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A study on bipolar soft metric spaces

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Abstract. In this paper, as a new modification of metric spaces, we introduce a bipolar soft metric space which is built by two different soft point sets. After that we give some considerable properties of this new concept as convergence, completeness etc., we prove some fixed point theorems of bipolar soft contractive mappings on bipolar soft metric spaces.

1. Introduction

Metric spaces have a major role in mathematics and other sciences. The concept of metric space was initiated by Fréchet at the beginning of 20th year [7]. Since that day a great many generalizations of metric space were obtained by different authors. Firstly in 1963, $2-$metric space was studied by Gahler [8]. In 1984, $D-$metric space was introduced [5] by using basic modifications in the definition of $2-$metric space. After that Mustafa and Sims introduced $G-$metric space since they found various mistakes in the definition of open set in $D-$metric spaces [12]. In 2012, Sedghi et al introduced the notion of $S-$metric space modifying some conditions in the definition of $D-$metric space in [15]. In this flow, after partial metric space was introduced by [10], Abbas et al examined $A-$metric space as a generalization of $S-$metric space [2]. A. Mutlu and U. Gurdal defined bipolar metric space as a type of partial distance and proved some fixed point theorems [13]. Soft set theory was presented as a significant tool by Molodtsov for dealing with uncertanities [11]. More researchers applied this new concept for their own studies in [1, 3, 6, 9, 14].

Today a great many researchers have defined new generalizations of metric spaces and proved fixed point type theorems in these spaces. In this paper, as a new modification of metric spaces, we introduce concept of bipolar soft metric space which is built by two different soft point sets. After that we give some considerable properties of this new concept as convergence, completeness etc. Finally some important fixed point theorems are studied on bipolar soft metric space.

2. Preliminaries

In this section, we recall some important concepts introduced soft set theory which serve a background to this work. Throughout this paper X, E and $P(X)$ respectively denote the initial universe, a set of all parameters, and the power set of X.

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Definition 2.1. ([11]) A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$.

In other words, the soft set is a parameterized family of subsets of the set X. For $e \in E$, $F(e)$ may be considered as the set of $e-$elements of the soft set (F, E), or as the set of $e-$approximate elements of the soft set.

Definition 2.2. ([1]) For two soft sets (F, E) and (G, E) over X, (F, E) is called a soft subset of (G, E) if $\forall e \in E$, $F(e) \subseteq G(e)$. This relationship is denoted by $(F, E)\widetilde{\subseteq}(G, E)$.

Similarly, (F, E) is called a soft superset of (G, E) if (G, E) is a soft subset of (F, E). This relationship is denoted by $(F, E)\widetilde{\supseteq}(G, E)$. Two soft sets (F, E) and (G, E) over X are called soft equal if (F, E) is a soft subset of (G, E) and (G, E) is a soft subset of (F, E).

Definition 2.3. ([1]) The intersection of two soft sets (F, E) and (G, E) over X is the soft set (H, E), where $\forall e \in E$, $H(e) = F(e) \cap G(e)$. This is denoted by $(F, E)\widetilde{\cap}(G, E) = (H, E)$.

Definition 2.4. ([1]) The union of two soft sets (F, E) and (G, E) over X is the soft set (H, E), where $\forall e \in E$, $H(e) = F(e) \cup G(e)$. This is denoted by $(F, E)\widetilde{\cup}(G, E) = (H, E)$.

Definition 2.5. ([9]) A soft set (F, E) over X is said to be a null soft set denoted by Φ if for all $e \in E$, $F(e) = \varnothing$.

Definition 2.6. ([9]) A soft set (F, E) over X is said to be an absolute soft set denoted by $\widetilde{X}$ if for all $e \in E$, $F(e) = X$.

Definition 2.7. ([9]) The difference (H, E) of two soft sets (F, E) and (G, E) over X, denoted by $(F, E)\widetilde{\backslash}(G, E)$, is defined as $H(e) = F(e)\backslash G(e)$ for all $e \in E$.

Definition 2.8. ([9]) The complement of a soft set (F, E), denoted by $(F, E)^c$, is defined $(F, E)^c = (F^c, E)$, where $F^c : E \to P(X)$ is a mapping given by $F^c(e) = X\backslash F(e)$, $\forall e \in E$ and F^c is called the soft complement function of F.

Definition 2.9. ([3, 4]) Let (F, E) be a soft set over X. The soft set (F, E) is called a soft point, denoted by (x_e, E), if for the element $e \in E$, $F(e) = \{x\}$ and $F(e') = \varnothing$ for all $e' \in E - \{e\}$ (briefly denoted by x_e).

It is obvious that each soft set can be expressed as a union of soft points. For this reason, to give the family of all soft sets on X it is sufficient to give only soft points on X.

