Filomat 37:10 (2023), 3021–3028
https://doi.org/10.2298/FIL2310021C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized Cline's formula for the generalized Drazin inverse in rings

Huanyin Chen[a], Marjan Sheibani Abdolyousefi[b,*]

[a] *School of Mathematics, Hangzhou Normal University, Hangzhou, China*
[b] *Farzanegan Campus, Semnan University, Semnan, Iran*

Abstract. In this paper, we present new generalized Cline's formula for g-Drazin inverse in rings and Banach algebras. These extend many known results, e.g., Chen and Abdolyousefi (Cline's formula for g-Drazin inverses in a ring, Filomat, **33**(2019), 2249–2255), Miller and Zguitti (New extensions of Jacobson's lemma and Cline's formula, Rend. Circ. Mat. Palermo, II. Ser., *67*(2018), 105–114).

1. Introduction

Let R be an associative ring with an identity. The commutant of $a \in R$ is defined by $comm(a) = \{x \in R \mid xa = ax\}$. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ for all } y \in comm(a)\}$. An element $a \in R$ has g-Drazin inverse (i.e., generalized Drazin inverse) provided that there exists $b \in R$ such that $b = bab, b \in comm^2(a), a - a^2b \in R^{qnil}$. The preceding b is unique if exists, we denote it by a^d. Here, $R^{qnil} = \{a \in R \mid 1 + ax \in R^{-1} \text{ for every } x \in comm(a)\}$. The g-Drazin invertibility in rings and Banach algebras are very attractive. It has widespread applications in singular differential equations, Markov chains, and iterative methods. Many authors have studied such problems from many different views, e.g., [4, 5, 7, 11, 13].

Let $a, b \in R$. If $ab \in R^d$, then $ba \in R^d$ and $(ba)^d = b((ab)^d)^2a$ ([9, Theorem 2.1]). This was known as Cline's formula for g-Drazin inverse. Recently, the Cline's formula for g-Drazin inverse has been extensively studied. Let $a, b, c \in R$ with $abc = aca$. If $ac \in R^d$, then $ba \in R^d$ ([8, Theorem 2.3]). Let $a, b, c \in R$ with $(ac)^2a = acaba = abaca = a(ba)^2$. If $ac \in R^d$, then $ba \in R^d$ ([3, Theorem 2.2]). Let $a, b, c, d \in R$ with $aca = dba, acd = dbd$. If $ac \in R^d$, then $ba \in R^d$ ([10, Theorem 3.2]). Further extension of Cline's formula for generalized Drazin inverse could be found in [5, Theorem 2.2].

In this paper, we present new generalized Cline's formula for generalized Drazin inverse in rings and Banach algebras. These make the preceding mentioned results as our special cases. More simpler conditions are obtained in a Banach algebra. We then apply our results to bounded linear operators and obtain the common spectral properties.

Throughout the paper, all rings are associative with an identity and all Banach algebras are complex with an identity. We use R^{-1}, R^d and R^D to denote the sets of all invertible, g-Drazin invertible and Drazin invertible elements in a ring R, respectively. We use a^{π} to stand for the spectral idempotent $1 - aa^d$ of an element $a \in \mathcal{A}^d$.

2020 *Mathematics Subject Classification*. 15A09; Secondary 47A11.
Keywords. Generalized Cline's formula; Generalized Drazin inverse; Banach algebra; Common spectral property.
Received: 01 May 2022; Accepted: 22 July 2022
Communicated by Dragan S. Djordjević
Research supported by the Natural Science Foundation of Zhejiang Province, China (No. LY21A010018).
* Corresponding author: Marjan Sheibani Abdolyousefi
Email addresses: huanyinchen@aliyun.com (Huanyin Chen), m.sheibani@semnan.ac.ir (Marjan Sheibani Abdolyousefi)

2. Generalized Cline's Formula

As is well known, $ab \in R^{qnil}$ if and only if $ba \in R^{qnil}$ for any $a, b \in R$. This was extended by Lian and Zeng (see [8, Lemma 2.2.]), Miller and Zguitti ([10, Lemma 3.1]), Chen and Abdolyousefi ([5, Lemma 2.1]). We generalize these elementary facts as follows.

