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On some types of light-like submanifolds of poly-Norden semi-Riemannian manifolds

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Abstract. In this paper, we initiate the study of poly-Norden generalized CR-light-like submanifolds and poly-Norden screen transversal CR-light-like submanifolds of poly-Norden semi-Riemannian manifolds. We give examples for such types light-like submanifolds and investigate the conditions for both integrability and totally geodesic foliation descriptions of distributions.

1. Introduction

Studying light-like submanifolds of semi-Riemannian (briefly, s-Riemannian) manifolds, which have physical meanings in general relativity and black hole theory (see [1]), is often much more complex and difficult than in the Riemannian case because of the degeneracy of the induced metric. Although there are great similarities between the local and global geometries of submanifolds in s-Riemannian manifolds and submanifolds in Riemannian manifolds, some issues require very different approaches and solutions in light-like case. To overcome these and similar problems, two different ways have been followed until today.

Light-like geometry of submanifolds in s-Riemannian manifolds was initiated to study by Duggal and Bejancu [3], by introducing a non-degenerate distribution, namely screen distribution, to provide a good technique for a considereable amount of geometric results. Their approach is known as an extrinsic approach. For another approach which can be viewed as an intrinsic approach we refer to [2].

Duggal and Bejancu [3] defined CR-light-like submanifolds of indefinite Kaehler manifolds as a generalization of light-like real hypersurfaces of indefinite Kaehler manifolds. Since CR-light-like submanifolds do not contain invariant and totally real light-like submanifolds, Duggal and Şahin [4] introduced screen CR-light-like submanifolds of indefinite Kaehler manifolds which include invariant submanifolds as well as screen real submanifolds.

After then, to construct a relation between CR-light-like submanifolds and screen CR-light-like submanifolds, Duggal and Şahin [5, 8] defined a new class of light-like submanifolds called generalized CR-light-like submanifolds. It can be seen that real light-like curves are not included by CR-light-like, screen CR-light-like

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and generalized CR-light-like submanifolds. So, screen transversal light-like submanifolds of indefinite Kahler manifolds which contain real light-like curves were introduced in [6]. A new type of light-like submanifolds which is called screen transversal CR-lightlike submanifolds was defined by Doğan et al. [19] and it was shown that this class can be viewed as an umbrella for CR-light-like submanifolds and screen transversal light-like submanifolds. There are many paper in the literature related to this subject, (see [7, 9, 20]).

For any positive integer p, q, the positive solutions of the equation

$$x^2 - px - q = 0,$$

are called as metallic means and they are also known $(p, q)-$metallic numbers which are denoted by [11]

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

By use of metallic means, metallic Riemannian manifolds and their submanifolds were defined by Hretcanu, Crasmereanu and Blaga [11, 14]. Recently, different types of submanifolds of golden and metallic (semi) Riemannian manifolds have been published (see [4, 10–14]).

On the other hand, a new bronze mean have been introduced by Kalia [15]. Also, unlike the bronze mean given in [16], new bronze mean given in [15] can not be expressed with $\sigma_{p,q}$ for any positive integer p, q.

In [17], a new type of manifold which is called almost poly-Norden manifolds has been investigated by Şahin. Recently Perktaş studied submanifolds of almost poly-Norden Riemannian manifolds [18].

In the present paper, motivated by the above studies, we define poly-Norden generalized CR-light-like submanifolds and poly-Norden screen transversal CR-light-like submanifolds in a poly-Norden s-Riemannian manifold. We give examples. Also, we investigate conditions for the distributions involved in the definitions of such light-like submanifolds to be integrable and define totally geodesic foliations.

2. Preliminaries

The positive solution of $x^2 - mx + 1 = 0$, is called bronze mean [15], which is given by

$$B_m = \frac{m + \sqrt{m^2 - 4}}{2}. \tag{1}$$

In [17], using (1) Şahin defined a new type of manifold equipped with the bronze structure. A differentiable manifold $\hat{M}$ equipped with a $(1, 1)$-type tensor field $\hat{\Phi}$ and a s-Riemannian metric $\hat{g}$ satisfying [17]

$$\hat{\Phi}^2 = m\hat{\Phi} - I, \tag{2}$$

$$\hat{g}(\hat{\Phi}U, \hat{\Phi}V) = m\hat{g}(\hat{\Phi}U, V) - \hat{g}(U, V), \tag{3}$$

is said to be an almost poly-Norden s-Riemannian manifold. In this case, $\hat{\Phi}$ is known as an almost poly-Norden structure. If $\hat{\Phi}$ is parallel with respect to the Levi-Civita connection $\hat{\nabla}$ on $\hat{M}$, then $\hat{M}$ is said to be a poly-Norden s-Riemannian manifold. Using (3), we can write

$$\hat{g}(\hat{\Phi}U, V) = \hat{g}(U, \hat{\Phi}V). \tag{4}$$

From now on, throughout the paper, we will suppose that m is different from zero.

Now we recall same basic notations and formulas in [3]. A n-dimensional submanifold $\tilde{M}$ of a $(n + s)$-dimensional s-Riemannian manifold $\hat{M}$ is called a light-like submanifold if the induced metric $\tilde{g}$ is degenerate on $\tilde{M}$ and $rank(Rad\, T\tilde{M}) = r$, $1 \leq r \leq n$. Here $Rad\, T\tilde{M}$ is the radical distribution and $T\tilde{M}^{\perp}$ is the normal bundle of $T\tilde{M}$ which are defined by

$$Rad\, T\tilde{M} = T\tilde{M} \cap T\tilde{M}^{\perp},$$

and

$$T\tilde{M}^{\perp} = \bigcup_{p\in\tilde{M}} \{V_p \in T_p\hat{M} : g_p(U_p, V_p) = 0, \forall U \in \Gamma(T_p\tilde{M})\},$$

respectively. Degeneracy of $T\tilde{M}$ and $T\tilde{M}^{\perp}$ implies that there exist complementary non-degenerate distributions $S(T\tilde{M})$ and $S(T\tilde{M}^{\perp})$ of $Rad\,T\tilde{M}$ in $T\tilde{M}$ and $T\tilde{M}^{\perp}$, respectively, which are known as the screen distribution and screen transversal bundle of $\tilde{M}$ with

$$T\tilde{M} = S(T\tilde{M}) \perp Rad\,T\tilde{M},$$

$$T\tilde{M}^{\perp} = S(T\tilde{M}^{\perp}) \perp Rad\,T\tilde{M}.$$

Also, in view of an orthogonal complementary subbundle $S(T\tilde{M})^{\perp}$ to $S(T\tilde{M})$ in $T\hat{M}$ we have

$$S(T\tilde{M})^{\perp} = S(T\tilde{M}^{\perp}) \perp S(T\tilde{M}^{\perp})^{\perp},$$

where $S(T\tilde{M}^{\perp})^{\perp}$ is the orthogonal complementary to $S(T\tilde{M}^{\perp})$ in $S(T\tilde{M})^{\perp}$.

From Lemma 1.2 (see [3], page 142), we see that a complementary (not orthogonal) vector bundle $tr(T\tilde{M})$, which is called transversal vector bundle, to $T\tilde{M}$ in $T\hat{M}$ exists, and following equations are hold:

$$tr(T\tilde{M}) = ltr(T\tilde{M}) \perp S(T\tilde{M}^{\perp}), \tag{5}$$

$$S(T\tilde{M}^{\perp})^{\perp} = Rad\,T\tilde{M} \oplus ltr(T\tilde{M}). \tag{6}$$

So, in view of equations (5) and (6), we write

$$\begin{aligned} T\hat{M} &= S(T\tilde{M}) \perp S(T\tilde{M})^{\perp} \\ &= S(T\tilde{M}) \perp \{Rad\,T\tilde{M} \oplus ltr(T\tilde{M})\} \perp S(T\tilde{M})^{\perp} \\ &= T\tilde{M} \oplus tr(T\tilde{M}). \end{aligned} \tag{7}$$

The Gauss and Weingarten formulas of $\tilde{M}$ are given by

$$\hat{\nabla}_U V = \tilde{\nabla}_U V + \hbar(U, V), \qquad \forall U, V \in \Gamma(T\tilde{M}), \tag{8}$$

$$\hat{\nabla}_U N = -A_N U + \tilde{\nabla}^t_U N, \qquad \forall U \in \Gamma(T\tilde{M}),\ \ N \in \Gamma(ltr(T\tilde{M})), \tag{9}$$

where $\nabla_U V, A_N U \in \Gamma(T\tilde{M})$ and $\hbar(U, V), \nabla^t_U N \in \Gamma(tr(T\tilde{M}))$.

