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Refinements of some numerical radius inequalities for Hilbert space operators

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Abstract. In this article, we present some refinements of numerical radius inequalities for operators on Hilbert space. Further, we also obtain some new upper bounds for the numerical radius of the Cartesian decomposition of operators which improves the existing bounds.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|.\|$. Let $\mathcal{L}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators from $\mathcal{H}$ into itself. In the case when $dim\mathcal{H} = n$, we identify $\mathcal{L}(\mathcal{H})$ with the matrix algebra $\mathbb{M}_n$ of all $n \times n$ complex matrices. An operator $A \in \mathcal{L}(\mathcal{H})$ is said to be positive, and denoted $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and is called positive definite, denoted $A > 0$, if $\langle Ax, x\rangle > 0$ for all non zero vectors $x \in \mathcal{H}$.

The *numerical range* of $T \in \mathcal{L}(\mathcal{H})$ is defined as

$$W(T) = \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$$

and the *numerical radius* of T, denoted by $w(T)$, is defined by $w(T) = \sup\{|z| : z \in W(T)\}$.

It is known that the set $W(T)$ is a convex subset of the complex plane and that the numerical radius $w(\cdot)$ is a norm on $\mathcal{L}(\mathcal{H})$; being equivalent to the usual operator norm $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\}$. In fact, for every $T \in \mathcal{L}(\mathcal{H})$,

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1}$$

The inequalities in (1) are sharp. If $T^2 = 0$, then the first inequality becomes an equality, on the other hand the second inequality becomes an equality if T is normal. In fact, for a nilpotent operator T with $T^n = 0$, Haagerup and Harpe [4] showed that $w(T) \leq \|T\| \cos(\pi/(n+1))$. In particular, when $n = 2$, we get the reverse inequality of the first inequality in (1). For basic information about numerical radius one can

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refer [3]. The author's of [9, 11] improved the inequality (1) which is stated in the inequality (2). If $T \in \mathcal{L}(\mathcal{H})$, then

$$w(T) \leq \frac{1}{2}\big\||T| + |T^*|\big\| \leq \frac{1}{2}\big(\|T\| + \|T^2\|^{1/2}\big), \tag{2}$$

where $|T| = (T^*T)^{1/2}$ is the absolute value of T, and

$$\frac{1}{4}\big\|T^*T + TT^*\big\| \leq w^2(T) \leq \frac{1}{2}\big\|T^*T + TT^*\big\|. \tag{3}$$

The inequalities in (2) refines the second inequality in (1). For applications of these inequalities one can refer [9, 10]. Let $T = P + iQ$ be the Cartesian decomposition of T, then P and Q are self-adjoint, and $T^*T + TT^* = 2(P^2 + Q^2)$. Thus the inequality (3) can be written as

$$\frac{1}{2}\big\|P^2 + Q^2\big\| \leq w^2(T) \leq \big\|P^2 + Q^2\big\|, \tag{4}$$

or equivalently,

$$\frac{1}{4}\big\|(P+Q)^2 + (P-Q)^2\big\| \leq w^2(T) \leq \frac{1}{2}\big\|(P+Q)^2 + (P-Q)^2\big\|. \tag{5}$$

Further generalizations of the first inequality in (2) and the second inequality in (3) have been proved in [1]. It has been shown that if $T, S \in \mathcal{L}(\mathcal{H})$, $0 < \mu < 1$ and $r \geq 1$. Then

$$w^r(T) \leq \frac{1}{2}\big\||T|^{2\mu r} + |T^*|^{2(1-\mu)r}\big\|. \tag{6}$$

$$w^r(T+S) \leq 2^{r-2}\big\||T|^{2r\mu} + |T^*|^{2r(1-\mu)} + |S|^{2r\mu} + |S^*|^{2r(1-\mu)}\big\|. \tag{7}$$

Kittaneh [11] proved a general numerical radius inequality which states that if $A, B, C, D, S, T \in \mathcal{L}(\mathcal{H})$, $0 < \mu < 1$, then

$$w(ATB + CSD) \leq \frac{1}{2}\big\|A|T^*|^{2(1-\mu)}A^* + B^*|T|^{2\mu}B + C|S^*|^{2(1-\mu)}C^* + D^*|S|^{2\mu}D\big\| \tag{8}$$

In particular,

$$w(AB \pm BA) \leq \frac{1}{2}\big\|A^*A + AA^* + B^*B + BB^*\big\|. \tag{9}$$

We refer the reader to the recent articles [1, 11–15, 17] for different generalizations, refinements and applications of numerical radius inequalities.