Lemma 2.1. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$dbaca = (db)^2a;$$
$$acdbd = (db)^2d.$$

If $ac \in R^{qnil}$, then $bd \in R^{qnil}$.

Proof. Let $x \in comm(bd)$. Then we verify that

$$\begin{array}{rcl} (dbdx^5bdbac)ac & = & dbdx^5b(dbaca)c \\ & = & dbdx^5b(dbdba)c \\ & = & (dbdbd)x^5bdbac \\ & = & (acdbd)x^5bdbac \\ & = & ac(dbdx^5bdbac) \end{array}$$

This implies that $dbdx^5bdbac \in comm(ac)$, and then $1 - dbd(x^5bdbacac) = (dbdx^5bdbac)ac \in R^{-1}$. In view of Jacobson's Lemma, we have

$$\begin{array}{rcl} 1 - x^5bdbdbdbdbd & = & 1 - x^5bdb(dbdbd)bd \\ & = & 1 - x^5bdbac(dbdbd) \\ & = & 1 - (x^5bdbacac)dbd \\ & \in & R^{-1}. \end{array}$$

Then

$$\begin{array}{rl} & (1 - xbd)(1 + xbd + x^2bdbd + x^3bdbdbd + x^4bdbdbdbd) \\ = & (1 + xbd + x^2bdbd + x^3bdbdbd + x^4bdbdbdbd)(1 - xbd) \\ = & 1 - x^5bdbdbdbdbd \\ \in & R^{-1}, \end{array}$$

therefore $bd \in R^{qnil}$, as asserted. □

We come now to the main result of this paper.

Theorem 2.2. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$(ac)^2a = acdba = dbaca = (db)^2a;$$
$$(ac)^2d = acdbd = dbacd = (db)^2d.$$

If $ac \in R^d$, then $bd \in R^d$ and $(bd)^d = b((ac)^d)^2d$.

Proof. Suppose that ac has g-Drazin inverse and $(ac)^d = h$. Let $e = bh^2d$ and $t \in comm(bd)$. We check that

$$\begin{array}{rcl} ac(dtbdbdbac) & = & (acdbdbd)(tbac) \\ & = & (dbdbdbd)(tbac) \\ & = & dtbdb(dbdba)c \\ & = & (dtbdbdbac)ac. \end{array}$$

Thus $dtbdbdbac \in comm(ac)$, and so $(dtbdbdbac)h = h(dtbdbdbac)$. We compute that

$$\begin{array}{rcl} et & = & (bh^6(ac)^4d)t \\ & = & bh^6acacdbdbdt \\ & = & bh^6(dbdbdbdbd)t \\ & = & bh^6(dtbdbdbacd) \\ & = & b(dtbdbdbac)h^6d \\ & = & tbdbdbdbach^6d \\ & = & tb(ac)^4h^6d \\ & = & tbh^2d \\ & = & te \end{array}$$

Hence $e \in comm^2(bd)$.

Since $bd \in comm(bd)$, by the preceding discussion, we prove that $dbdbdbdbac \in comm(ac)$, and so

$$\begin{array}{rcl} (db)^5(ac) & = & (db)^3(dbdba)c \\ & = & (db)^4(ac)^2 \\ & = & (dbdbd)b(dbaca)c \\ & = & (acdbd)b(dbdba)c \\ & = & (ac)(db)^4(ac) \\ & = & (ac)(db)^5. \end{array}$$

Hence, we have $(db)^5h = h(db)^5$. Also we get

$$\begin{array}{rcl} e(bd)e & = & bh^2d(bd)bh^2d = b(h^5acacac)dbdbh^2d = bh^5(db)^5h^2d \\ & = & bh^7(db)^5d = bh^7(ac)^5d = bh^2d = e. \end{array}$$