By using (7) with projection morphisms defined by

$$L : tr(T\tilde{M}) \to ltr(T\tilde{M}), \qquad S : tr(T\tilde{M}) \to S(T\tilde{M}^{\perp}),$$

we can state,

$$\hat{\nabla}_U V = \tilde{\nabla}_U V + \hbar^l(U, V) + \hbar^s(U, V), \tag{10}$$

$$\hat{\nabla}_U N = -A_N U + \tilde{\nabla}^l_U N + \Omega^s(U, N), \tag{11}$$

$$\hat{\nabla}_U W = -A_W U + \Omega^l(U, W) + \tilde{\nabla}^s_U W, \tag{12}$$

for any $U, V \in \Gamma(T\tilde{M})$, $N \in \Gamma(ltr(T\tilde{M}))$ and $W \in \Gamma(S(T\tilde{M}^{\perp}))$, where $\tilde{\nabla}^l_U N$, $\Omega^l(U, W) \in \Gamma(ltr(T\tilde{M}))$, $\Omega^s(U, N)$, $\tilde{\nabla}^s_U W \in \Gamma(S(T\tilde{M}^{\perp}))$ and

$$\hbar^l(U, V) = Ł\hbar(U, V) \in \Gamma(ltr(T\tilde{M})),$$

$$\hbar^s(U, V) = S\hbar(U, V) \in \Gamma(S(T\tilde{M}^{\perp})).$$

If we denote the projection of $T\tilde{M}$ on $S(T\tilde{M})$ with $\tilde{P}$, then from (8), (10), (11) and (12), we obtain

$$\tilde{g}(\hbar^s(U,V),W) + \tilde{g}(Y,\Omega^l(U,W)) = \tilde{g}(A_W U,Y), \tag{13}$$

$$\tilde{g}(\Omega^s(U,N),W) = \tilde{g}(N,A_W U), \tag{14}$$

$$\tilde{\nabla}_U \tilde{P}V = \nabla^*_U \tilde{P}V + \hbar^*(U,\tilde{P}V), \tag{15}$$

$$\tilde{\nabla}_U E = -A^*_E U + \nabla^{*t}_U E, \tag{16}$$

for any $U,V \in \Gamma(T\tilde{M})$, $E \in \Gamma(Rad\,T\tilde{M})$. We note that ∇^* is a metric connection of $S(T\tilde{M})$.

Moreover, the induced connection $\tilde{\nabla}$ on $\tilde{M}$ is not a metric connection in general and satisfies

$$(\tilde{\nabla}_U \tilde{g})(V,Y) = \tilde{g}(\hbar^l(U,V),Y) + \tilde{g}(\hbar^l(U,Y),V), \tag{17}$$

whereas ∇^* is a metric connection on $S(T\tilde{M})$.

3. POLY-NORDEN GENERALIZED CR-LIGHT-LIKE SUBMANIFOLDS

Now, we introduce poly-Norden generalized CR-light-like submanifolds in poly-Norden s-Riemannian manifolds by adapting the definition given in [8] for indefinite Kaehler manifolds to our case:

Definition 3.1. *A light-like submanifold $(\tilde{M},\tilde{g})$ of an almost poly-Norden s-Riemannian manifold $(\hat{M},\hat{\Phi},\hat{g})$ is called a poly-Norden generalized CR-light-like submanifold if the following conditions are satisfied:*

i) There exist subbundles D_α and D_β of $Rad\,T\tilde{M}$ such that

$$Rad\,T\tilde{M} = D_\alpha \oplus D_\beta, \quad \hat{\Phi}(D_\alpha) = D_\alpha, \quad \hat{\Phi}(D_\beta) = S(T\tilde{M}).$$

ii) There exist subbundles $\mathring{D}$ and $\tilde{D}$ of $S(T\tilde{M})$ such that

$$S(T\tilde{M}) = \{\hat{\Phi}(D_\beta) \oplus \tilde{D}\} \perp \mathring{D}, \quad \hat{\Phi}(\mathring{D}) = \mathring{D}, \quad \hat{\Phi}(\tilde{L} \perp \tilde{S}) = \tilde{D},$$

where $\mathring{D}$ is a non-degenerate distribution, $\tilde{L}$ and $\tilde{S}$ are vector subbundles of $ltr(T\tilde{M})$ and $S(T\tilde{M}^\perp)$, respectively.

If we consider

$$\hat{\Phi}(\tilde{L}) = L_1 \quad \text{and} \quad \hat{\Phi}(\tilde{S}) = S_1,$$

then we can state

$$\tilde{D} = \hat{\Phi}(\tilde{L}) \perp \hat{\Phi}(\tilde{S}) = L_1 \perp S_1.$$

So, we arrive at

$$T\tilde{M} = \tilde{D} \oplus D, \tag{18}$$

where $D = Rad\,T\tilde{M} \perp \mathring{D} \perp \hat{\Phi}(D_\beta)$.

Example 3.2. *Let $\hat{M} = \mathbb{R}^{12}_4$ be a 12-dimensional semi-Euclidean space with coordinate system $(x_1, x_2, ..., x_{12})$ and signature $(-,-,+,+,-,-,+,+,+,+,+,+)$. Taking*

$$\hat{\Phi}(x_1,...,x_{12}) = \begin{pmatrix} B_m x_1, \bar{B}_m x_2, B_m x_3, \bar{B}_m x_4, m x_5 + x_6, -x_5, \\ m x_7 + x_8, -x_7, m x_9 + x_{10}, -x_9, m x_{11} + x_{12}, -x_{11} \end{pmatrix},$$

then we can say $\hat{\Phi}$ is an almost poly-Norden structure on $\hat{M}$.

Assume that $\tilde{M}$ is a submanifold of $\mathbb{R}^{12}_4$ defined by

$$x_1 = x_3 = u_1 + B_m u_2, \quad x_2 = x_4 = u_1 + \bar{B}_m u_2,$$

$$x_5 = -u_4 - u_5, \quad x_6 = B_m u_3,$$

$$x_7 = -u_4 + u_5, \quad x_8 = B_m u_3,$$

$$x_9 = -u_7 - B_m u_8, \quad x_{10} = -\bar{B}_m u_6,$$

$$x_{11} = u_7 + B_m u_8, \quad x_{12} = \bar{B}_m u_6,$$

Then $T\tilde{M}$ is given by

$$\tilde{\Psi}_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4},$$

$$\tilde{\Psi}_2 = B_m \frac{\partial}{\partial x_1} + \bar{B}_m \frac{\partial}{\partial x_2} + B_m \frac{\partial}{\partial x_3} + \bar{B}_m \frac{\partial}{\partial x_4},$$

$$\tilde{\Psi}_3 = B_m \frac{\partial}{\partial x_6} + B_m \frac{\partial}{\partial x_8},$$

$$\tilde{\Psi}_4 = -\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7},$$

$$\tilde{\Psi}_5 = -\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7},$$

$$\tilde{\Psi}_6 = -\bar{B}_m \frac{\partial}{\partial x_{10}} + \bar{B}_m \frac{\partial}{\partial x_{12}},$$

$$\tilde{\Psi}_7 = -\frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{11}},$$

$$\tilde{\Psi}_8 = B_m \frac{\partial}{\partial x_9} + B_m \frac{\partial}{\partial x_{111}},$$