The objective of the paper is to present some refinement's of Kittaneh's inequality. The organization of the article is as follows. After presenting some results from the literature which are required to prove our main results; which include certain generalized refinements of numerical radius inequalities for operators on Hilbert space. In particular, we generalize and refine the inequalities (3) and (5).

2. Preliminaries

The first needed inequality is the following generalization of the mixed Cauchy-Schwarz inequality [8, Theorem 1].

Lemma 2.1. *Let $A \in \mathcal{L}(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying the identity $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle| \leq \|f(|A|x)\|\|g(|A^*|)y\|$$

for all x, y in $\mathcal{H}$.

In 1952, Kato [6] showed the mixed Schwarz inequality, which asserts

Lemma 2.2. *Let $A \in \mathcal{L}(\mathcal{H})$, and $0 \le \mu \le 1$. Then*

$$|\langle Ax, y\rangle|^2 \le \left\langle |A|^{2\mu} x, x\right\rangle \left\langle |A^*|^{2(1-\mu)} y, y\right\rangle \tag{10}$$

for the vectors $x, y \in \mathcal{H}$, where $|A| = (A^*A)^{1/2}$.

When dealing with inner product inequalities, the following inequality becomes handy [2, Theorem 1.2]:

$$f(\langle Ax, x\rangle) \le \langle f(A)x, x\rangle, \tag{11}$$

valid for the convex function $f : J \to \mathbb{R}$, the self adjoint operator A with spectrum in J and the unit vector $x \in \mathcal{H}$. The inequality (11) is reversed when f is concave. As a consequence of this inequality, we obtain the following celebrated McCarthy inequality.

Lemma 2.3. *Let $A \in \mathcal{L}(\mathcal{H})$, $A \ge 0$ and $x \in \mathcal{H}$ be a unit vector. Then*
(i) $\langle Ax, x\rangle^r \le \langle A^r x, x\rangle$ for $r \ge 1$;
(ii) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ for $0 < r \le 1$.

The following lemma is an immediate consequence of the spectral theorem for self-adjoint operators. For generalizations of this lemma, one can refer [8].

Lemma 2.4. *Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint, and let $x \in \mathcal{H}$ be any vector. Then*

$$|\langle Ax, x\rangle| \le \langle |A|x, x\rangle.$$

The following lemma proved by Kian [7] for positive operators and $r \ge 2$.

Lemma 2.5. *Let $A \in \mathcal{L}(\mathcal{H})$ be a positive operator and $x \in \mathcal{H}$ be any unit vector. Then*

$$\langle Ax, x\rangle^r \le \langle A^r x, x\rangle - \langle |A - \langle Ax, x\rangle|^r x, x\rangle, \text{ for } r \ge 2. \tag{12}$$

The following lemma is a consequence of the convexity of the function $f(t) = t^r, r \ge 1$.

Lemma 2.6. *Let a_i be a positive real number, $1 \le i \le n$. Then for each $r \ge 1$*

$$\left(\sum_{i=1}^{n} a_i\right)^r \le n^{r-1} \sum_{i=1}^{n} a_i^r \text{ for all } r \ge 1.$$

The following lemma is a consequence of the classical Jensen inequality concerning the convexity or the concavity of certain power functions [5].