Let $p = 1 - (ac)h$. Then $(pa)c = ac - achac = ac - (ac)^2h \in R^{qnil}$. Moreover, we have

$$\begin{array}{rcl} bd - (bd)^2e & = & bd - bdbdbh^2d \\ & = & bd - b(dbdba)ch^3d \\ & = & bd - b(ac)^3h^3d \\ & = & b(1 - ach)d \\ & = & b(pd). \end{array}$$

We directly compute that

$$\begin{array}{rcl} (pa)c(pd)b(pd) & = & pacdbpd \\ & = & acdbd - acdbachd - h(ac)^2dbd + h(ac)^2dbachd \\ (pd)b(pd)bd & = & p(db)p(db)d \\ & = & dbdbd - dbhacdbd - hac(db)^2d + hacdbhacdbd. \end{array}$$

Clearly,

$$acdbd = dbdbd.$$

$$\begin{array}{rcl} acdbachd & = & (acdba)c(ac)^dd \\ & = & (dbaca)c(ac)^dd \\ & = & db(ac)^d(ac)^2d \\ & = & db(ac)^d(acdbd) \\ & = & dbhacdbd. \end{array}$$

$$\begin{array}{rcl} h(ac)^2dbd & = & h(ac)^2dbd \\ & = & (ac)^d(ac)^2dbd \\ & = & (ac)^d(acdbd)bd \\ & = & hac(db)^2d. \end{array}$$

$$\begin{aligned} h(ac)^2dbachd &= hac(acdbachd) \\ &= hac(acdbac(ac)^d d) \\ &= hac(acdbac((ac)^d)^3(ac)^2d) \\ &= hac(dbacac((ac)^d)^3acdbd) \\ &= hac(db(ac)^d acdbd) \\ &= hac(dbhacdbd) \\ &= hacdbhacdbd. \end{aligned}$$

Therefore $(pa)c(pd)b(pd) = (pd)b(pd)bd$. Likewise, we verify that $(pd)b(pa)c(pa) = (pd)b(pd)ba$. Then by Lemma 2.1, $b(pd) \in R^{qnil}$. Hence bd has g-Drazin inverse e. That is, $e = bh^2a = (bd)^d$, as desired. □

In the case that $c = b$ and $d = a$, we recover the known generalized Cline's formula for g-Drazin inverse.

Corollary 2.3. *(see [3, Theorem 2.2]) Let R be a ring, and let $a, b, c \in R$ satisfying*

$$(ac)^2a = acaba = abaca = a(ba)^2.$$

If $ac \in R^d$, then $ba \in R^d$ and $(ba)^d = b((ab)^d)^2a$.

Corollary 2.4. *(see [10, Theorem 3.2]) Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$\begin{aligned} aca &= dba; \\ acd &= dbd. \end{aligned}$$

If $ac \in R^d$, then $bd \in R^d$ and $(bd)^d = b((ac)^d)^2d$.

Proof. This is obvious by Theorem 2.2. □

The following example shows that the preceding theorem is independent from [10, Theorem 3.2] and [16, Theorem 2.7].

Example 2.5.

Let $R = \mathbb{C}^{3\times 3}$. Choose

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, d = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R.$$

Then we check that

$$\begin{aligned} (ac)^2a = acdba = dbaca = (db)^2a; \\ (ac)^2d = acdbd = dbacd = (db)^2d. \end{aligned}$$

But $(ac)^2a \neq a(ba)^2$ and $aca \neq dba$.

3. Extensions in Banach algebras

In this section, we investigate the generalized Cline's formula in a Banach algebra. We now delieve

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$\begin{aligned} (ac)^2a &= (db)^2a; \\ (ac)^2d &= (db)^2d. \end{aligned}$$

If $ac \in \mathcal{A}^d$, then $bd \in \mathcal{A}^d$. In this case, $(bd)^d = b[(ac)^d]^2d$.