Hence $\tilde{M}$ is a light-like submanifold with $Rad\,T\tilde{M} = Span\{\tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{\Psi}_3\}$. Since $\hat{\Phi}\tilde{\Psi}_1 = \tilde{\Psi}_2$ and $\hat{\Phi}\tilde{\Psi}_3 = B_m\tilde{\Psi}_4 \in S(T\tilde{M})$ then $D_\alpha = Span\{\tilde{\Psi}_1, \tilde{\Psi}_2\}$ and $D_\beta = Span\{\tilde{\Psi}_3\}$. Also $\hat{\Phi}\tilde{\Psi}_6 = \bar{B}_m\tilde{\Psi}_7$ thus $\mathring{D} = Span\{\tilde{\Psi}_6, \tilde{\Psi}_7\}$. On the other hand we have $ltr(T\tilde{M}) = Span\{N_1, N_2, N_3\}$ and $S(T\tilde{M}^\perp) = Span\{W\}$ where

$$N_1 = -\frac{1}{4}\{\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\},$$

$$N_2 = -\frac{1}{2(B_m^2 + \bar{B}_m^2)} \left\{ \begin{array}{c} B_m \frac{\partial}{\partial x_1} + \bar{B}_m \frac{\partial}{\partial x_2} \\ -B_m \frac{\partial}{\partial x_3} - \bar{B}_m \frac{\partial}{\partial x_4} \end{array} \right\},$$

$$N_3 = -\frac{1}{2B_m}\{\frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_8}\},$$

and

$$W = \frac{\partial}{\partial x_{10}} + \frac{\partial}{\partial x_{12}}.$$

So, $Span\{N_1, N_2\}$ invariant and $\hat{\Phi}N_3 = \frac{1}{2B_m}\tilde{\Psi}_5$ and $W = -\tilde{\Psi}_8$. Then we have $\tilde{L} = Sp\{N_3\}$ and $\tilde{S} = Sp\{W\}$. As a result, $\tilde{M}$ is a poly-Norden generalized CR-light-like submanifold.

To avoid repetition in the remain part of this section $(\hat{M}, \hat{\Phi}, \hat{g})$ will be considered as a poly-Norden s-Riemannian manifold. So, for $U \in \Gamma(T\tilde{M})$, we write

$$\hat{\Phi}U = \varphi U + fU, \tag{19}$$

where φU is tangential part of $\hat{\Phi}U$ and fU is transversal part of $\hat{\Phi}U$.
For $V \in \Gamma(tr(T\tilde{M}))$, we state

$$\hat{\Phi}V = \omega V + tV, \tag{20}$$

where ωV is tangential part of $\hat{\Phi}V$ and tV is transversal part of $\hat{\Phi}V$.

Theorem 3.3. *Let $(\tilde{M}, \tilde{g})$ be a poly-Norden generalized CR-light-like submanifold of $(\hat{M}, \hat{\Phi}, \hat{g})$. Then D is integrable if and only if*
i) $\tilde{g}(\hbar^l(U, \hat{\Phi}V), \zeta) = \tilde{g}(\hbar^l(V, \hat{\Phi}U), \zeta)$,
ii) $\tilde{g}(\hbar^s(U, \hat{\Phi}V), W) = \tilde{g}(\hbar^s(V, \hat{\Phi}U), W)$,
for any $U, V \in \Gamma(D)$, $\zeta \in \Gamma(Rad\, T\tilde{M})$ *and* $W \in \Gamma(\tilde{S})$.

Proof. If we consider the definiton of D, then D is integrable if and only if

$$\tilde{g}([U, V], \hat{\Phi}\zeta) = 0, \quad \tilde{g}([U, V], \hat{\Phi}W) = 0.$$

In view of (10), we have

$$\begin{aligned}\tilde{g}([U, V], \hat{\Phi}\zeta) &= \hat{g}(\hat{\nabla}_U V, \hat{\Phi}\zeta) - \hat{g}(\hat{\nabla}_V U, \hat{\Phi}\zeta)\\ &= \hat{g}(\hat{\nabla}_U \hat{\Phi}V, \zeta) - \hat{g}(\hat{\nabla}_V \hat{\Phi}U, \zeta)\\ &= \tilde{g}(\hbar^l(U, \hat{\Phi}V) - \hbar^l(V, \hat{\Phi}U), \zeta),\end{aligned}$$

and

$$\begin{aligned}\tilde{g}([U, V], \hat{\Phi}W) &= \hat{g}(\hat{\nabla}_U V, \hat{\Phi}W) - \hat{g}(\hat{\nabla}_V U, \hat{\Phi}W)\\ &= \hat{g}(\hat{\nabla}_U \hat{\Phi}V, W) - \hat{g}(\hat{\nabla}_V \hat{\Phi}U, W)\\ &= \tilde{g}(\hbar^s(U, \hat{\Phi}V) - \hbar^s(V, \hat{\Phi}U), W),\end{aligned}$$

which give our assertion. □

Theorem 3.4. *Let $(\tilde{M}, \tilde{g})$ be a poly-Norden generalized CR-light-like submanifold of $(\hat{M}, \hat{\Phi}, \hat{g})$. Then $\tilde{D}$ is integrable if and only if*
i) $\tilde{g}(A_{V_1}Y, \hat{\Phi}Z) = \tilde{g}(A_{V_2}X, \hat{\Phi}Z)$,
ii) $\tilde{g}(\hbar^*(X, Y), \tilde{N}) = \tilde{g}(\hbar^*(Y, X), \tilde{N})$ *and* $\tilde{g}(A_{V_2}X, \tilde{N}) = \tilde{g}(A_{V_1}Y, \tilde{N})$,
iii) $\tilde{g}(A_N X, Y) = \tilde{g}(A_N Y, X)$,
for $X, Y \in \Gamma(\tilde{D})$, $Z \in \Gamma(\mathring{D})$, $\tilde{N} \in \Gamma(\tilde{L})$ *and* $V_1, V_2 \in \Gamma(\tilde{L} \perp \tilde{S})$.

Proof. If we consider the definiton of $\tilde{D}$, then $\tilde{D}$ is integrable if and only if

$$\tilde{g}([X, Y], Z) = 0, \quad \tilde{g}([X, Y], \hat{\Phi}\tilde{N}) = 0, \quad \tilde{g}([X, Y], N) = 0.$$

Taking $X, Y \in \Gamma(\tilde{D})$, then we can write

$$X = \hat{\Phi}V_1 \text{ and } Y = \hat{\Phi}V_2,$$

such that $V_1, V_2 \in \Gamma(\tilde{L} \perp \tilde{S})$. So, from (3), (11) and (15), we get

$$\begin{aligned}\tilde{g}([X, Y], Z) &= \hat{g}(\hat{\nabla}_X Y, Z) - \hat{g}(\hat{\nabla}_Y X, Z)\\ &= \hat{g}(\hat{\nabla}_X V_2, \hat{\Phi}Z) - \hat{g}(\hat{\nabla}_Y V_1, \hat{\Phi}Z)\\ &= \tilde{g}(A_{V_1}Y - A_{V_2}X, \hat{\Phi}Z),\end{aligned}$$

which gives (i). Also, we have

$$\begin{aligned}
\tilde{g}([X,Y],\hat{\Phi}\tilde{N}) &= \hat{g}(\hat{\nabla}_X Y,\hat{\Phi}\tilde{N}) - \hat{g}(\hat{\nabla}_Y X,\hat{\Phi}\tilde{N}) \\
&= \hat{g}(\hat{\nabla}_X \hat{\Phi}V_2,\hat{\Phi}\tilde{N}) - \hat{g}(\hat{\nabla}_Y \hat{\Phi}V_1,\hat{\Phi}\tilde{N}) \\
&= m\hat{g}(\hat{\Phi}\hat{\nabla}_X V_2,\tilde{N}) - \hat{g}(\hat{\nabla}_X V_2,\tilde{N}) \\
&\quad -m\hat{g}(\hat{\Phi}\hat{\nabla}_Y V_1,\tilde{N}) + \hat{g}(\hat{\nabla}_Y V_1,\tilde{N}) \\
&= m\tilde{g}(\tilde{\nabla}_X Y,\tilde{N}) + \tilde{g}(A_{V_2}X,\tilde{N}) \\
&\quad -m\tilde{g}(\tilde{\nabla}_Y X,\tilde{N}) - \tilde{g}(A_{V_1}Y,\tilde{N}) \\
&= m\tilde{g}(\hbar^*(X,Y) - \hbar^*(Y,X),\tilde{N}) \\
&\quad +\tilde{g}(A_{V_2}X - A_{V_1}Y,\tilde{N}),
\end{aligned}$$

which gives (*ii*). Finally, we obtain

$$\begin{aligned}
\tilde{g}([X,Y],N) &= \hat{g}(\hat{\nabla}_X Y,N) - \hat{g}(\hat{\nabla}_Y X,N) \\
&= \hat{g}(X,\hat{\nabla}_Y N) - \hat{g}(Y,\hat{\nabla}_X N) \\
&= \tilde{g}(A_N X,Y) - \tilde{g}(A_N Y,X),
\end{aligned}$$

which gives (*iii*). □

Theorem 3.5. *Let $(\tilde{M},\tilde{g})$ be a poly-Norden generalized CR-light-like submanifold of $(\hat{M},\hat{\Phi},\hat{g})$. Then the distribution D defines a totally geodesic foliation on $\tilde{M}$ if and only if*

$$\omega\hbar(U,\hat{\Phi}V) = 0,$$

for $U,V \in \Gamma(D)$.

