = ve$. Accordingly, $n + ue$ is uniquely π-regular.
(2) Follows from (1) by letting $n = 0$.
(3) Follows from (2) by letting $e = 1$.
(4) Follows from (1) by letting $e = 0$. □

Example 3.3. *Set $R = \mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. We have $\mathrm{Idem}(R) = \{(0, \bar{0}), (0, \bar{1}), (1, \bar{0}), (1, \bar{1})\}$, $\mathrm{nil}(R) = \{0\} \times (\bar{2})$, and $\mathrm{ann}(\mathrm{nil}(R)) = \mathbb{Z} \times (\bar{2})$ ($\mathrm{ann}(\mathrm{nil}(R))$ is the annihilator of $\mathrm{nil}(R)$). Thus $\mathrm{Idem}(R) \cap \mathrm{ann}(\mathrm{nil}(R)) = \{(0, \bar{0}), (1, \bar{0})\}$.*

1. *Using Proposition 3.2(1), the set of uniquely π-regular elements is:*

$$\{n + ue \mid u \in \mathrm{U}(R),\ n \in \mathrm{nil}(R),\ e \in \{(0, \bar{0}), (1, \bar{0})\}\} = \{(0, \bar{0}), (1, \bar{0}), (-1, \bar{0}), (0, \bar{2}), (1, \bar{2}), (-1, \bar{2})\}.$$

2. *The element $(0, -\bar{1}) = (0, \bar{2}) + (1, \bar{1})(0, \bar{1}) = (0, \bar{0}) + (1, -\bar{1})(0, \bar{1})$ is π-regular but not uniquely π-regular.*

Corollary 3.4. *Let R be a ring. The following are equivalent:*

1. *Every π-regular element of R is uniquely π-regular.*
2. *Every von Neumann regular element of R is uniquely π-regular.*
3. *Every idempotent element of R is uniquely π-regular.*
4. *R is reduced.*

Proof. (1) ⇒ (2) ⇒ (3) Clear.
(3) ⇒ (4) Follows from Proposition 3.2 since 1 is an idempotent element.
(4) ⇒ (1) Since R is reduced, we have $\pi\mathrm{-r}(R) = \mathrm{vnr}(R) = \{ue \mid u \in \mathrm{U}(R) \text{ and } e \in \mathrm{Idem}(R)\}$ (by [2, Theorems 2.2 and 4.2]). Hence, the desired implication follows from Proposition 3.2. □

Corollary 3.5. *Let R be a ring. The following are equivalent:*

1. *Every element of R is uniquely π-regular.*
2. *R is a von Neumann regular ring.*

Proof. (1) ⇒ (2) The ring R is clearly π-regular, and therefore it is zero-dimensional. Moreover, by Corollary 3.4, R is reduced. Hence, R is a von Neumann regular ring.
(2) ⇒ (1) Follows easily from the fact that a von Neumann regular ring is a reduced π-regular ring. □

It is proved in [2, Theorem 4.4] that, for a ring R, $\pi-\mathrm{r}(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ if and only if R is reduced or indecomposable. Next, we give a new characterization of such rings by using the notion of uniquely π-regularity.

Theorem 3.6. *Let R be a ring. The following are equivalent:*

1. *Every non unit π-regular element of R is uniquely π-regular.*
2. *Every non unit von Neumann regular ring element of R is uniquely π-regular.*
3. *Every non trivial idempotent element of R is uniquely π-regular.*
4. *R is reduced or indecomposable.*
5. $\pi-\mathrm{r}(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$.

Proof. (1) ⇒ (2) ⇒ (3) Clear.
(3) ⇒ (4) Suppose that R is decomposable and let e be a non trivial idempotent of R. Since e is uniquely π-regular, by Proposition 3.2, we get $e\mathrm{nil}(R) = \{0\}$. Similarly, $(1 - e)\mathrm{nil}(R) = \{0\}$. Thus, $\mathrm{nil}(R) = \{0\}$.
(4) ⇔ (5) is [2, Theorem 4.4].
(4) ⇒ (1) If R is a reduced, then every (non unit) π-regular element of R is uniquely π-regular (by Corollary 3.4). Next, suppose that R is indecomposable. Using [2, Theorems 2.2 and 4.2], we get that $\pi-\mathrm{r}(R) = \mathrm{vnr}(R) + \mathrm{nil}(R) = \mathrm{U(R)} \cup \{0\} + \mathrm{nil}(R) = \mathrm{U(R)} \cup \mathrm{nil}(R)$. Let $x \in R$ be a non unit π-regular element. Then, x is nilpotent, and so is uniquely π-regular (by Proposition 3.2). □

It is proved in [2, Theorem 2.4] that, for a ring R, $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ if and only if R is a von Neumann regular ring or R is a *UN*-ring. Next, we give a new characterization of such rings by using the notion of uniquely π-regularity.