Proof. Let $aca = a^{'}, c = c^{'}, dbd = d^{'}$ and $b = b^{'}$. Then we have

$$\begin{array}{lll} a^{'}c^{'}a^{'} & = & ((ac)^2aca \\ & = & (db)^2aca \\ & = & d^{'}b^{'}a^{'}; \\ a^{'}c^{'}d^{'} & = & (ac)^2dbd \\ & = & (dbd)b(dbd) \\ & = & d^{'}b^{'}d'. \end{array}$$

Since $ac \in \mathcal{A}^d$, it follows by [11, Corollary 2.2] that $a'c' = (ac)^2 \in \mathcal{A}^d$, in light of Corollary 2.4, $b'd' = (bd)^2 \in \mathcal{A}^d$. Therefore $bd \in \mathcal{A}^d$ by [11, Corollary 2.2]. Moreover, we have

$$\begin{array}{lll} (bd)^d & = & [(bd)^2]^d bd \\ & = & (b'd')^d bd = b'[(a'c')^d]^2 d'bd \\ & = & b[(ac)^d]^4(db)^2d \\ & = & b[(ac)^d]^4(ac)^2d \\ & = & b[(ac)^d]^2d, \end{array}$$

as required. □

In the case that $c = b$ and $d = a$, we now derive

Corollary 3.2. *Let R be a Banach algebra, and let $a, b, c \in R$ with $(ac)^2a = a(ba)^2$. If $ac \in R^d$, then $ba \in R^d$ and $(ba)^d = b((ab)^d)^2a$.*

Corollary 3.3. *(see [8, Theorem 2.3]) Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c \in \mathcal{A}$ with $aba = aca$. If $ac \in \mathcal{A}$ has g-Drazin inverse, then $ba \in \mathcal{A}$ has g-Drazin inverse. In this case, $(ba)^d = b[(ac)^d]^2a$.*

Proof. Since $aba = aca$, we have $(ac)^2a(ac(aca) = (aca)ba = (ab)^2a$. This completes the proof by Corollary 3.2. □

In particular, $ab \in \mathcal{A}$ has g-Drazin inverse if and only if $ba \in \mathcal{A}$ has g-Drazin inverse (see [9, Theorem 2.1]).

Corollary 3.4. *Let $\mathcal{A}$ be a Banach algebra, and let $a, b, c, d \in \mathcal{A}$ satisfying*

$$(ac)^2a = (db)^2a;$$
$$(ac)^2d = (db)^2d.$$

If $ac \in \mathcal{A}^D$, then $bd \in \mathcal{A}^D$. In this case, $(bd)^D = b[(ac)^D]^2d$.

Proof. Since $ac \in \mathcal{A}^D$, we have $ac \in \mathcal{A}^d$. In light of Theorem 2.2, $(bd)^d = b[(ac)^D]^2d$. Clearly, $bd = (bd)(bd)^d(bd)$ and $(bd)(bd)^d = (bd)^d(bd)$. One easily checks that

$$\begin{array}{lll} [bd - (bd)^d(bd)^2](bd)^2 & = & [bd - b[(ac)^D]^2d(bd)^2](bd)^2 \\ & = & [bd - b[(ac)^D]^2(ac)^2d](bd)^2 \\ & = & b[1 - (ac)(ac)^D](db)^2d \\ & = & b[1 - (ac)(ac)^D](ac)^2d. \end{array}$$

Hence, we have

$$\begin{array}{lll} [bd - (bd)^d(bd)^2]^3 & = & [bd - (bd)^d(bd)^2](bd)^2[1 - (bd)^d(bd)] \\ & = & b[ac - (ac)^2(ac)^D]acd[1 - (bd)^d(bd)]. \end{array}$$