Proof. The distribution D defines a totally geodesic foliation if and only if

$$\tilde{g}(\tilde{\nabla}_U V,\hat{\Phi}\zeta) = 0, \quad \tilde{g}(\tilde{\nabla}_U V,\hat{\Phi}W) = 0,$$

for any $U,V \in \Gamma(D)$, $\zeta \in \Gamma(D_\beta)$ and $W \in \Gamma(\tilde{S})$.

By use of (2) and (10), we obtain

$$\begin{aligned}
\tilde{g}(\tilde{\nabla}_U V,\hat{\Phi}\zeta) &= \hat{g}(\hat{\nabla}_U V,\hat{\Phi}\zeta) \\
&= \hat{g}(\hat{\nabla}_U \hat{\Phi}V,\zeta) \\
&= \tilde{g}(\hbar^l(U,\hat{\Phi}V),\zeta),
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\tilde{g}(\tilde{\nabla}_U V,\hat{\Phi}W) &= \hat{g}(\hat{\nabla}_U V,\hat{\Phi}W) \\
&= \hat{g}(\hat{\nabla}_U \hat{\Phi}V,W) \\
&= \tilde{g}(\hbar^s(U,\hat{\Phi}V),W).
\end{aligned} \tag{22}$$

From (21) with (22), we can say that D defines a totally geodesic foliation if and only if $\hbar^l(U,\hat{\Phi}V)$ has no component in $\tilde{L}$ and $\hbar^s(U,\hat{\Phi}V)$ has no component in $\tilde{S}$. So, the proof is completed. □

Theorem 3.6. *Let $(\tilde{M},\tilde{g})$ be a poly-Norden generalized CR-light-like submanifold of $(\hat{M},\hat{\Phi},\hat{g})$. Then $\tilde{D}$ defines totally geodesic foliation on $\tilde{M}$ if and only if*

i) $A_V X$ has no component in $\mathring{D}$,

ii) $m\tilde{g}(\hbar^*(X,Y),\tilde{N}) = -\tilde{g}(A_V X,\tilde{N})$,

iii) $A_N X$ has no component in $\hat{\Phi}D_\beta \perp \Phi\tilde{S}$,

for $X,Y \in \Gamma(\tilde{D})$, $\tilde{N} \in \Gamma(\tilde{L})$, $V \in \Gamma(\tilde{L}\perp\tilde{S})$ and $N \in \Gamma(ltr(T\tilde{M}))$.

Proof. The distribution $\tilde{D}$ defines a totally geodesic foliation if and only if

$$\tilde{g}(\tilde{\nabla}_X Y, Z) = 0, \;\; \tilde{g}(\tilde{\nabla}_X Y, \hat{\Phi}\tilde{N}) = 0, \;\; \tilde{g}(\tilde{\nabla}_X Y, \tilde{N}) = 0,$$

for any $X, Y \in \Gamma(\tilde{D})$, $\tilde{N} \in \Gamma(\tilde{L})$, $Z \in \Gamma(\mathring{D})$ and $N \in \Gamma(ltr(T\tilde{M}))$.

If we choose Y from the distribution $\tilde{D}$, we write $Y = \hat{\Phi}V$ such that $V \in \Gamma(\tilde{L}\perp\tilde{S})$. Using (3) with (10), we have

$$\begin{aligned} \tilde{g}(\tilde{\nabla}_X Y, Z) &= \hat{g}(\hat{\nabla}_X Y, Z) \\ &= \hat{g}(\hat{\nabla}_X V, \hat{\Phi}Z) \\ &= -\tilde{g}(A_V X, \hat{\Phi}Z). \end{aligned} \tag{23}$$

Again, using (3) and (10) with (15), we have

$$\begin{aligned} \tilde{g}(\tilde{\nabla}_X Y, \hat{\Phi}\tilde{N}) &= \hat{g}(\hat{\nabla}_X Y, \hat{\Phi}\tilde{N}) \\ &= \hat{g}(\hat{\Phi}\hat{\nabla}_X V, \hat{\Phi}\tilde{N}) \\ &= m\hat{g}(\hat{\nabla}_X \hat{\Phi}V, \tilde{N}) - \hat{g}(\hat{\nabla}_X V, \tilde{N}) \\ &= m\tilde{g}(\tilde{\nabla}_X Y, \tilde{N}) + \tilde{g}(A_V X, \tilde{N}) \\ &= m\tilde{g}(\hbar^*(X, Y), \tilde{N}) + \tilde{g}(A_V X, \tilde{N}). \end{aligned} \tag{24}$$

Since $\hat{\nabla}$ is a metric connection, if we use (11), we arrive at

$$\begin{aligned} \hat{g}(\hat{\nabla}_X Y, N) &= -\hat{g}(Y, \hat{\nabla}_X N) \\ &= \tilde{g}(A_N X, Y). \end{aligned} \tag{25}$$

In view of (23)-(25), we correct our assertion. □

Theorem 3.7. *Let $(\tilde{M}, \tilde{g})$ be a poly-Norden generalized CR-light-like submanifold of $(\hat{M}, \hat{\Phi}, \hat{g})$. Then $\mathring{D}$ is integrable if and only if*

i) $\tilde{g}(\nabla^*_U V, \hat{\Phi}\zeta) = \tilde{g}(\nabla^*_V U, \hat{\Phi}\zeta)$,

ii) $\tilde{g}(\hbar^s(U, \hat{\Phi}V), W) = \tilde{g}(\hbar^s(V, \hat{\Phi}U), W)$,

iii) $\tilde{g}(\hbar^*(U, \hat{\Phi}V), \tilde{N}) = \tilde{g}(\hbar^*(V, \hat{\Phi}U), \tilde{N})$,

iv) $\tilde{g}(\hbar^*(U, V), N) = \tilde{g}(\hbar^*(V, U), N)$,

for $U, V \in \Gamma(\mathring{D})$, $\tilde{N} \in \Gamma(\tilde{L})$, $\zeta \in \Gamma(D_\beta)$ *and* $N \in \Gamma(ltr(T\tilde{M}))$.

Proof. If we consider the definiton of $\mathring{D}$, then $\mathring{D}$ is integrable if and only if

$$\tilde{g}([U, V], \hat{\Phi}\zeta) = 0, \;\; \tilde{g}([U, V], \hat{\Phi}W) = 0,$$

$$\tilde{g}([U, V], \hat{\Phi}\tilde{N}) = 0, \;\; \tilde{g}([U, V], N) = 0.$$

In view of (2) with (10), we have

$$\begin{aligned} \tilde{g}([U, V], \hat{\Phi}\zeta) &= \hat{g}(\hat{\nabla}_U V, \hat{\Phi}\zeta) - \hat{g}(\hat{\nabla}_V U, \hat{\Phi}\zeta) \\ &= \tilde{g}(\tilde{\nabla}_U V, \hat{\Phi}\zeta) - \tilde{g}(\tilde{\nabla}_V U, \hat{\Phi}\zeta) \\ &= \tilde{g}(\nabla^*_U V - \nabla^*_V U, \hat{\Phi}\zeta), \end{aligned} \tag{26}$$

$$\begin{aligned} \tilde{g}([U, V], \hat{\Phi}W) &= \hat{g}(\hat{\nabla}_U V, \hat{\Phi}W) - \hat{g}(\hat{\nabla}_V U, \hat{\Phi}W) \\ &= \tilde{g}(\tilde{\nabla}_U \hat{\Phi}V, W) - \tilde{g}(\tilde{\nabla}_V \hat{\Phi}U, W) \\ &= \tilde{g}(\hbar^s(U, \hat{\Phi}V) - \hbar^s(V, \hat{\Phi}U), W), \end{aligned} \tag{27}$$