Definition 2.10. ([3]) Two soft points x_e and $y_{e'}$ over a common universe X, we say that the soft points are different if $x \neq y$ or $e \neq e'$.

Definition 2.11. ([3]) The soft point x_e is said to be belonging to the soft set (F, E), denoted by $x_e\widetilde{\in}(F, E)$, if $x_e(e) \in F(e)$,i.e., $\{x\} \subseteq F(e)$.

Definition 2.12. ([3]) Let $(X, \widetilde{\tau}, E)$ be a soft topological space over X. A soft set $(F, E)\widetilde{\subseteq}(X, E)$ is called a soft neighborhood of the soft point $x_e\widetilde{\in}(F, E)$ if there exists a soft open set (G, E) such that $x_e\widetilde{\in}(G, E)\widetilde{\subset}(F, E)$.

Definition 2.13. ([4]) Let $\mathbb{R}$ be the set of all real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E be taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. It is denoted by (F, E). If (F, E) is a singleton soft set, then it will be called a soft real number and denoted by $\widetilde{r}, \widetilde{s}, \widetilde{t}$ etc. Here $\widetilde{r}, \widetilde{s}, \widetilde{t}$ will denote a particular type of soft real numbers such that $\widetilde{r}(e) = r$ for all $e \in E$. $\widetilde{0}$ and $\widetilde{1}$ are the soft real numbers, where $\widetilde{0}(e) = 0$, $\widetilde{1}(e) = 1$ for all $e \in E$, respectively.

The following definition is about a partial ordering on the set of soft real numbers.

Definition 2.14. ([4]) Let $\widetilde{r}, \widetilde{s}$ be two soft real numbers, then the following statements hold:

(i) $\widetilde{r}\widetilde{\leq}\widetilde{s}$, if $\widetilde{r}(e) \leq \widetilde{s}(e)$ for all $e \in E$,
(ii) $\widetilde{r}\widetilde{\geq}\widetilde{s}$, if $\widetilde{r}(e) \geq \widetilde{s}(e)$ for all $e \in E$,
(iii) $\widetilde{r}\widetilde{<}\widetilde{s}$, if $\widetilde{r}(e) < \widetilde{s}(e)$ for all $e \in E$,
(iv) $\widetilde{r}\widetilde{>}\widetilde{s}$, if $\widetilde{r}(e) > \widetilde{s}(e)$ for all $e \in E$.

Definition 2.15. ([13]) Let X and Y be two non-empty sets and $d : X \times Y \to [0, \infty)$ be a function satisfying the following conditions:

(1) If $d(x, y) = 0$, then $x = y$ for all $(x, y) \in X \times Y$,
(2) If $x = y$, then $d(x, y) = 0$ for all $(x, y) \in X \times Y$,
(3) $d(x, y) = d(y, x)$ for all $(x, y) \in X \cap Y$,
(4) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.
Then d is called a bipolar metric on (X, Y) and the pair (X, Y, d) is called a bipolar metric space.

3. Contractive mapping on bipolar soft metric spaces

In this section, we introduce concept of bipolar soft metric space. Later we define some important concepts such as bipolar soft bisequence, Cauchy bisequence and complete bipolar soft metric space and prove some fixed point theorems. Let $\widetilde{X}$ be the absolute soft set, E be a non-empty set of parameters.

Definition 3.1. Let $SP(\widetilde{X})$, $SP(\widetilde{Y})$ be two non-empty soft sets of soft points on X and Y, respectively and $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers, $\widetilde{d} : SP(\widetilde{X}) \times SP(\widetilde{Y}) \to \mathbb{R}(E)^*$ be a function . Consider the following properties:

($SB0$) If $\widetilde{d}(x_e, y_{e'}) = \widetilde{0}$, then $x_e = y_{e'}$ for all $(x_e, y_{e'}) \in SP(\widetilde{X}) \times SP(\widetilde{Y})$,
($SB1$) If $x_e = y_{e'}$, then $\widetilde{d}(x_e, y_{e'}) = \widetilde{0}$ for all $(x_e, y_{e'}) \in SP(\widetilde{X}) \times SP(\widetilde{Y})$,
($SB2$) $\widetilde{d}(x_e, y_{e'}) = \widetilde{d}(y_{e'}, x_e)$ for all $(x_e, y_{e'}) \in SP(\widetilde{X}) \cap SP(\widetilde{Y})$,
($SB3$) $\widetilde{d}\left(x^1_{e_1}, y^2_{e'_2}\right) \leq \widetilde{d}\left(x^1_{e_1}, y^1_{e'_1}\right) + \widetilde{d}\left(x^2_{e_2}, y^1_{e'_1}\right) + \widetilde{d}\left(x^2_{e_2}, y^2_{e'_2}\right)$ for all $x^1_{e_1}, x^2_{e_2} \in SP(\widetilde{X})$ and $y^1_{e'_1}, y^2_{e'_2} \in SP(\widetilde{Y})$.
(i) If ($SB1$) and ($SB2$) hold, then $\widetilde{d}$ is called a bipolar soft pseudo-semimetric on $\left(SP(\widetilde{X}), SP(\widetilde{Y})\right)$.