Theorem 3.7. *Let R be a ring. The following are equivalent:*

1. *Every non unit element of R is uniquely π-regular.*
2. *R is a von Neumann regular ring or R is a UN-ring.*
3. $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$.

Proof. (1) ⇒ (3) It is clear that every element of R is π-regular (including unit elements of course). So, R is π-regular. Thus, $R = \pi-\mathrm{r}(R)$. Using Theorem 3.6, we get also that $\pi-\mathrm{r}(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$. Hence, $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$.
(2) ⇔ (3) [2, Theorem 2.4].
(2) ⇒ (1) If R is a von Neumann regular ring, then every (non unit) element of R is uniquely π-regular (by Corollary 3.5). Next, suppose that R is a *UN*-ring. Let $x \in R$ be a non unit element. Then, x is nilpotent, and so is uniquely π-regular (by Proposition 3.2). □

Corollary 3.8. *Let R_1 and R_2 be two rings and set $R = R_1 \times R_2$. Then,*

1. $\pi-\mathrm{r}(R) = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ *if and only if R is reduced.*
2. $R = \mathrm{vnr}(R) \cup \mathrm{nil}(R)$ *if and only if R is a von Neumann regular ring.*

Proof. Follows from Theorems 3.6 and 3.7 since R is decomposable. □

4. Discussion and open question

Section 2 gives a connection between weakly nil-clean and π-regular rings in the commutative aspect. However, the class of weakly nil-clean rings is defined over associative (not necessarily commutative) rings (see [3]), and a complete description of this notion has been given in [7], and independently in [15]. Let us also recall that the notion of π-regular rings is also defined in associative (not necessarily commutative) rings . A ring R is π-regular if for all $a \in R$ there exists a positive integer n such that $a^n \in a^n R a^n$. It is therefore logical to ask the following question:

Question: What is the relation between the notion of weakly nil-clean rings and the π-regularity in the case of an arbitrary associative ring?

A work analogous to that done in Section 2 may encounter several problems. Indeed, by passing to the non-commutative case, the notions in question lose many of their properties.
Note that this question was the subject of part of the paper [10]. But do not confuse the notion of weakly nil clean defines in [3] and that defines in [10].

Acknowledgement

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through the project (R.G.P.1/101/43). The authors would like also to thank the referee for the careful review and the valuable comments which improved the presentation of the paper.

References

[1] M.S. Ahn, D.D.Anderson, Weakly clean rings and almost clean rings, The Rocky Mountain Journal of Mathematics 36 (2006) 783–798.
[2] D. F. Anderson, A. Badawi, Von Neumann regular and related elements in commutative rings, Algebra Colloquium 19(1) (2012) 1017–1040.
[3] S. Breaz, P. Danchev, Y. Zhou, Rings in which every element is either a sum or a difference of a nilpotent and an idempotent, Journal of Algebra and its Applications 15(8) (2016) 1650148 (11 pages).
[4] G. Calugareanu, *UN*-rings, Journal of Algebra and its Applications 15(9) (2016), 1650182.
[5] M. D'Anna, M. Fontana, An amalgamated duplication of a ring along an ideal: the basic properties, Journal of Algebra and its Applications 6 (3) (2007) 443–459.
[6] M. D'Anna, M. Fontana, The amalgamated duplication of a ring along a multiplicative-canonical ideal, Arkiv för Matematik 45 (2) (2007) 241–252.
[7] P.V. Danchev, Weakly UU rings, Tsukuba Journal of Mathematics 40(1) (2016) 101–118.
[8] P.V. Danchev, W.Wm. McGovern, Commutative weakly nil clean unital rings, Journal of Algebra 425 (2015) 410–422.
[9] P. Danchev, M. Samiei, Commutative weakly nil neat rings, Novi Sad Journal of Mathematics 50(2), (2020) 51–59.
[10] P. Danchev, J. Šter, Generalizing π-regular rings, Taiwanese Journal of Mathematics 19(6) (2015) 1577–1592.
[11] A. J. Diesl, Nil clean rings, Journal of Algebra 383 (2013) 197–211.
[12] W.Wm. McGovern, Neat rings, Journal of Pure and Applied Algebra 205 (2006) 243–265.
[13] M. Nagata, Local rings, Interscience Tracts in Pure and Applied Mathematics, 13, Wiley Interscience, New York, 1962.
[14] W. K. Nicholson, Lifting idempotents and exchange rings. Transactions of the American Mathematical Society 229 (1977) 269–278.
[15] J. Šter, Nil-clean quadratic elements, Journal of Algebra and its Applications 16(10) (2017) 1750197 (8 pages).
[16] U. Tekir, S. Koc and K. H. Oral, *n*-Ideals of commutative rings, Filomat 31(10) (2017) 2933–2941.