$$\begin{aligned}\tilde{g}([U,V],\hat{\Phi}\tilde{N}) &= \hat{g}(\hat{\nabla}_U V,\hat{\Phi}\tilde{N}) - \hat{g}(\hat{\nabla}_V U,\hat{\Phi}\tilde{N})\\ &= \tilde{g}(\tilde{\nabla}_U \hat{\Phi}V,\tilde{N}) - \tilde{g}(\tilde{\nabla}_V \hat{\Phi}U,\tilde{N})\\ &= \tilde{g}(\hbar^*(U,\hat{\Phi}V) - \hbar^*(V,\hat{\Phi}U),\tilde{N}),\end{aligned} \tag{28}$$

and

$$\begin{aligned}\tilde{g}([U,V],N) &= \hat{g}(\hat{\nabla}_U V,N) - \hat{g}(\hat{\nabla}_V U,N)\\ &= \tilde{g}(\tilde{\nabla}_U V,N) - \tilde{g}(\tilde{\nabla}_V U,N)\\ &= \tilde{g}(\hbar^*(U,V) - \hbar^*(V,U),N).\end{aligned} \tag{29}$$

The results follows from (26)−(29). □

Theorem 3.8. *Let $(\tilde{M},\tilde{g})$ be a poly-Norden generalized CR-lightlike submanifold of $(\hat{M},\hat{\Phi},\hat{g})$. Then $Rad\,T\tilde{M}$ is integrable if and only if*
i) $\tilde{g}(E_1,\hbar^l(E_2,\hat{\Phi}\zeta)) = \tilde{g}(E_2,\hbar^l(E_1,\hat{\Phi}\zeta))$,
ii) $\tilde{g}(E_1,\hbar^l(E_2,\hat{\Phi}\tilde{N})) = \tilde{g}(E_2,\hbar^l(E_1,\hat{\Phi}\tilde{N}))$,
iii) $\tilde{g}(E_1,\hbar^s(E_2,\hat{\Phi}W)) = \tilde{g}(E_2,\hbar^s(E_1,\hat{\Phi}W))$,
iv) $\tilde{g}(A^*_{E_1}E_2,Z) = (A^*_{E_2}E_1,Z)$
for $E_1,E_2 \in \Gamma(Rad\,T\tilde{M})$, $\tilde{N} \in \Gamma(\tilde{L})$, $\zeta \in \Gamma(D_\beta)$, $W \in \Gamma(\tilde{S})$ *and* $Z \in \Gamma(\mathring{D})$.

Proof. From the definition of $Rad\,T\tilde{M}$, we state that is $Rad\,T\tilde{M}$ integrable if and only if

$$\tilde{g}([E_1,E_2],\hat{\Phi}\zeta) = 0,\ \tilde{g}([E_1,E_2],\hat{\Phi}\tilde{N}) = 0,$$

$$\tilde{g}([E_1,E_2],\hat{\Phi}W) = 0,\ \tilde{g}([E_1,E_2],Z) = 0.$$

In view of (2) and (10), we get

$$\begin{aligned}\tilde{g}([E_1,E_2],\hat{\Phi}\zeta) &= \hat{g}(\hat{\nabla}_{E_1}E_2,\hat{\Phi}\zeta) - \hat{g}(\hat{\nabla}_{E_2}E_1,\hat{\Phi}\zeta)\\ &= \hat{g}(E_1,\hat{\nabla}_{E_2}\hat{\Phi}\zeta) - \hat{g}(E_2,\hat{\nabla}_{E_1}\hat{\Phi}\zeta)\\ &= \tilde{g}(E_1,\hbar^l(E_2,\hat{\Phi}\zeta)) - \tilde{g}(E_2,\hbar^l(E_1,\hat{\Phi}\zeta)),\end{aligned} \tag{30}$$

$$\begin{aligned}\tilde{g}([E_1,E_2],\hat{\Phi}\tilde{N}) &= \hat{g}(\hat{\nabla}_{E_1}E_2,\hat{\Phi}\tilde{N}) - \hat{g}(\hat{\nabla}_{E_2}E_1,\hat{\Phi}\tilde{N})\\ &= \hat{g}(E_1,\hat{\nabla}_{E_2}\hat{\Phi}\tilde{N}) - \hat{g}(E_2,\hat{\nabla}_{E_1}\hat{\Phi}\tilde{N})\\ &= \tilde{g}(E_1,\hbar^l(E_2,\hat{\Phi}\tilde{N})) - \tilde{g}(E_2,\hbar^l(E_1,\hat{\Phi}\tilde{N})),\end{aligned} \tag{31}$$

$$\begin{aligned}\tilde{g}([E_1,E_2],\hat{\Phi}W) &= \hat{g}(\hat{\nabla}_{E_1}E_2,\hat{\Phi}W) - \hat{g}(\hat{\nabla}_{E_2}E_1,\hat{\Phi}W)\\ &= \hat{g}(E_1,\hat{\nabla}_{E_2}\hat{\Phi}W) - \hat{g}(E_2,\hat{\nabla}_{E_1}\hat{\Phi}W)\\ &= \tilde{g}(E_1,\hbar^s(E_2,\hat{\Phi}W)) - \tilde{g}(E_2,\hbar^s(E_1,\hat{\Phi}W)),\end{aligned} \tag{32}$$

and

$$\begin{aligned}\tilde{g}([E_1,E_2],Z) &= \hat{g}(\hat{\nabla}_{E_1}E_2,Z) - \hat{g}(\hat{\nabla}_{E_2}E_1,Z)\\ &= \tilde{g}(\tilde{\nabla}_{E_1}E_2,Z) - \tilde{g}(\tilde{\nabla}_{E_2}E_1,Z)\\ &= \tilde{g}(A^*_{E_1}E_2,Z) - (A^*_{E_2}E_1,Z),\end{aligned} \tag{33}$$

So from (30)−(33), the result follows. □

4. POLY-NORDEN SCREEN TRANSVERSAL CR-LIGHT-LIKE SUBMANIFOLDS

In this section, we introduce poly-Norden screen transversal CR-light-like submanifolds in poly-Norden s-Riemannian manifolds by adapting the definition given in [19] for indefinite Kaehler manifolds to our case:

Definition 4.1. *Let $(\tilde{M}, \tilde{g})$ be a light-like submanifold of an almost poly-Norden s-Riemanian manifold $(\hat{M}, \hat{\Phi}, \hat{g})$. Then $\tilde{M}$ is called a poly-Norden screen transversal CR-light-like submanifold if the following conditions are satisfied:*
i) There exist D_α and D_β subbundles of $Rad\, T\tilde{M}$ such that

$$Rad\, T\tilde{M} = D_\alpha \oplus D_\beta, \quad \hat{\Phi}(D_\alpha) \subset S(T\tilde{M}), \quad \hat{\Phi}(D_\beta) \subset S(T\tilde{M}^\perp), \tag{34}$$

ii) There exist $\mathring{D}$ and $D^{'}$ subbundles of $S(T\tilde{M})$ such that

$$S(T\tilde{M}) = \{\hat{\Phi}(D_\alpha) \oplus D^{'}\} \perp \mathring{D}, \quad \hat{\Phi}(\mathring{D}) = \mathring{D}, \quad \hat{\Phi}(D^{'}) = \tilde{L}, \tag{35}$$

where $\mathring{D}$ is a non degenerate distribution on M, $\tilde{L}$ and $\tilde{S}$ are vector subbundles of $ltr(T\tilde{M})$ and $S(T\tilde{M}^\perp)$, respectively.