(ii) If $\widetilde{d}$ is a bipolar soft pseudo-semimetric satisfying the condition ($SB3$) , then $\widetilde{d}$ is called a bipolar soft pseudo-metric.

(iii) If $\widetilde{d}$ is a bipolar soft pseudo-metric satisfying the condition ($SB0$) , then $\widetilde{d}$ is called a bipolar soft metric.

A bipolar soft metric space is denoted by $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$.

Example 3.2. Let $E = \mathbb{N}$ be a non-empty set of parameters, $X = \mathbb{R}$ and $Y = [-5, 5]$. Then let us define a mapping

$$\widetilde{d} : SP(\widetilde{X}) \times SP(\widetilde{Y}) \to \mathbb{R}(E)^*$$

by

$$\widetilde{d}(x_m, y_k) = \left|x^2 - y^2\right| + |m - k|$$

for all $x_m \in SP(\widetilde{X})$ and $y_k \in SP(\widetilde{Y})$.It is clear that $\widetilde{d}$ is a bipolar soft metric on the pair $SP(\widetilde{X}) \times SP(\widetilde{Y})$.

Definition 3.3. If $SP(\widetilde{X}) \cap SP(\widetilde{Y}) = \Phi$, then $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$ is said to be a disjoint space. Otherwise $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$ is said to be a joint. Also, the soft set $SP(\widetilde{X})$ is said to be left pole and $SP(\widetilde{Y})$ is said to be right pole of bipolar soft metric space $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$. Here the soft points of $SP(\widetilde{X})$ and $SP(\widetilde{Y})$ are called as left and right soft points, respectively. Any soft sequence which is consisted of only left (right) soft points is said to be left (right) soft sequence on $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$.

Definition 3.4. Let $\left(\widetilde{X},\widetilde{Y},\widetilde{d},E\right)$ be a bipolar soft pseudo-semimetric space. A left soft sequence $\left\{x^n_{e_n}\right\}_{n\in\mathbb{N}}$ converges to a right soft point y_e if and only if for every $\widetilde{\varepsilon}>\widetilde{0}$, there exists a $n_0\in\mathbb{N}$ such that $\widetilde{d}\left(x^n_{e_n},y_{e'}\right)<\widetilde{\varepsilon}$ for each $n\geq n_0$, denoted by $x^n_{e_n}\to y_{e'}$. This means that $\widetilde{d}\left(x^n_{e_n},y_{e'}\right)\to\widetilde{0}$.

Definition 3.5. Let $\left(\widetilde{X},\widetilde{Y},\widetilde{d},E\right)$ be a bipolar soft pseudo-semimetric space. A right soft sequence $\left\{y^n_{e'_n}\right\}_{n\in\mathbb{N}}$ converges to a left soft point x_e if and only if for every $\widetilde{\varepsilon}>\widetilde{0}$, there exists a $n_0\in\mathbb{N}$ such that $\widetilde{d}\left(x_e,y^n_{e'_n}\right)<\widetilde{\varepsilon}$ for each $n\geq n_0$ and denoted by $y^n_{e'_n}\to x_e$. This means that $\widetilde{d}\left(x_e,y^n_{e'_n}\right)\to\widetilde{0}$.

Definition 3.6. Let $\left(\widetilde{X},\widetilde{Y},\widetilde{d},E\right)$ be a bipolar soft pseudo-semimetric space.

(a) A soft sequence $\left\{\left(x^n_{e_n},y^n_{e'_n}\right)\right\}$ on the soft set $SP(\widetilde{X})\times SP(\widetilde{Y})$ is said to be a soft bisequence on $\left(\widetilde{X},\widetilde{Y},\widetilde{d},E\right)$.

(b) If both $\left\{x^n_{e_n}\right\}$ and $\left\{y^n_{e'_n}\right\}$ convergent, then the soft bisequence $\left\{\left(x^n_{e_n},y^n_{e'_n}\right)\right\}$ is said to be a soft convergent.

(c) If $\left\{x^n_{e_n}\right\}$ and $\left\{y^n_{e'_n}\right\}$ both convergent to a same soft point $z_e\in SP(\widetilde{X})\cap SP(\widetilde{Y})$, then $\left\{\left(x^n_{e_n},y^n_{e'_n}\right)\right\}$ is said to be a biconvergent.