Since $ac - (ac)^2(ac)^D$ is nilpotent, by induction, $bd - (bd)^d(bd)^2$ is nilpotent. Therefore we complete the proof. □

4. Applications

Let X be Banach space, and let $\mathcal{L}(X)$ denote the set of all bounded linear operators from Banach space to itself, and let $a \in \mathcal{L}(X)$. The Drazin spectrum $\sigma_D(a)$ and g-Drazin spectrum $\sigma_d(a)$ are defined by

$$\begin{array}{c} \sigma_D(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin A^D\}; \\ \sigma_d(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin A^d\}. \end{array}$$

The purpose of this section is to present new common spectrum properties of $\mathcal{L}(X)$. We now extend [3, Lemma 3.1] as follows.

Lemma 4.1. *Let R be a ring, and let $a, b, c, d \in R$ satisfying*

$$\begin{array}{c} (ac)^2a = acdba = dbaca = (db)^2a; \\ (ac)^2d = acdbd = dbacd = (db)^2d. \end{array}$$

Then $1 - ac \in R^{-1}$ if and only if $1 - bd \in R^{-1}$. In this case,

$$(1 - bd)^{-1} = [1 - b(1 - ac)^{-1}(acd - dbd)][1 + b(1 - ac)^{-1}d].$$

Proof. Let $s = (1 - ac)^{-1}$. Then $s(1 - ac) = 1$, and so $1 - s = -sac$. We check that

$$\begin{array}{rcl} (1 + bsd)(1 - bd) & = & 1 - b(1 - s)d - bsdbd \\ & = & 1 + bsacd - bsdbd \\ & = & 1 + bs(acd - dbd). \end{array}$$

Hence,

$$\begin{array}{rl} & [1 - bs(acd - dbd)](1 + bsd)(1 - bd) \\ = & 1 - b(1 - s)d - bsdbd \\ = & 1 + bsacd - bsdbd \\ = & 1 - bs(acd - dbd)bs(acd - dbd) \\ = & 1 - bs(acd - dbd)b(1 + sac)(acd - dbd) \\ = & 1 - bs(ac - db)db(acd - dbd) \\ = & 1. \end{array}$$

Also we check that

$$\begin{array}{rcl} (1 - bd)(1 + bd + bacsd) & = & 1 - bdbd + b(1 - db)acsd \\ & = & 1 - b[db(1 - ac) - (1 - db)ac]sd \\ & = & 1 - b(db - ac)sd; \end{array}$$

hence, we have

$$\begin{array}{rl} & (1 - bd)(1 + bd + bacsd)[1 + b(db - ac)sd] \\ = & 1 - b(db - ac)sdb(db - ac)sd \\ = & 1 - b(db - ac)sdb(db - ac)(1 + acs)d \\ = & 1 - b(db - ac)sdb(db - ac)d \\ = & 1. \end{array}$$

Therefore $1 - bd$ is right and left invertible. In this case,

$$(1 - bd)^{-1} = [1 - bs(acd - dbd)](1 + bsd).$$

The converse is symmetric. □

We now ready to prove the following.

Theorem 4.2. *Let $A, B, C, D \in \mathcal{L}(X)$ such that*

$$(AC)^2A = ACDBA = DBACA = (DB)^2A;$$
$$(AC)^2D = ACDBD = DBACD = (DB)^2D.$$

then $\sigma_d(BD) = \sigma_d(AC)$.

Proof. Case 1. $0 \in \sigma_d(BD)$. Then $BD \notin A^d$. In view of Theorem 2.2, $AC \notin A^d$. Thus $0 \in \sigma_d(AC)$.