From the definition, we obtain that

$$T\tilde{M} = D \oplus \hat{D}, \tag{36}$$

where

$$D = \mathring{D} \oplus D_\alpha \oplus \hat{\Phi}(D_\alpha), \quad \hat{D} = D_\beta \oplus \hat{\Phi}(\tilde{L}) \oplus \hat{\Phi}(\tilde{S}). \tag{37}$$

Also, we have

$$ltr(T\tilde{M}) = \tilde{L} \oplus \mathring{L}, \quad \hat{\Phi}(\tilde{L}) \subset S(T\tilde{M}), \quad \hat{\Phi}(\mathring{L}) \subset S(T\tilde{M}^\perp),$$

$$S(T\tilde{M}^\perp) = \{\hat{\Phi}(\mathring{L}) \oplus \hat{\Phi}(D_\beta)\} \perp \tilde{S}.$$

Example 4.2. *Let $\hat{M} = \mathbb{R}^{12}_4$ be a semi-Euclidean space with coordinate system $(x_1, x_2, ..., x_{12})$ and signature $(-, +, -, +, +, +, +, +, -, +, -, +)$. If we take*

$$\hat{\Phi}(x_1, x_2, ..., x_{12}) = \begin{pmatrix} \bar{B}_m x_1, \bar{B}_m x_2, B_m x_3, B_m x_4, m x_5 + x_6, -x_5, \\ m x_7 + x_8, -x_7, B_m x_9, B_m x_{10}, \bar{B}_m x_{11}, \bar{B}_m x_{12} \end{pmatrix},$$

then we say that $\hat{\Phi}$ is an almost poly-Norden structure on $\hat{M}$.
Suppose that $\tilde{M}$ is a submanifold of $\mathbb{R}^{12}_4$ defined by

$$x_1 = B_m u_1 + u_2 - \bar{B}_m u_4,$$

$$x_2 = B_m u_1 + u_2 + \bar{B}_m u_4,$$

$$x_3 = x_4 = \bar{B}_m u_3,$$

$$x_5 = -u_6, \quad x_6 = u_5,$$

$$x_7 = -u_6 + u_7, \quad x_8 = u_5 + u_7,$$

$$x_9 = x_{10} = u_3, \quad x_{11} = x_{12} = u_1 + \bar{B}_m u_2.$$

Then $T\tilde{M}$ is given by $\{\tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{\Psi}_3, \tilde{\Psi}_4, \tilde{\Psi}_5, \tilde{\Psi}_6, \tilde{\Psi}_7\}$ where

$$\tilde{\Psi}_1 = B_m \frac{\partial}{\partial x_1} + B_m \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_{11}} + \frac{\partial}{\partial x_{12}},$$

$$\tilde{\Psi}_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \bar{B}_m\frac{\partial}{\partial x_{11}} + \bar{B}_m\frac{\partial}{\partial x_{12}},$$

$$\tilde{\Psi}_3 = \bar{B}_m\frac{\partial}{\partial x_3} + \bar{B}_m\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{10}},$$

$$\tilde{\Psi}_4 = -\bar{B}_m\frac{\partial}{\partial x_1} + \bar{B}_m\frac{\partial}{\partial x_2},$$

$$\tilde{\Psi}_5 = \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8},$$

$$\tilde{\Psi}_6 = -\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7},$$

$$\tilde{\Psi}_7 = \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}.$$

Therefore, $\tilde{M}$ is a light-like submanifold with $Rad\,T\tilde{M} = Span\{\tilde{\Psi}_1, \tilde{\Psi}_3\}$ and $S(T\tilde{M}) = Span\{\tilde{\Psi}_2, \tilde{\Psi}_4, \tilde{\Psi}_5, \tilde{\Psi}_6, \tilde{\Psi}_7\}$. One can see that $\hat{\Phi}\tilde{\Psi}_1 = \tilde{\Psi}_2 \in S(T\tilde{M})$ so $D_\alpha = Span\{\tilde{\Psi}_1\}$ and $D_\beta = Span\{\tilde{\Psi}_3\}$. Also, since $\hat{\Phi}\tilde{\Psi}_5 = \tilde{\Psi}_6 \in S(T\tilde{M})$, we have $\mathring{D} = Span\{\tilde{\Psi}_5, \tilde{\Psi}_6\}$ and we get $ltr(T\tilde{M})$ is spanned by

$$N_1 = \frac{1}{2B_m}\{-\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\},$$

and

$$N_2 = \frac{1}{2\bar{B}_m}\{-\frac{\partial}{\partial x_{11}} + \frac{\partial}{\partial x_{12}}\}.$$

Then we have $\tilde{L} = Span\{N_1\}, \mathring{L} = Span\{N_2\}$, $S(T\tilde{M}^\perp) = Span\{\hat{\Phi}(\tilde{\Psi}_3), \hat{\Phi}N_2, \hat{\Phi}(\tilde{\Psi}_7)\}$, $\tilde{S} = Span\{\hat{\Phi}\tilde{\Psi}_7 = W\}$ and $D' = Span\{\hat{\Phi}N_1 = \tilde{\Psi}_4, W\}$. Hence, $\tilde{M}$ is a poly-Norden screen transversal CR-light-like submanifold of $\hat{M}$.

To avoid repetition in the remain part of this section $(\hat{M}, \hat{\Phi}, \hat{g})$ will be considered a poly-Norden s-Riemannian manifold. If we denote the projection morphisms from $T\tilde{M}$ to $\mathring{D}$, $\hat{\Phi}(D_\alpha)$, $\hat{\Phi}\tilde{L}$, $\hat{\Phi}\tilde{S}$, D_α and D_β by $\mathring{T}$, T_α, T_β, $\hat{T}$, S_α and S_β, respectively, then, we can write for any $U \in \Gamma(T\tilde{M})$,

$$\begin{aligned} U &= TU + SU \\ &= \mathring{T}U + T_\alpha U + T_\beta U + \hat{T}U + S_\alpha U + S_\beta U, \end{aligned} \tag{38}$$

and

$$\hat{\Phi}U = PU + wU, \tag{39}$$

where $TU \in \Gamma(D)$ and $SU \in \Gamma(\hat{D})$ and PU and wU are tangential and transversal parts of $\hat{\Phi}U$, respectively.

Applying $\hat{\Phi}$ to (38) and denoting $\hat{\Phi}\mathring{T}, \hat{\Phi}T_\alpha, \hat{\Phi}T_\beta, \hat{\Phi}\hat{T}, \hat{\Phi}S_\alpha$ and $\hat{\Phi}S_\beta$ by $\mathring{Q}, Q_\alpha, Q_\beta, w_{\tilde{L}}, w_{\tilde{S}}, \tilde{T}_\alpha, w_\beta$, respectively, we arrive at

$$\begin{aligned} \hat{\Phi}U &= \mathring{Q}U + Q_\alpha U + Q_\beta U + w_{\tilde{L}}U \\ &\quad + w_{\tilde{S}}U + \tilde{T}_\alpha U + w_\beta U, \end{aligned} \tag{40}$$

where $\mathring{Q}U \in \Gamma(\mathring{D})$, $Q_\alpha U \in \Gamma(D_\alpha)$, $\tilde{T}_\alpha U \in (\hat{\Phi}D_\alpha)$, $w_{\tilde{L}}U \in \Gamma(\tilde{L})$, $w_{\tilde{S}}U \in \Gamma(\tilde{S})$ and $w_\beta U \in \Gamma(\hat{\Phi}D_\beta)$.

Similarly, if we denote projections from $tr(TM)$ to $\hat{\Phi}Ł$, $\hat{\Phi}\mathring{L}$, $\hat{\Phi}\tilde{S}$, $\hat{\Phi}\tilde{L}$ and $\mathring{L}$ by R_α, R_β, $\hat{R}$, Q_α, Q_β, respectively, then for $V \in \Gamma(tr(T\tilde{M}))$, we write

$$V = R_\alpha V + R_\beta V + \hat{R}V + Q_\alpha V + Q_\beta V, \tag{41}$$

and denoting $\hat{\Phi}R_{\alpha}, \hat{\Phi}R_{\beta}, \hat{\Phi}\hat{R}, \hat{\Phi}Q_{\alpha}$ and $\hat{\Phi}Q_{\beta}$ by $K_{\beta}, F_{\check{L}}, \tilde{K}_{S}, \tilde{K}_{Ł}$ and $\tilde{F}_{Ł}$, respectively, we state

$$\hat{\Phi}V = K_{\beta}V + F_{\check{L}}V + \tilde{K}_{S}V + \tilde{K}_{Ł}V + \tilde{F}_{Ł}V, \tag{42}$$

where KV and FV are sections of $T\tilde{M}$ and $tr(T\tilde{M})$, respectively.