(d) The soft bisequence $\left\{\left(x^n_{e_n},y^n_{e'_n}\right)\right\}$ on the soft set $SP(\widetilde{X})\times SP(\widetilde{Y})$ is said to be a Cauchy soft bisequence, if for $\widetilde{\varepsilon}>\widetilde{0}$, there exists $n_0\in\mathbb{N}$ such that for each $n,m\geq n_0$, $\widetilde{d}\left(x^n_{e_n},y^m_{e'_m}\right)<\widetilde{\varepsilon}$, i.e. $\widetilde{d}\left(x^n_{e_n},y^m_{e'_m}\right)\to\widetilde{0}$ as $n,m\to\infty$.

Proposition 3.7. *Every biconvergent soft bisequence in a bipolar soft pseudo-metric space is also Cauchy soft bisequence.*

Proof. Let $\left(\widetilde{X},\widetilde{Y},\widetilde{d},E\right)$ be a bipolar soft pseudo-metric space and $\left\{\left(x^n_{e_n},y^n_{e'_n}\right)\right\}$ be a biconvergent soft bisequence to a soft point $z_e\in SP(\widetilde{X})\cap SP(\widetilde{Y})$. Then for all positive integers n,m, we obtain

$$\widetilde{d}\left(x^n_{e_n},y^m_{e'_m}\right)\leq\widetilde{d}\left(x^n_{e_n},z_e\right)+\widetilde{d}\left(z_e,y^m_{e'_m}\right),$$

which implies that $\left\{\left(x^n_{e_n},y^n_{e'_n}\right)\right\}$ is a Cauchy soft bisequence. □

Definition 3.8. If every Cauchy soft bisequence is convergent, then bipolar soft metric space $\left(\widetilde{X},\widetilde{Y},\widetilde{d},E\right)$ is said to be complete.

Definition 3.9. Let $\left(\widetilde{X},\widetilde{Y},\widetilde{d},E\right)$ be a bipolar soft metric space. $f:\left(\widetilde{X},\widetilde{Y},\widetilde{d},E\right)\to\left(\widetilde{X},\widetilde{Y},\widetilde{d},E\right)$ is called a bipolar soft contraction mapping if there exists a soft real number $\widetilde{\alpha}\in\mathbb{R}(E)$, $\widetilde{0}\leq\widetilde{\alpha}<\widetilde{1}$ ($\mathbb{R}(E)$ denotes the soft real numbers set) such that

$$\widetilde{d}\left(f\left(x_e\right),f\left(y_{e'}\right)\right)\leq\widetilde{\alpha}\widetilde{d}\left(x_e,y_{e'}\right)$$

for every soft points $x_e\in SP(\widetilde{X})$ and $y_{e'}\in SP(\widetilde{Y})$.

Proposition 3.10. *Every bipolar soft contraction mapping is a soft continuous mapping.*

Proof. Let $x_e\in SP(\widetilde{X})$ and $y_{e'}\in SP(\widetilde{Y})$ be any soft points and $\widetilde{\varepsilon}>\widetilde{0}$ be an arbitrary. If we choose $\widetilde{d}\left(x_e,y_{e'}\right)<\widetilde{\delta}<\widetilde{\varepsilon}$, then since f is a bipolar soft contraction mapping we obtain

$$\widetilde{d}\left(f\left(x_e\right),f\left(y_{e'}\right)\right)\leq\widetilde{\alpha}\widetilde{d}\left(x_e,y_{e'}\right)<\widetilde{\alpha}\widetilde{\delta}<\widetilde{\varepsilon}$$

and so f is a soft contraction mapping. □

Definition 3.11. Let $\left(\widetilde{X_1}, \widetilde{Y_1}, \widetilde{d_1}, E\right)$ and $\left(\widetilde{X_2}, \widetilde{Y_2}, \widetilde{d_2}, E\right)$ be two bipolar soft pseudo-semimetric spaces and $f : \left(\widetilde{X_1} \cup \widetilde{Y_1}, E\right) \to \left(\widetilde{X_2} \cup \widetilde{Y_2}, E\right)$ be a soft function. If $f\left(\widetilde{X_1}\right) \subset \left(\widetilde{X_2}\right)$ and $f\left(\widetilde{Y_1}\right) \subset \left(\widetilde{Y_2}\right)$ are satisfied, then the soft mapping f is called a covariant soft mapping from $\left(\widetilde{X_1}, \widetilde{Y_1}, \widetilde{d_1}, E\right)$ to $\left(\widetilde{X_2}, \widetilde{Y_2}, \widetilde{d_2}, E\right)$. Also, if $f : \left(\widetilde{X_1}, \widetilde{Y_1}, \widetilde{d_1}, E\right) \to \left(\widetilde{Y_2}, \widetilde{X_2}, \widetilde{d_2}, E\right)$ is a soft mapping, then f is called contravariant soft mapping from $\left(\widetilde{X_1}, \widetilde{Y_1}, \widetilde{d_1}, E\right)$ to $\left(\widetilde{X_2}, \widetilde{Y_2}, \widetilde{d_2}, E\right)$.