Lemma 2.7. *For $a, b \ge 0$ and $0 < \mu < 1$ and $r \ne 0$,*

(i) $(\mu a^r + (1-\mu)b^r)^{\frac{1}{r}} \le (\mu a^s + (1-\mu)b^s)^{\frac{1}{s}}$, *for $r \le s$,*
(ii) $a^\mu b^{1-\mu} \le \mu a + (1-\mu)b \le (\mu a^r + (1-\mu)b^r)^{\frac{1}{r}}$ *for $r > 0$,*
(iii) $(a^s + b^s)^{\frac{1}{s}} \le (a^r + b^r)^{\frac{1}{r}}$ *for $0 < r \le s$.*

3. Main Results

Due to the theme of the results, we will split our main results into two subsections.

3.1. Refinements of numerical radius inequalities

The first main result is the following.

Theorem 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$, $0 < \mu < 1$ and $r \geq 2$. Then*

$$w^r(T) \leq \frac{1}{2}\left\| |T|^{2\mu r} + |T^*|^{2(1-\mu)r} \right\| - \inf_{\|x\|=1} \xi(x),$$

where $\xi(x) = \frac{1}{2}\left\langle \left(\left| |T|^{2\mu} - \left\langle |T|^{2\mu}x, x\right\rangle \right|^r + \left| |T^*|^{2(1-\mu)} - \left\langle |T^*|^{2(1-\mu)}x, x\right\rangle \right|^r \right)x, x \right\rangle.$

Proof. For every unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
|\langle Tx, x\rangle| &\leq \langle |T|^{2\mu}x, x\rangle^{\frac{1}{2}} \langle |T^*|^{2(1-\mu)}x, x\rangle^{\frac{1}{2}} \\
&\leq \left(\frac{\langle |T|^{2\mu}x, x\rangle^r + \langle |T^*|^{2(1-\mu)}x, x\rangle^r}{2} \right)^{\frac{1}{r}} \\
&\leq \left[\frac{1}{2}\left(\langle |T|^{2\mu r}x, x\rangle - \left\langle \left| |T|^{2\mu} - \langle |T|^{2\mu}x, x\rangle \right|^r x, x\right\rangle + \langle |T^*|^{2(1-\mu)r}x, x\rangle - \left\langle \left| |T^*|^{2(1-\mu)} - \langle |T^*|^{2(1-\mu)}x, x\rangle \right|^r x, x\right\rangle \right) \right]^{\frac{1}{r}} \\
&= \frac{1}{2^{\frac{1}{r}}}\left[\left(\left\langle \left(|T|^{2\mu r} + |T^*|^{2(1-\mu)r} \right)x, x\right\rangle - \left\langle \left| |T|^{2\mu} - \left\langle |T|^{2\mu}x, x\right\rangle \right|^r x, x\right\rangle - \left\langle \left| |T^*|^{2(1-\mu)} - \left\langle |T^*|^{2(1-\mu)}x, x\right\rangle \right|^r x, x\right\rangle \right) \right]^{\frac{1}{r}} \\
&= \frac{1}{2^{\frac{1}{r}}}\left[\left\langle \left(|T|^{2\mu r} + |T^*|^{2(1-\mu)r} \right)x, x\right\rangle - \left\langle \left(\left| |T|^{2\mu} - \left\langle |T|^{2\mu}x, x\right\rangle \right|^r + \left| |T^*|^{2(1-\mu)} - \left\langle |T^*|^{2(1-\mu)}x, x\right\rangle \right|^r \right)x, x\right\rangle \right]^{\frac{1}{r}},
\end{aligned}$$

where the first inequality follows from Lemma 2.2, the second inequality follows from Lemma 2.7, and the third inequality follows from Lemma 2.5.