If we differentiate (40) and using the definition of poly-Norden s-Riemannian manifold i.e., $\tilde{\nabla}\hat{\Phi} = 0$ with (10)-(12) and (42), for $U, V \in \Gamma(T\tilde{M})$, we get

$$\begin{aligned}(\tilde{\nabla}_{U}P)V &= A_{w_{\check{L}}V}U + A_{w_{\check{S}}V}U + A_{w_{\beta}V}U \\ &\quad + K\hbar(U, V),\end{aligned} \tag{43}$$

$$\begin{aligned}\Omega^{l}(U, w_{\check{S}}V) + \Omega^{l}(U, w_{\beta}V) &= w_{\check{L}}(\tilde{\nabla}_{U}V) - \tilde{\nabla}^{l}_{U}(w_{\check{L}}V) \\ &\quad - \hbar^{l}(U, PV) + F\hbar^{l}(U, V),\end{aligned} \tag{44}$$

and

$$\begin{aligned}\Omega^{s}(U, w_{\check{L}}V) &= w_{\check{S}}(\tilde{\nabla}_{U}V) + w_{\beta}(\tilde{\nabla}_{U}V) - \tilde{\nabla}^{s}_{U}(w_{\check{S}}V) \\ &\quad - \tilde{\nabla}^{s}_{U}(w_{\beta}V) - \hbar^{s}(U, PV) + F\hbar^{s}(U, V).\end{aligned} \tag{45}$$

Now, we give our main results of this section;

Theorem 4.3. *Let $(\tilde{M}, \tilde{g})$ be a poly-Norden screen transversal CR-light-like submanifold of $(\hat{M}, \hat{\Phi}, \hat{g})$. Then the distribution D is integrable if and only if*
i) $\hbar^{s}(U, V) = \hbar^{s}(V, U)$ and $\hbar^{s}(U, \hat{\Phi}V) = \hbar^{s}(V, \hat{\Phi}U)$,
ii) $\hbar^{l}(U, \hat{\Phi}V) = \hbar^{l}(V, \hat{\Phi}U)$,
for any $U, V \in \Gamma(D)$.

Proof. From the definition of the distribution D, it is integrable if and only if

$$\hat{g}([U, V], N_2) = 0,$$

$$\hat{g}([U, V], \hat{\Phi}\zeta] = 0,$$

$$\hat{g}([U, V], \hat{\Phi}W] = 0,$$

for $U, V \in \Gamma(D)$, $N_2 \in \Gamma(\mathring{L})$, $\zeta \in \Gamma(D_{\alpha})$ and $W \in \Gamma(\tilde{S})$.

In view of (10), we get

$$\begin{aligned}\hat{g}([U, V], N_2] &= \hat{g}(\hat{\nabla}_{U}V - \hat{\nabla}_{V}U, N_2) \\ &= m\hat{g}(\hat{\Phi}\hat{\nabla}_{U}V, N_2) - \hat{g}(\hat{\Phi}\hat{\nabla}_{U}V, \hat{\Phi}N_2) \\ &\quad - m\hat{g}(\hat{\Phi}\hat{\nabla}_{V}U, N_2) + \hat{g}(\hat{\Phi}\hat{\nabla}_{V}U, \hat{\Phi}N_2) \\ &= m\hat{g}(\hat{\nabla}_{U}V, \hat{\Phi}N_2) - \hat{g}(\hat{\nabla}_{U}\hat{\Phi}V, \hat{\Phi}N_2) \\ &\quad - m\hat{g}(\hat{\nabla}_{V}U, \hat{\Phi}N_2) + \hat{g}(\hat{\nabla}_{V}\hat{\Phi}U, \hat{\Phi}N_2) \\ &= m\tilde{g}(\hbar^{s}(U, V) - \hbar^{s}(V, U), \hat{\Phi}N_2) \\ &\quad - m\tilde{g}(\hbar^{s}(U, \hat{\Phi}V) - \hbar^{s}(V, \hat{\Phi}U), \hat{\Phi}N_2),\end{aligned}$$

$$\begin{aligned}\hat{g}([U, V], \hat{\Phi}\zeta] &= \hat{g}(\hat{\nabla}_{U}V - \hat{\nabla}_{V}U, \hat{\Phi}\zeta) \\ &= \hat{g}(\hat{\Phi}\hat{\nabla}_{U}V, \zeta) - \hat{g}(\hat{\Phi}\hat{\nabla}_{V}U, \zeta) \\ &= \hat{g}(\hat{\nabla}_{U}\hat{\Phi}V, \zeta) - \hat{g}(\hat{\nabla}_{V}\hat{\Phi}U, \zeta) \\ &= \tilde{g}(\hbar^{l}(U, \hat{\Phi}V) - \hbar^{l}(V, \hat{\Phi}U), \zeta),\end{aligned}$$

and

$$\begin{aligned}\hat{g}([U,V],\hat{\Phi}W] &= \hat{g}(\hat{\nabla}_U V - \hat{\nabla}_V U, \hat{\Phi}W)\\ &= \hat{g}(\hat{\Phi}\hat{\nabla}_U V, W) - \hat{g}(\hat{\Phi}\hat{\nabla}_V U, W)\\ &= \hat{g}(\hat{\nabla}_U \hat{\Phi}V, W) - \hat{g}(\hat{\nabla}_V \hat{\Phi}U, W)\\ &= \tilde{g}(\hbar^s(U,\hat{\Phi}V) - \hbar^s(V,\hat{\Phi}U), W),\end{aligned}$$

which complete the proof. □

Theorem 4.4. *Let $(\tilde{M},\tilde{g})$ be a poly-Norden screen transversal CR-light-like submanifold of $(\hat{M},\hat{\Phi},\hat{g})$. Then the distribution $\hat{D}$ is integrable if and only if*

$$A_{\hat{\Phi}U}V = A_{\hat{\Phi}V}U,$$

for any $U,V \in \Gamma(\hat{D})$.

Proof. By using the definition of $\hat{D}$, we know that $\hat{D}$ is integrable if and only if

$$\hat{g}([U,V],N_1] = 0,$$

$$\hat{g}([U,V],\hat{\Phi}N_1] = 0,$$

$$\hat{g}([U,V],Z] = 0,$$

for $U,V \in \Gamma(\hat{D})$, $N_1 \in \Gamma(\tilde{L})$ and $Z \in \Gamma(\mathring{D})$.

By use of (10), we get

$$\begin{aligned}\hat{g}([U,V],N_1] &= \hat{g}(\hat{\nabla}_U V - \hat{\nabla}_V U, N_1)\\ &= m\hat{g}(\hat{\Phi}\hat{\nabla}_U V, N_1) - \hat{g}(\hat{\Phi}\hat{\nabla}_U V, \hat{\Phi}N_1)\\ &\quad -m\hat{g}(\hat{\Phi}\hat{\nabla}_V U, N_1) + \hat{g}(\hat{\Phi}\hat{\nabla}_V U, \hat{\Phi}N_1)\\ &= m\hat{g}(\hat{\nabla}_U \hat{\Phi}V, N_1) - \hat{g}(\hat{\nabla}_U \hat{\Phi}V, \hat{\Phi}N_1)\\ &\quad -m\hat{g}(\hat{\nabla}_V \hat{\Phi}U, N_1) + \hat{g}(\hat{\nabla}_V \hat{\Phi}U, \hat{\Phi}N_1)\\ &= m\tilde{g}(A_{\hat{\Phi}U}V - A_{\hat{\Phi}V}U, \hat{\Phi}N_1)\\ &\quad -\tilde{g}(A_{\hat{\Phi}U}V - A_{\hat{\Phi}V}U, \hat{\Phi}N_1),\end{aligned}$$

$$\begin{aligned}\hat{g}([U,V],\hat{\Phi}N_1] &= \hat{g}(\hat{\nabla}_U V - \hat{\nabla}_V U, \hat{\Phi}N_1)\\ &= \hat{g}(\hat{\Phi}\hat{\nabla}_U V, N_1) - \hat{g}(\hat{\Phi}\hat{\nabla}_V U, N_1)\\ &= \hat{g}(\hat{\nabla}_U \hat{\Phi}V, N_1) - \hat{g}(\hat{\nabla}_V \hat{\Phi}U, N_1)\\ &= \tilde{g}(A_{\hat{\Phi}U}V - A_{\hat{\Phi}V}U, N_1),\end{aligned}$$

and

$$\begin{aligned}\hat{g}([U,V],\hat{\Phi}Z] &= \hat{g}(\hat{\nabla}_U V - \hat{\nabla}_V U, \hat{\Phi}Z)\\ &= \hat{g}(\hat{\Phi}\hat{\nabla}_U V, Z) - \hat{g}(\hat{\Phi}\hat{\nabla}_V U, Z)\\ &= \hat{g}(\hat{\nabla}_U \hat{\Phi}V, Z) - \hat{g}(\hat{\nabla}_V \hat{\Phi}U, Z)\\ &= \tilde{g}(A_{\hat{\Phi}U}V - A_{\hat{\Phi}V}U, Z).\end{aligned}$$