Theorem 3.12. *Let $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$ be a complete bipolar soft metric space and $f : \left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right) \to \left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$ be a bipolar soft contraction mapping. Then the soft function $f : \left(\widetilde{X} \cup \widetilde{Y}, E\right) \to \left(\widetilde{X} \cup \widetilde{Y}, E\right)$ has a unique soft fixed point.*

Proof. Since f is a bipolar soft contraction mapping, there exists $\widetilde{\alpha} \in \left(\widetilde{0}, \widetilde{1}\right)$ such that

$$\widetilde{d}\left(f\left(x_e\right), f\left(y_{e'}\right)\right) \leq \widetilde{\alpha}\widetilde{d}\left(x_e, y_{e'}\right)$$

for all $(x_e, y_{e'}) \in SP(\widetilde{X}) \times SP(\widetilde{Y})$. Let $x^0_{e_0} \in SP(\widetilde{X})$, $y^0_{e'_0} \in SP(\widetilde{Y})$. For each $n \in \mathbb{N}$, define $f\left(x^n_{e_n}\right) = x^{n+1}_{e_{n+1}}$, $f\left(y^n_{e'_n}\right) = y^{n+1}_{e'_{n+1}}$. Then $\left\{\left(x^n_{e_n}, y^n_{e'_n}\right)\right\}$ is a soft bisequence on $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$. Then for each positive integer n and p, we have

$$\begin{aligned}
\widetilde{d}\left(x^n_{e_n}, y^n_{e'_n}\right) &= \widetilde{d}\left(f\left(x^{n-1}_{e_{n-1}}\right), f\left(y^{n-1}_{e'_{n-1}}\right)\right) \\
&\leq \widetilde{\alpha}\widetilde{d}\left(x^{n-1}_{e_{n-1}}, y^{n-1}_{e'_{n-1}}\right) \leq \ldots \leq \widetilde{\alpha}^n \widetilde{d}\left(x^0_{e_0}, y^0_{e'_0}\right),
\end{aligned}$$

and also, define $P := \widetilde{d}\left(x^0_{e_0}, y^0_{e'_0}\right) + \widetilde{d}\left(x^0_{e_0}, y^1_{e'_1}\right)$. Then

$$\begin{aligned}
\widetilde{d}\left(x^n_{e_n}, y^{n+1}_{e'_{n+1}}\right) &= \widetilde{d}\left(f\left(x^{n-1}_{e_{n-1}}\right), f\left(y^n_{e'_n}\right)\right) \\
&\leq \widetilde{\alpha}\widetilde{d}\left(x^{n-1}_{e_{n-1}}, y^n_{e'_n}\right) \leq \ldots \leq \widetilde{\alpha}^n \widetilde{d}\left(x^0_{e_0}, y^1_{e'_1}\right).
\end{aligned}$$

Thus we get

$$\begin{aligned}
\widetilde{d}\left(x^{n+p}_{e_n+p}, y^n_{e'_n}\right) &\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+1}_{e'_{n+1}}\right) + \widetilde{d}\left(x^n_{e_n}, y^{n+1}_{e'_{n+1}}\right) + \widetilde{d}\left(x^n_{e_n}, y^n_{e'_n}\right) \\
&\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+1}_{e'_{n+1}}\right) + \widetilde{\alpha}^n \left(\widetilde{d}\left(x^0_{e_0}, y^1_{e'_1}\right) + \widetilde{d}\left(x^0_{e_0}, y^0_{e'_0}\right)\right) \\
&\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+1}_{e'_{n+1}}\right) + \widetilde{\alpha}^n P \\
&\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+2}_{e'_{n+2}}\right) + \widetilde{d}\left(x^{n+1}_{e_{n+1}}, y^{n+2}_{e'_{n+2}}\right) + \widetilde{d}\left(x^{n+1}_{e_{n+1}}, y^{n+1}_{e'_{n+1}}\right) + \widetilde{\alpha}^n P \\
&\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+2}_{e'_{n+2}}\right) + \left(\widetilde{\alpha}^{n+1} + \widetilde{\alpha}^n\right) P \leq \ldots \\
&\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+p}_{e'_{n+p}}\right) + \left(\widetilde{\alpha}^{n+p-1} + \ldots + \widetilde{\alpha}^{n+1} + \widetilde{\alpha}^n\right) P \\
&\leq \left(\widetilde{\alpha}^{n+p} + \ldots + \widetilde{\alpha}^{n+1} + \widetilde{\alpha}^n\right) P \\
&\leq \widetilde{\alpha}^n P \sum_{k=0}^{\infty} \widetilde{\alpha}^k = \frac{\widetilde{\alpha}^n . P}{1 - \widetilde{\alpha}} = A_n.
\end{aligned}$$