Thus,

$$|\langle Tx, x\rangle|^r \leq \frac{1}{2}\left\langle \left(|T|^{2\mu r} + |T^*|^{2(1-\mu)r} \right)x, x\right\rangle - \xi(x),$$

where $\xi(x) = \frac{1}{2}\left\langle \left(\left| |T|^{2\mu} - \left\langle |T|^{2\mu}x, x\right\rangle \right|^r + \left| |T^*|^{2(1-\mu)} - \left\langle |T^*|^{2(1-\mu)}x, x\right\rangle \right|^r \right)x, x \right\rangle.$

Taking supremum over all unit vector in $\mathcal{H}$, we have

$$w^r(T) \leq \frac{1}{2}\left\| |T|^{2\mu r} + |T^*|^{2(1-\mu)r} \right\| - \inf_{\|x\|=1} \xi(x).$$

□

Remark 3.2. *In Theorem 3.1,* $\inf_{\|x\|=1} \xi(x) = 0$ *if and only if*

$$0 \in \overline{W\left(\left| |T|^{2\mu} - \left\langle |T|^{2\mu}x, x\right\rangle \right|^r + \left| |T^*|^{2(1-\mu)} - \left\langle |T^*|^{2(1-\mu)}x, x\right\rangle \right|^r \right)}.$$

As a special case of Theorem 3.1, which leads to sharper then the inequality (3) for $\xi(x) > 0$. By putting $\mu = \frac{1}{2}$ and $r = 2$ in Theorem 3.1 we have the following corollary which is the refinement of Kittaneh's inequality (3).

Corollary 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$, then*

$$w^2(T) \leq \frac{1}{2}\left\| T^*T + TT^* \right\| - \inf_{\|x\|=1} \xi(x),$$

where $\xi(x) = \frac{1}{2}\left\langle \left(\left| |T| - \left\langle |T|x, x\right\rangle \right|^2 + \left| |T^*| - \left\langle |T^*|x, x\right\rangle \right|^2 \right)x, x \right\rangle.$

The following theorem is another generalizations and refinement of Kittaneh's inequality (3) for $\xi(x) > 0$.

Theorem 3.4. *Let $T \in \mathcal{L}(\mathcal{H})$, $0 < \mu < 1$ and $r \geq 2$. Then*

$$w^{2r}(T) \leq \left\| \mu|T|^{2r} + (1-\mu)|T^*|^{2r} \right\| - \inf_{\|x\|=1} \xi(x),$$

where $\xi(x) = \langle \left(\mu \left| |T|^2 - \langle |T|^2 x, x \rangle \right|^r + (1-\mu) \left| |T^*|^2 - \langle |T^*|^2 x, x \rangle \right|^r \right) x, x \rangle$.

Proof. For every unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned}
&|\langle Tx, x \rangle|^2 \\
&\leq \langle |T|^{2\mu} x, x \rangle \langle |T^*|^{2(1-\mu)} x, x \rangle \\
&\leq \langle |T|^2 x, x \rangle^{\mu} \langle |T^*|^2 x, x \rangle^{(1-\mu)} \\
&\leq \left(\mu \langle |T|^2 x, x \rangle^r + (1-\mu) \langle |T^*|^2 x, x \rangle^r \right)^{\frac{1}{r}} \\
&\leq \left[\mu \left(\langle |T|^{2r} x, x \rangle - \langle \left| |T|^2 - \langle |T|^2 x, x \rangle \right|^r x, x \rangle \right) + (1-\mu) \left(\langle |T^*|^{2r} x, x \rangle - \langle \left| |T^*|^2 - \langle |T^*|^2 x, x \rangle \right|^r x, x \rangle \right) \right]^{\frac{1}{r}} \\
&= \left[\mu \langle |T|^{2r} x, x \rangle + (1-\mu) \langle |T^*|^{2r} x, x \rangle - \mu \langle \left| |T|^2 - \langle |T|^2 x, x \rangle \right|^r x, x \rangle - (1-\mu) \langle \left| |T^*|^2 - \langle |T^*|^2 x, x \rangle \right|^r x, x \rangle \right]^{\frac{1}{r}} \\
&= \left[\langle \left(\mu |T|^{2r} + (1-\mu) |T^*|^{2r} \right) x, x \rangle - \langle \left(\mu \left| |T|^2 - \langle |T|^2 x, x \rangle \right|^r + (1-\mu) \left| |T^*|^2 - \langle |T^*|^2 x, x \rangle \right|^r \right) x, x \rangle \right]^{\frac{1}{r}},
\end{aligned}$$

where the first inequality follows from Lemma 2.2, the second inequality follows from Lemma 2.3, the third inequality follows from Lemma 2.7, and the fourth inequality follows from Lemma 2.5.
Thus,

$$|\langle Tx, x \rangle|^{2r} \leq \langle \left(\mu |T|^{2r} + (1-\mu) |T^*|^{2r} \right) x, x \rangle - \inf_{\|x\|=1} \xi(x).$$