So, the proof is completed. □

Theorem 4.5. *Let $(\tilde{M},\tilde{g})$ be a poly-Norden screen transversal CR-light-like submanifold of $(\hat{M},\hat{\Phi},\hat{g})$. Then the distribution D defines a totally geodesic foliation if and only if $\hat{\Phi}\hbar(U,V)$ has no component in $\Gamma(T\tilde{M})$, i.e.,*

$$K\hbar(U,\hat{\Phi}) = 0,$$

for any $U,V \in \Gamma(D)$.

Proof. The distribution D defines a totally geodesic foliation if and only if

$$\tilde{\nabla}_U V \in \Gamma(D),$$

for any $U, V \in \Gamma(D)$.

In view of the definition of the distribution D, we can write

$$\tilde{g}(\tilde{\nabla}_U V, N_2) = 0, \quad \tilde{g}(\tilde{\nabla}_U V, \hat{\Phi}\zeta) = 0, \quad \tilde{g}(\tilde{\nabla}_U V, \hat{\Phi} W) = 0,$$

for all $U, V \in \Gamma(D), N_2 \in \Gamma(\mathring{L}), \zeta \in \Gamma(D_\alpha)$ and $W \in \Gamma(\tilde{S})$.

Therefore, using (10) with (3), we have

$$\begin{aligned}\tilde{g}(\nabla_U V, N_2) &= \hat{g}(\hat{\nabla}_U V, N_2)\\ &= m\hat{g}(\hat{\Phi}\hat{\nabla}_U V, N_2) - \hat{g}(\hat{\Phi}\hat{\nabla}_U V, \hat{\Phi} N_2)\\ &= m\hat{g}(\hat{\nabla}_U V, \hat{\Phi} N_2) - \hat{g}(\hat{\nabla}_U \hat{\Phi} V, \hat{\Phi} N_2)\\ &= m\tilde{g}(\hbar^s(U, V), \hat{\Phi} N_2) - m\tilde{g}(\hbar^s(U, \hat{\Phi} V), \hat{\Phi} N_2),\end{aligned} \tag{46}$$

$$\begin{aligned}\tilde{g}(\nabla_U V, \hat{\Phi}\zeta) &= \hat{g}(\hat{\nabla}_U V, \hat{\Phi}\zeta)\\ &= \hat{g}(\hat{\Phi}\hat{\nabla}_U V, \zeta)\\ &= \hat{g}(\hat{\nabla}_U \hat{\Phi} V, \zeta) = \tilde{g}(\hbar^l(U, \hat{\Phi}), \zeta),\end{aligned} \tag{47}$$

and

$$\begin{aligned}\tilde{g}(\nabla_U V, \hat{\Phi} W) &= \hat{g}(\hat{\nabla}_U V, \hat{\Phi} W)\\ &= \hat{g}(\hat{\Phi}\hat{\nabla}_U V, W)\\ &= \hat{g}(\hat{\nabla}_U \hat{\Phi} V, W) = \tilde{g}(\hbar^s(U, \hat{\Phi}), W).\end{aligned} \tag{48}$$

It follows from (46)-(48) that D defines totally geodesic foliation if and only if $\hbar^l(U, \hat{\Phi})$ in $\Gamma(\tilde{L})$ and $\hbar^s(U, \hat{\Phi} V)$ has no component in $W \in \Gamma(\mathring{L} \cup \tilde{S})$. So the proof is completed. □

Theorem 4.6. *Let $(\tilde{M}, \tilde{g})$ be a poly-Norden screen transversal CR-light-like submanifold of $(\hat{M}, \hat{\Phi}, \hat{g})$. Then the distribution $\hat{D}$ defines a totally geodesic foliation if and only if*

$$A_{wV} U \in \Gamma(\hat{D}),$$

for any $U, V \in \Gamma(\hat{D})$.

Proof. The distribution $\hat{D}$ defines a totally geodesic foliation if and only if

$$\tilde{\nabla}_U V \in \Gamma(\hat{D}),$$

for any $U, V \in \Gamma(\hat{D})$. Because of $V \in \Gamma(\hat{D})$ and $\nabla_U V \in \Gamma(\hat{D})$, we say that PV and $P\nabla_U V$ are zero.

By use of (43), we get

$$\begin{aligned}\tilde{\nabla}_U PV - P\tilde{\nabla}_U V &= A_{w_{\tilde{L}} V} U + A_{w_{\tilde{S}} V} U + A_{w_\beta V} U\\ &\quad + K\hbar(U, V),\end{aligned}$$

which gives

$$-K\hbar(U, V) = A_{wV} U \in \Gamma(\hat{D}).$$

Conversely, if $A_{wV} U \in \Gamma(\hat{D})$, from (43), we arrive at

$$P\tilde{\nabla}_U V = 0,$$

which completes the proof. □

Definition 4.7. *Let $(\tilde{M}, \tilde{g})$ be a poly-Norden screen transversal CR-light-like submanifold of $(\hat{M}, \hat{\Phi}, \hat{g})$. If the second fundamental form $\hbar$ of M satisfies*

$$\hbar(U, V) = 0, \qquad \forall U, V \in \Gamma(D), \tag{49}$$

then we say that $\tilde{M}$ is a $D-$geodesic screen transversal CR-light-like submanifold.

From (49), one can observe that $\tilde{M}$ is $D-$geodesic if

$$\hbar^l(U, V) = 0, \quad \hbar^s(U, V) = 0, \quad \forall U, V \in \Gamma(D). \tag{50}$$

Theorem 4.8. *Let $(\tilde{M}, \tilde{g})$ be a poly-Norden screen transversal CR-light-like submanifold of $(\hat{M}, \hat{\Phi}, \hat{g})$. Then the distribution D defines a totally geodesic foliation on $\hat{M}$ if and only if $\tilde{M}$ is $D-$geodesic.*

Proof. Suppose that D defines a totally geodesic foliation on $\hat{M}$. Then from (10), we have

$$\hat{g}(\tilde{\nabla}_U V, E) = \tilde{g}(\hbar^l(U, V), E), \tag{51}$$

$$\hat{g}(\tilde{\nabla}_U V, W) = \tilde{g}(\hbar^s(U, V), W), \tag{52}$$

for every $U, V \in \Gamma(D), E \in \Gamma(Rad\, T\tilde{M})$ and $W \in \Gamma(S(T\tilde{M}^{\perp}))$.

If D defines a totally geodesic foliation from (51) and (52), we find that $\hbar(U, V) = 0$ and M is $D-$geodesic.

Conversely we suppose that $\tilde{M}$ is $D-$geodesic. From (49) with (10), we have

$$\hat{g}(\tilde{\nabla}_U V, \hat{\Phi}\xi) = \hat{g}(\tilde{\nabla}_U \hat{\Phi} V, \xi) = \tilde{g}(\hbar^l(U, \hat{\Phi} V), \xi),$$

$$\hat{g}(\tilde{\nabla}_U V, \hat{\Phi} W) = \hat{g}(\tilde{\nabla}_U \hat{\Phi} V, W) = \tilde{g}(\hbar^s(U, \hat{\Phi} V), W),$$

for any $U, V \in \Gamma(D), \xi \in \Gamma(D_\beta)$ and $W \in \Gamma(S(T\tilde{M}^{\perp}))$. From last two equations, we obtain $\tilde{\nabla}_U V \in \Gamma(D)$ and the proof is completed. □

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