Case 2. $0 \notin \lambda \in \sigma_d(BD)$. Then $\lambda \in acc\sigma(BD)$. Thus, we see that

$$\lambda = \lim_{n\to\infty}\{\lambda_n \mid \lambda_n I - BD \notin \mathcal{L}(X)^{-1}\}.$$

For $\lambda_n \neq 0$, it follows by Lemma 4.1 that $I - (\frac{1}{\lambda_n}A)C \in \mathcal{L}(X)^{-1}$ implies $I - B(\frac{1}{\lambda_n}D) \in \mathcal{L}(X)^{-1}$. Therefore

$$\lambda = \lim_{n\to\infty}\{\lambda_n \mid \lambda_n I - AC \notin \mathcal{L}(X)^{-1}\} \in acc\sigma(AC) = \sigma_d(AC).$$

Therefore $\sigma_d(BD) \subseteq \sigma_d(AC)$. By the symmetry, $\sigma_d(AC) \subseteq \sigma_d(BD)$, as required. □

Corollary 4.3. *Let $A, B \in \mathcal{L}(X)$, then $\sigma_d(AB) = \sigma_d(BA)$.*

Proof. By choosing $C = B$ and $D = A$ in Theorem 4.2, we see that $\sigma_d(BA) = \sigma_d(AB)$. By the symmetry, we have $\sigma_d(AB) = \sigma_d(BA)$, as desired. □

For the Drazin spectrum $\sigma_D(a)$, we now derive

Theorem 4.4. *Let $A, B, C, D \in \mathcal{L}(X)$ such that*

$$(AC)^2A = ACDBA = DBACA = (DB)^2A;$$
$$(AC)^2D = ACDBD = DBACD = (DB)^2D.$$

then $\sigma_D(BD) = \sigma_D(AC)$.

Proof. In view of Corollary 3.4, $AC \in \mathcal{L}(X)^D$ implies that $BD \in \mathcal{L}(X)^D$, and therefore we complete the proof by [15, Theorem 3.1]. □

A bounded linear operator $T \in \mathcal{L}(X)$ is a Fredholm operator if $dimN(T)$ and $codimR(T)$ are finite, where $N(T)$ and $R(T)$ are the null space and the range of T respectively. $T \in \mathcal{L}(X)$ is a B-Fredholm operator if $R(T^n)$ is closed and $T_{[n]}$ is a Fredholm operator for some nonnegative integer n. $T \in \mathcal{L}(X)$ is a B-Weyl operator if $T_{[n]}$ is a Fredholm operator of index zero. Also we note that an operator T is B-Weyl if and only if $\pi(T)$ is Drazin invertible in the Calkin algebra (see [1, 2]). The B-Fredholm and B-Weyl spectrums of T are defined by

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a } B - \text{Fredholm operator}\};$$
$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not a } B - \text{Weyl operator}\}.$$

(see [3]).

Corollary 4.5. *Let $A, B, C, D \in \mathcal{L}(X)$ such that*

$$(AC)^2A = ACDBA = DBACA = (DB)^2A;$$
$$(AC)^2D = ACDBD = DBACD = (DB)^2D,$$

then $\sigma_{BF}(BD) = \sigma_{BF}(AC)$.

Proof. Let $\pi : \mathcal{L}(X) \to \mathcal{L}(X)/F(X)$ be the canonical map and $F(X)$ be the ideal of finite rank operators in $\mathcal{L}(X)$. As in well known, $T \in \mathcal{L}(X)$ is B-Fredholm if and only if $\pi(T)$ has Drazin inverse. By hypothesis, we see that

$$\begin{array}{l} (\pi(A)\pi(C))^2\pi(A) = \pi(A)\pi(C)\pi(D)\pi(B)\pi(A) \\ = \pi(D)\pi(B)\pi(A)\pi(C)\pi(A) = (\pi(D)\pi(B))^2\pi(A); \\ (\pi(A)\pi(C))^2\pi(D) = \pi(A)\pi(C)\pi(D)\pi(B)\pi(D) \\ = \pi(D)\pi(B)\pi(A)\pi(C)\pi(D) = (\pi(D)\pi(B))^2\pi(D), \end{array}$$

By virtue of Theorem 4.4, for every scalar λ, we have

$$\lambda I - \pi(AC) \text{ has Drazin inverse } \Longrightarrow \lambda I - \pi(BD) \text{ has Drazin inverse},$$

hence the result. □

Corollary 4.6. *Let $A, B, C, D \in \mathcal{L}(X)$ such that*

$$\begin{array}{c} (AC)^2A = ACDBA = DBACA = (DB)^2A; \\ (AC)^2D = ACDBD = DBACD = (DB)^2D, \end{array}$$

then $\sigma_{BW}(BD) = \sigma_{BW}(AC)$.