Taking supremum over $x \in \mathcal{H}$, we have

$$w^{2r}(T) \leq \left\| \mu|T|^{2r} + (1-\mu)|T^*|^{2r} \right\| - \inf_{\|x\|=1} \xi(x),$$

where $\xi(x) = \langle \left(\mu \left| |T|^2 - \langle |T|^2 x, x \rangle \right|^r + (1-\mu) \left| |T^*|^2 - \langle |T^*|^2 x, x \rangle \right|^r \right) x, x \rangle$. □

3.2. Numerical radius inequalities for Cartesian decomposition of operators

In this subsection we present some numerical radius inequalities but with the Cartesian decomposition of operators.

Theorem 3.5. *Let $T \in \mathcal{L}(\mathcal{H})$ with Cartesian decomposition $T = P + \mathrm{i}Q$ and let $r \geq 2$. Then*

$$w^r(T) \leq 2^{\frac{r}{2}-1} \left\| |P|^r + |Q|^r \right\| - \inf_{\|x\|=1} \xi(x),$$

where $\xi(x) = 2^{\frac{r}{2}-1} \left\langle \left(\left| |P| - \langle |P| x, x \rangle \right|^r + \left| |Q| - \langle |Q| x, x \rangle \right|^r \right) x, x \right\rangle$.

Proof. For every unit vector $x \in \mathcal{H}$, using Lemma 2.7, Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned}
\frac{|\langle Tx,x\rangle|}{\sqrt{2}} &= \left(\frac{\langle Px,x\rangle^2+\langle Qx,x\rangle^2}{2}\right)^{\frac{1}{2}}\\
&\le \left(\frac{|\langle Px,x\rangle|^r+|\langle Qx,x\rangle|^r}{2}\right)^{\frac{1}{r}}\\
&\le \frac{1}{2^{1/r}}\left(\langle |P|x,x\rangle^r+\langle |Q|x,x\rangle^r\right)^{\frac{1}{r}}\\
&\le \frac{1}{2^{1/r}}\left[\left(\langle |P|^r x,x\rangle-\langle \left||P|-\langle |P|x,x\rangle\right|^r x,x\rangle+\langle |Q|^r x,x\rangle-\langle \left||Q|-\langle |Q|x,x\rangle\right|^r x,x\rangle\right)\right]^{\frac{1}{r}}\\
&= \frac{1}{2^{1/r}}\left[\langle (|P|^r+|Q|^r)x,x\rangle-\langle \left(\left||P|-\langle |P|x,x\rangle\right|^r+\left||Q|-\langle |Q|x,x\rangle\right|^r\right)x,x\rangle\right]^{\frac{1}{r}}.
\end{aligned}$$

Thus,

$$|\langle Tx,x\rangle|^r \le 2^{\frac{r}{2}-1}\left\langle\left(|P|^r+|Q|^r\right)x,x\right\rangle-2^{\frac{r}{2}-1}\left\langle\left(\left||P|-\left\langle|P|x,x\right\rangle\right|^r+\left||Q|-\left\langle|Q|x,x\right\rangle\right|^r\right)x,x\right\rangle.$$

Taking supremum over $x \in \mathcal{H}$ with $\|x\|=1$, we have

$$w^r(T) \le 2^{\frac{r}{2}-1}\left\||P|^r+|Q|^r\right\|-\inf_{\|x\|=1}\xi(x),$$

where $\xi(x)=2^{\frac{r}{2}-1}\left\langle\left(\left||P|-\left\langle|P|x,x\right\rangle\right|^r+\left||Q|-\left\langle|Q|x,x\right\rangle\right|^r\right)x,x\right\rangle$. □

The following result is the generalization and refinement of second inequality (5) for $\xi(x)>0$.