Similarly, we obtain $\widetilde{d}\left(x^n_{e_n}, y^{n+p}_{e'_{n+p}}\right) \leq A_n$. Since $A_n \to \widetilde{0}$, then for each $\widetilde{\varepsilon} > \widetilde{0}$, there exists $n_0 \in \mathbb{N}$ such that

$A_{n_0} < \frac{\widetilde{\varepsilon}}{3}$. Then

$$\begin{aligned}\widetilde{d}\left(x_{e_n}^n, y_{e'_m}^m\right) &\leq \widetilde{d}\left(x_{e_n}^n, y_{e'_{n_0}}^{n_0}\right) + \widetilde{d}\left(x_{e_{n_0}}^{n_0}, y_{e'_{n_0}}^{n_0}\right) + \widetilde{d}\left(x_{e_{n_0}}^{n_0}, y_{e'_m}^m\right) \\ &\leq 3A_{n_0} \leq \widetilde{\varepsilon}\end{aligned}$$

and hence $\left\{\left(x_{e_n}^n, y_{e'_n}^n\right)\right\}$ is a Cauchy soft bisequence. Since $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$ is a complete bipolar soft metric space, $\left\{\left(x_{e_n}^n, y_{e'_n}^n\right)\right\}$ converges. Thus soft bisequence biconverges to a soft point $z_e \in SP(\widetilde{X}) \cap SP(\widetilde{Y})$ and $f\left(y_{e'_n}^n\right) = y_{e'_{n+1}}^{n+1} \to z_e \in SP(\widetilde{X}) \cap SP(\widetilde{Y})$.Hence $\left\{f\left(y_{e'_n}^n\right)\right\}$ has a unique soft limit. Since f is a soft continuous, $\left\{f\left(y_{e'_n}^n\right)\right\} \to f(z_e)$. So $f(z_e) = z_e$. Therefore z_e is a soft fixed point of f.

If u_e is any soft fixed point of f, then $f(u_e) = u_e$ implies that $u_e \in SP(\widetilde{X}) \cap SP(\widetilde{Y})$ and we obtain

$$\widetilde{d}(z_e, u_e) = \widetilde{d}(f(z_e), f(u_e)) \leq \widetilde{\alpha}.\widetilde{d}(z_e, u_e),$$

which implies that $\widetilde{d}(z_e, u_e) = \widetilde{0}$, so $z_e = u_e$. □

Theorem 3.13. *Let $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$ be a complete bipolar soft metric space and $f : \left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right) \to \left(\widetilde{Y}, \widetilde{X}, \widetilde{d}, E\right)$ be a contravariant soft contraction mapping. Then the soft function $f : \left(\widetilde{X} \cup \widetilde{Y}, E\right) \to \left(\widetilde{X} \cup \widetilde{Y}, E\right)$ has a unique soft fixed point.*

Proof. Since f is a bipolar soft contraction mapping, there exists $\widetilde{\alpha} \in \left(\widetilde{0}, \widetilde{1}\right)$ such that

$$\widetilde{d}(f(y_{e'}), f(x_e)) \leq \widetilde{\alpha}\widetilde{d}(x_e, y_{e'})$$

for all $(x_e, y_{e'}) \in SP(\widetilde{X}) \times SP(\widetilde{Y})$. Let $x_{e_0}^0 \in SP(\widetilde{X})$, $y_{e'_0}^0 \in SP(\widetilde{Y})$. For each $n \in \mathbb{N}$, define $f\left(x_{e_n}^n\right) = y_{e'_n}^n$, $f\left(y_{e'_n}^n\right) = x_{e_{n+1}}^{n+1}$. Then $\left\{\left(x_{e_n}^n, y_{e'_n}^n\right)\right\}$ is a soft bisequence on $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$. Then for each positive integer n and p, we have

$$\begin{aligned}\widetilde{d}\left(x_{e_n}^n, y_{e'_n}^n\right) &= \widetilde{d}\left(f\left(y_{e'_{n-1}}^{n-1}\right), f\left(x_{e_n}^n\right)\right) \\ &\leq \widetilde{\alpha}\widetilde{d}\left(x_{e_n}^n, y_{e'_{n-1}}^{n-1}\right) = \widetilde{\alpha}\widetilde{d}\left(f\left(y_{e'_{n-1}}^{n-1}\right), f\left(x_{e_{n-1}}^{n-1}\right)\right) \\ &\leq \widetilde{\alpha}^2\widetilde{d}\left(x_{e_{n-1}}^{n-1}, y_{e'_{n-1}}^{n-1}\right) \leq \ldots \leq \widetilde{\alpha}^{2n}\widetilde{d}\left(x_{e_0}^0, y_{e'_0}^0\right)\end{aligned}$$