Proof. If T is B-Fredholm then for $\lambda \neq 0$ small enough, $T - \lambda I$ is Fredholm and $ind(T) = ind(T - \lambda I)$. Similarly to [14, Lemma 2.3 and Lemma 2.4], $I - AC$ is Fredholm implies $I - BD$ is Fredholm. This completes the proof by Corollary 4.5. □

References

[1] M. Berkani, Index of B-Fredholm operators and generalization of a Weyl theorem, *Proc. Amer. Math. Soc.*, **130**(2002), 1717–1723.

[2] M. Berkani and M. Sarih, An Atkinson-type theorem for B-Fredholm operators, *Studia Math.*, **148**(2001), 251–257.

[3] H. Chen and M.S. Abdolyousefi, Cline's formula for g-Drazin inverses in a ring, *Filomat*, **33**(2019), 2249–2255.

[4] H. Chen and M.S. Abdolyousefi, Generalized Cline's formula and commonl spectral property, *J. Algebra Appl.*, **20**(2021), 2150094. http://dx.doi.org/10.1142/S0219498 821500948.

[5] H. Chen and M.S. Abdolyousefi, Generalized Cline's formula for g-Drazin inverse in a ring, *Filomat*, **35**(2021), 2573–2583.

[6] R.E. Cline, An application of representation for the generalized inverse of a matrix, *MRC Technical Report*, **592**, 1965.

[7] D.S. Djordjevic and P.S. Stanimirovic, On the generalized Drazin inverse and generalized resolvent, *Czechoslovak Math. J.*, **51(126)** (2001), 617–634.

[8] Y. Lian and Q. Zeng, An extension of Cline's formula for generalized Drazin inverse, *Turk. Math. J.*, **40**(2016), 161–165.

[9] Y. Liao; J. Chen and J. Cui, Cline's formula for the generalized Drazin inverse, *Bull. Malays. Math. Sci. Soc.*, **37**(2014), 37–42.

[10] V.G. Miller and H. Zguitti, New extensions of Jacobson's lemma and Cline's formula, *Rend. Circ. Mat. Palermo, II. Ser.*, *67*(2018), 105–114.

[11] D. Mosic, A note on Cline's formula for the generalized Drazin inverse, *Linear Multilinear Algebra*, **63**(2014), 1106–1110.

[12] D. Mosic, On Jacobson's lemma and Cline's formula for Drazin inverses, , *Revista de la Unión Matemática Argentina*, **61**(2020), 267–276.

[13] Z. Wu and Q. Zeng, Extensions of Cline's formula for some new generalized inverses, *Filomat*, **35**(2021), 477–483.

[14] K. Yang and X. Fang, Common properties of the operator products in spectral theory, *Ann. Funct. Anal.*, **6**(2015), 60–69.

[15] K. Yan; Q. Zeng and Y. Zhu, On Drazin spectral equation for the operator products, *Complex Anal. Oper. Theory*, **14**, 12(2020). https://doi.org/10.1007/s11785-019-00979-y.

[16] Q.P. Zeng; Z. Wu and Y. Wen, New extensions of Cline's formula for generalized inverses, *Filomat*, **31**(2017), 1973–1980.

[17] G. Zhuang; J. Chen and J. Cui, Jacobson's lemma for the generalized Drazin inverse, *Linear Algebra Appl.*, **436**(2012), 742–746.