Theorem 3.6. *Let $T \in \mathcal{L}(\mathcal{H})$ with Cartesian decomposition $T=P+\mathrm{i}Q$ and let $r\ge 2$. Then*

$$w^r(T) \le \frac{1}{2}\left\||P+Q|^r+|P-Q|^r\right\|-\inf_{\|x\|=1}\xi(x),$$

where $\xi(x)=\frac{1}{2}\left\langle\left(\left||P+Q|-\left\langle|P+Q|x,x\right\rangle\right|^r+\left||P-Q|-\left\langle|P-Q|x,x\right\rangle\right|^r\right)x,x\right\rangle$.

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}
&|\langle Tx,x\rangle|^r\\
&= \left(\langle Px,x\rangle^2+\langle Qx,x\rangle^2\right)^{\frac{r}{2}}\\
&= \frac{1}{2^{\frac{r}{2}}}\left(\langle (P+Q)x,x\rangle^2+\langle (P-Q)x,x\rangle^2\right)^{\frac{r}{2}}\\
&\le \frac{1}{2^{\frac{r}{2}}}\times 2^{\frac{r}{2}-1}\left(|\langle (P+Q)x,x\rangle|^r+|\langle (P-Q)x,x\rangle|^r\right)\\
&\quad \text{by the convexity of the function } f(t)=t^{\frac{r}{2}} \textit{ on } [0,\infty)\\
&\le \frac{1}{2}\left(\langle |P+Q|x,x\rangle^r+\langle |P-Q|x,x\rangle^r\right)\\
&\le \frac{1}{2}\left[\langle |P+Q|^r x,x\rangle-\langle \left||P+Q|-\langle |P+Q|x,x\rangle\right|^r x,x\rangle+\langle |P-Q|^r x,x\rangle-\langle \left||P-Q|-\langle |P-Q|x,x\rangle\right|^r x,x\rangle\right]\\
&= \frac{1}{2}\left[\left\langle\left(|P+Q|^r+|P-Q|^r\right)x,x\right\rangle-\left\langle\left(\left||P+Q|-\left\langle|P+Q|x,x\right\rangle\right|^r+\left||P-Q|-\left\langle|P-Q|x,x\right\rangle\right|^r\right)x,x\right\rangle\right],
\end{aligned}$$

where the second inequality follows from Lemma 2.4, and the third inequality follows from Lemma 2.5. Taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ on both sides we have

$$w^r(T) \le \frac{1}{2}\left\||P+Q|^r + |P-Q|^r\right\| - \inf_{\|x\|=1} \xi(x),$$

where $\xi(x) = \frac{1}{2}\left\langle \left(\left| |P+Q| - \left\langle |P+Q|x,x\right\rangle \right|^r + \left| |P-Q| - \left\langle |P-Q|x,x\right\rangle \right|^r \right)x,x\right\rangle$. $\square$

In Theorem 3.7, we present an upper bound for numerical radius of sum of operators but with the Cartesian decomposition.

Theorem 3.7. *Let $T_i \in \mathcal{L}(\mathcal{H})$ with Cartesian decomposition $T_i = P_i + \mathrm{i}Q_i$ for $i = 1, 2, \ldots, n$ and let $r \ge 2$. Then*

$$w^r\Big(\sum_{i=1}^n T_i\Big) \le n^{r-1} 2^{\frac{r-1}{2}} \sum_{i=1}^n \left[\left\||P_i|^{2r} + |Q_i|^{2r}\right\| - \inf_{\|x\|=1} \xi(x) \right]^{\frac{1}{2}},$$

where $\xi(x) = \langle \left(\left||P_i|^2 - \langle |P_i|^2 x,x\rangle\right|^r + \left||Q_i|^2 - \langle |Q_i|^2 x,x\rangle\right|^r \right)x,x\rangle$.