and also, define $A_n = \frac{\widetilde{\alpha}^{2n}}{1-\widetilde{\alpha}}$. Then

$$\widetilde{d}\left(x_{e_n}^n, y_{e'_n}^n\right) \leq \widetilde{\alpha}^{2n}\widetilde{d}\left(x_{e_0}^0, y_{e'_0}^0\right) = \left(\widetilde{1} - \widetilde{\alpha}\right) A_n \leq A_n$$

is obtained.

$$\begin{aligned}\widetilde{d}\left(x_{e_{n+1}}^{n+1}, y_{e'_n}^n\right) &= \widetilde{d}\left(f\left(y_{e'_n}^n\right), f\left(x_{e_n}^n\right)\right) \\ &\leq \widetilde{\alpha}\widetilde{d}\left(x_{e_n}^n, y_{e'_n}^n\right) \\ &\leq \widetilde{\alpha}^{2n+1}\widetilde{d}\left(x_{e_0}^0, y_{e'_0}^0\right),\end{aligned}$$

and so we obtain

$$\begin{aligned}
\widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n}_{e'_n}\right) &\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+1}_{e'_{n+1}}\right) + \widetilde{d}\left(x^{n+1}_{e_{n+1}}, y^{n+1}_{e'_{n+1}}\right) + \widetilde{d}\left(x^{n+1}_{e_{n+1}}, y^{n}_{e'_n}\right) \\
&\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+1}_{e'_{n+1}}\right) + \left(\widetilde{\alpha}^{2n+2} + \widetilde{\alpha}^{2n+1}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) \\
&\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+2}_{e'_{n+2}}\right) + \widetilde{d}\left(x^{n+2}_{e_{n+2}}, y^{n+2}_{e'_{n+2}}\right) + \widetilde{d}\left(x^{n+2}_{e_{n+2}}, y^{n+1}_{e'_{n+1}}\right) + \left(\widetilde{\alpha}^{2n+2} + \widetilde{\alpha}^{2n+1}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) \\
&\leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+2}_{e'_{n+2}}\right) + \left(\widetilde{\alpha}^{2n+4} + \widetilde{\alpha}^{2n+3} + \widetilde{\alpha}^{2n+2} + \widetilde{\alpha}^{2n+1}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) \\
&\leq \ldots \leq \widetilde{d}\left(x^{n+p}_{e_n+p}, y^{n+p-1}_{e'_{n+p-1}}\right) + \left(\widetilde{\alpha}^{2n+2p-2} + \ldots + \widetilde{\alpha}^{2n+1}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) \\
&\leq \left(\widetilde{\alpha}^{2n+2p-1} + \widetilde{\alpha}^{2n+2p-2} + \widetilde{\alpha}^{2n+2p-3} + \ldots + \widetilde{\alpha}^{2n+1}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) \\
&\leq \widetilde{\alpha}^{2n+1}\sum_{k=0}^{\infty}\widetilde{\alpha}^{k}\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) = \widetilde{\alpha}A_n \leq A_n.
\end{aligned}$$

Hence,

$$\begin{aligned}
\widetilde{d}\left(x^{n}_{e_n}, y^{n+p}_{e'_{n+p}}\right) &\leq \widetilde{d}\left(x^{n}_{e_n}, y^{n}_{e'_n}\right) + \widetilde{d}\left(x^{n+1}_{e_{n+1}}, y^{n}_{e'_n}\right) + \widetilde{d}\left(x^{n+1}_{e_{n+1}}, y^{n+p}_{e'_{n+p}}\right) \\
&\leq \left(\widetilde{\alpha}^{2n} + \widetilde{\alpha}^{2n+1}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) + \widetilde{d}\left(x^{n+1}_{e_{n+1}}, y^{n+p}_{e'_{n+p}}\right) \\
&\leq \left(\widetilde{\alpha}^{2n} + \widetilde{\alpha}^{2n+1}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) + \widetilde{d}\left(x^{n+1}_{e_{n+1}}, y^{n+1}_{e'_{n+1}}\right) + \widetilde{d}\left(x^{n+2}_{e_{n+2}}, y^{n+1}_{e'_{n+1}}\right) + \widetilde{d}\left(x^{n+2}_{e_{n+2}}, y^{n+p}_{e'_{n+p}}\right) \\
&\leq \left(\widetilde{\alpha}^{2n} + \widetilde{\alpha}^{2n+1} + \widetilde{\alpha}^{2n+2} + \widetilde{\alpha}^{2n+3}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) + \widetilde{d}\left(x^{n+2}_{e_{n+2}}, y^{n+p}_{e'_{n+p}}\right) \\
&\leq \ldots \leq \left(\widetilde{\alpha}^{2n} + \widetilde{\alpha}^{2n+1} + \ldots + \widetilde{\alpha}^{2n+2p-1}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) + \widetilde{d}\left(x^{n+p}_{e_{n+p}}, y^{n+p}_{e'_{n+p}}\right) \\
&\leq \left(\widetilde{\alpha}^{2n} + \widetilde{\alpha}^{2n+1} + \ldots + \widetilde{\alpha}^{2n+2p-1} + \widetilde{\alpha}^{2n+2p}\right)\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) \\
&\leq \widetilde{\alpha}^{2n}\sum_{k=0}^{\infty}\widetilde{\alpha}^{k}\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) = A_n
\end{aligned}$$