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}
&\Big|\Big\langle \sum_{i=1}^n T_i x,x\Big\rangle\Big|^r \\
&\le \left(\sum_{i=1}^n \left(\langle P_i x,x\rangle^2 + \langle Q_i x,x\rangle^2 \right)^{\frac{1}{2}} \right)^r \\
&\le \left(\sum_{i=1}^n \left(\langle |P_i|^2 x,x\rangle + \langle |Q_i|^2 x,x\rangle \right)^{\frac{1}{2}} \right)^r \\
&\le n^{r-1} \sum_{i=1}^n \left(\langle |P_i|^2 x,x\rangle + \langle |Q_i|^2 x,x\rangle \right)^{\frac{r}{2}} \\
&\le (\sqrt{2}n)^{r-1} \sum_{i=1}^n \left(\langle |P_i|^2 x,x\rangle^r + \langle |Q_i|^2 x,x\rangle^r \right)^{\frac{1}{2}} \\
&\le (\sqrt{2}n)^{r-1} \sum_{i=1}^n \left[\langle |P_i|^{2r} x,x\rangle - \langle \left||P_i|^2 - \langle |P_i|^2 x,x\rangle\right|^r x,x\rangle + \langle |Q_i|^{2r} x,x\rangle - \langle \left||Q_i|^2 - \langle |Q_i|^2 x,x\rangle\right|^r x,x\rangle \right]^{\frac{1}{2}} \\
&= (\sqrt{2}n)^{r-1} \sum_{i=1}^n \left[\langle (|P_i|^{2r} + |Q_i|^{2r})x,x\rangle - \langle \left(\left||P_i|^2 - \langle |P_i|^2 x,x\rangle\right|^r + \left||Q_i|^2 - \langle |Q_i|^2 x,x\rangle\right|^r \right)x,x\rangle \right]^{\frac{1}{2}},
\end{aligned}$$

where the second inequality follows from Lemma 2.2, for $\mu = 1$, the third inequality follows from Lemma 2.6, the fourth inequality follows from Lemma 2.6, and the fifth inequality follows from Lemma 2.5. Taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ we have

$$w^r\Big(\sum_{i=1}^n T_i\Big) \le n^{r-1} 2^{\frac{r-1}{2}} \sum_{i=1}^n \left[\left\||P_i|^{2r} + |Q_i|^{2r}\right\| - \inf_{\|x\|=1} \xi(x) \right]^{\frac{1}{2}},$$

where $\xi(x) = \langle \left(\left||P_i|^2 - \langle |P_i|^2 x,x\rangle\right|^r + \left||Q_i|^2 - \langle |Q_i|^2 x,x\rangle\right|^r \right)x,x\rangle$. $\square$

For $n = 1$, we have the following result.

Corollary 3.8. *Let $T \in \mathcal{L}(\mathcal{H})$ with Cartesian decomposition $T = P + \mathrm{i}Q$ for $r \ge 2$. Then*

$$w^r(T) \le 2^{\frac{r-1}{2}} \left[\left\||P|^{2r} + |Q|^{2r}\right\| - \inf_{\|x\|=1} \xi(x) \right]^{\frac{1}{2}},$$

where $\xi(x) = \langle \left(\left||P|^2 - \langle |P|^2 x,x\rangle\right|^r + \left||Q|^2 - \langle |Q|^2 x,x\rangle\right|^r \right)x,x\rangle$.

In the following theorem, we obtain a different upper bound for numerical radius of sum of operators but with the Cartesian decomposition.

Theorem 3.9. *Let $T_i \in \mathcal{L}(\mathcal{H})$ with Cartesian decomposition $T_i = P_i + \mathrm{i}Q_i$ for $i = 1, 2, \ldots, n$ and let $r \geq 2$. Then*

$$w^r\Big(\sum_{i=1}^{n} T_i\Big) \leq n^{r-1} 2^{\frac{r}{2}-1} \sum_{i=1}^{n} \left[\big\| |P_i + Q_i|^{2r} + |P_i - Q_i|^{2r} \big\| - \inf_{\|x\|=1} \xi(x) \right]^{\frac{1}{2}},$$

where $\xi(x) = \langle \big(\big| |P_i + Q_i|^2 - \langle |P_i + Q_i|^2 x, x\rangle \big|^r + \big| |P_i - Q_i|^2 - \langle |P_i - Q_i|^2 x, x\rangle \big|^r \big) x, x\rangle$.

Proof. Let $x \in \mathcal{H}$ be any unit vector. Then

$$\begin{aligned}
&\Big|\Big\langle \sum_{i=1}^{n} T_i x, x\Big\rangle\Big|^r \\
&\leq \left(\sum_{i=1}^{n} \big(\langle P_i x, x\rangle^2 + \langle Q_i x, x\rangle^2 \big)^{\frac{1}{2}} \right)^r \\
&\leq \left(\sum_{i=1}^{n} \Big(\frac{1}{2} \big(\langle (P_i + Q_i) x, x\rangle^2 + \langle (P_i - Q_i) x, x\rangle^2 \big) \Big)^{\frac{1}{2}} \right)^r \\
&\leq n^{r-1} 2^{\frac{-r}{2}} \sum_{i=1}^{n} \big(\langle (P_i + Q_i) x, x\rangle^2 + \langle (P_i - Q_i) x, x\rangle^2 \big)^{\frac{r}{2}} \\
&\leq n^{r-1} 2^{\frac{-r}{2}} \sum_{i=1}^{n} \big(\langle |P_i + Q_i|^2 x, x\rangle + \langle |P_i - Q_i|^2 x, x\rangle \big)^{\frac{r}{2}} \\
&\leq n^{r-1} 2^{\frac{r}{2}-1} \sum_{i=1}^{n} \big(\langle |P_i + Q_i|^2 x, x\rangle^r + \langle |P_i - Q_i|^2 x, x\rangle^r \big)^{\frac{1}{2}} \\
&\leq n^{r-1} 2^{\frac{r}{2}-1} \sum_{i=1}^{n} \Big[\langle |P_i + Q_i|^{2r} x, x\rangle - \langle \big| |P_i + Q_i|^2 - \langle |P_i + Q_i|^2 x, x\rangle \big|^r x, x\rangle \\
&\qquad + \langle |P_i - Q_i|^{2r} x, x\rangle - \langle \big| |P_i - Q_i|^2 - \langle |P_i - Q_i|^2 x, x\rangle \big|^r x, x\rangle \Big]^{\frac{1}{2}} \\
&= n^{r-1} 2^{\frac{r}{2}-1} \sum_{i=1}^{n} \Big[\langle \big(|P_i + Q_i|^{2r} + |P_i - Q_i|^{2r} \big) x, x\rangle \\
&\qquad - \langle \big(\big| |P_i + Q_i|^2 - \langle |P_i + Q_i|^2 x, x\rangle \big|^r + \big| |P_i - Q_i|^2 - \langle |P_i - Q_i|^2 x, x\rangle \big|^r \big) x, x\rangle \Big]^{\frac{1}{2}},
\end{aligned}$$

where the third inequality follows from Lemma 2.6, the fourth inequality follows from Lemma 2.2, the fifth inequality follows from Lemma 2.6, and the sixth inequality follows from Lemma 2.5.
Taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ we have

$$w^r\Big(\sum_{i=1}^{n} T_i\Big) \leq n^{r-1} 2^{\frac{r}{2}-1} \sum_{i=1}^{n} \left[\big\| |P_i + Q_i|^{2r} + |P_i - Q_i|^{2r} \big\| - \inf_{\|x\|=1} \xi(x) \right]^{\frac{1}{2}},$$

where $\xi(x) = \langle \big(\big| |P_i + Q_i|^2 - \langle |P_i + Q_i|^2 x, x\rangle \big|^r + \big| |P_i - Q_i|^2 - \langle |P_i - Q_i|^2 x, x\rangle \big|^r \big) x, x\rangle$. □

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