is satisfied. Also, for each $\widetilde{\varepsilon} > \widetilde{0}$, there exists $n_0 \in \mathbb{N}$ such that $A_{n_0} = \frac{\widetilde{\alpha}^{2n_0+1}}{1-\widetilde{\alpha}}\widetilde{d}\left(x^{0}_{e_0}, y^{0}_{e'_0}\right) < \frac{\widetilde{\varepsilon}}{3}$. Then

$$\begin{aligned}
\widetilde{d}\left(x^{n}_{e_n}, y^{m}_{e'_m}\right) &\leq \widetilde{d}\left(x^{n}_{e_n}, y^{n_0}_{e'_{n_0}}\right) + \widetilde{d}\left(x^{n_0}_{e_{n_0}}, y^{n_0}_{e'_{n_0}}\right) + \widetilde{d}\left(x^{n_0}_{e_{n_0}}, y^{m}_{e'_m}\right) \\
&\leq 3A_{n_0} \leq \widetilde{\varepsilon}
\end{aligned}$$

and hence $\left\{\left(x^{n}_{e_n}, y^{n}_{e'_n}\right)\right\}$ is a Cauchy soft bisequence. Since $\left(\widetilde{X}, \widetilde{Y}, \widetilde{d}, E\right)$ is a complete bipolar soft metric space, $\left\{\left(x^{n}_{e_n}, y^{n}_{e'_n}\right)\right\}$ converges. Thus soft bisequence biconverges to a soft point $z_e \in SP(\widetilde{X}) \cap SP(\widetilde{Y})$ and the soft sequences $\left\{x^{n}_{e_n}\right\}$ and $\left\{y^{n}_{e'_n}\right\}$ have a unique soft limit point. Since f is a soft continuous, $x^{n}_{e_n} \to z_e$ implies that $\left\{y^{n}_{e'_n}\right\} = \left\{f\left(x^{n}_{e_n}\right)\right\} \to f(z_e)$. Since $y^{n}_{e'_n} \to z_e$ gives $f(z_e) = z_e$.Therefore z_e is a soft fixed point of f.

If v_e is any soft fixed point of f, then $f(v_e) = v_e$ implies that $v_e \in SP(\widetilde{X}) \cap SP(\widetilde{Y})$ and we obtain

$$\widetilde{d}(z_e, v_e) = \widetilde{d}(f(z_e), f(v_e)) \leq \widetilde{\alpha}.\widetilde{d}(z_e, v_e),$$

which implies that $\widetilde{d}(z_e, v_e) = \widetilde{0}$, so $z_e = v_e$. □

4. Conclusion

From the past to the present, there is the problem of calculating the distance between two points of a set in mathematics and many other sciences. At this point, the notion of metric help us in these problems. After understanding value of the metric, a great many generalizations of metric space were obtained. Besides the distance calculation, in real life problems and various scientific studies appear problems of calculating the distance between different points of different sets. Bipolar metric is the important structure that provides to overcome these problems and gets a solution in this kind distance problems. The concept of soft point which is presented a considerable and inspiring tool in mathematics is a special case of soft set. Considering the importance of both soft sets and bipolar metric spaces, this new metric structure, which is obtained by combining soft points and the structure of bipolar metric, has great importance. In this study, we have presented the concept of bipolar soft metric space. We have constructed this theory using both soft points of soft sets and the concept of bipolar metric spaces. Afterward, we have introduced bipolar soft metric space which is based on soft points of soft sets and prove some fixed point theorems. This new design are valuable since it is a new generalization of metric spaces and there are many possibilities for new studies in this concept. Also, it provides a new perspective in mathematics and encourages authors to studies in areas of mathematics such as fixed point theory, fixed circle theory, engineering and the other sciences.

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