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A Caristi type fixed point theorem which characterizes metric completeness

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Abstract. In this paper, we improve Caristi-Jachymski-SteinJr and Banach-Caristi type fixed point theorems by relaxing the strong continuity assumption of the mapping with some weaker continuity notions. As an application, we show that the weaker version of the Caristi-Jachymski-SteinJr fixed point theorem characterizes the completeness of the metric space and the Cantor intersection property.

1. Introduction and preliminaries

The Caristi (also known as Caristi-Kirk) fixed point theorem [10, 11] is considered one of the most important results in fixed point theory due to its applications in convex minimization problems, variational inequalities, generalized differential calculus, critical point theory, normal solvability and control theory through Ekeland's approach. Equivalent characterizations of the Caristi-Kirk theorem were studied by Ekeland [15]. For various other aspects related to the Caristi-Kirk theorem and its generalizations one may see the papers of [8, 14, 31] and references therein.

Fixed point theorems for contractive definitions that force the mappings to be discontinuous at the fixed point have emerged as one of the important areas of research in fixed point theory. The problem of discontinuity of contractive mappings at the fixed point was reiterated by Rhoades [28] as an open problem. However, the question of the existence of contractive mappings that admit discontinuity in the defined domain of definition was come up with the works of Kannan [18].

Theorem 1.1. *Let T be a self-mapping of a complete metric space (X, d) such that $d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)], 0 \leq k < 1/2$, for each $x, y \in X$. Then T has a unique fixed point. Moreover, the Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n, (n = 0, 1, 2, ...)$ converges to the unique fixed point $x_* \in X$ for any initial value $x_0 \in X$.*

Kannan's mapping not only allows discontinuity in its domain of definition but could be discontinuous everywhere except at the fixed point. The following example illustrates this fact:

Example 1.2. *Let $X = [0, 2]$ and d be the usual metric on X. Define $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x}{5}, & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} .$$

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Then T satisfies Kannan's condition

$$d(Tx,Ty) \leq \frac{1}{4}[d(x,Tx) + d(y,Ty)]$$

for each $x, y \in X$ and has a unique fixed point $x = 0$ at which T is continuous. However, T is discontinuous at all other points in X.

The first affirmative answer of Rhoades's open problem was given by Pant [22]. In recent works, some more solutions to the problem of discontinuity at the fixed point and their significance to neural networks with discontinuous activation functions have been reported in [3–5, 7, 23–25].

The following fixed point theorem was proved in [10, 11]:

Theorem 1.3. *Let T be a self-mapping of a complete metric space (X, d). Suppose there exists a lower semi-continuous function $\alpha : X \to \mathbb{R}^+$ such that for each $x \in X$*

$$d(x,Tx) \leq \alpha(x) - \alpha(Tx).$$

Then T possesses a fixed point.

In [17], the authors have shown that the weakened hypothesizes used in Generalized Banach Contraction Conjecture (GBCC) can be applied to the Caristi theorem. However, in [17], the mapping T is assumed to be continuous. Theorem 1.3 still holds if we drop the assumption of lower semi-continuity of α and assume the mapping T to be continuous or some weaker forms of continuity [1, 2, 9, 25]. One of the weaker versions of continuity, namely, orbital continuity was given in 1971 by Ćirić [12]. If T is a self-mapping of a metric space (X, d) then the set $O(x, T) = \{T^n x \mid n = 0, 1, 2, \ldots\}$ is called the orbit of T at $x \in X$. A mapping T is called orbitally continuous if $z \in X$ such that $z = \lim_{i\to\infty} T^{m_i}x$ for some $x \in X$, then $Tz = \lim_{i\to\infty} TT^{m_i}x$. Every continuous self-mapping is orbitally continuous, but not conversely. A space X is said to be $T-$orbitally complete if every Cauchy sequence in $O(x, T)$, $x \in X$, converges to a point in X.

In 2017, Pant ant Pant [24] coined the following weaker form of continuity which is stronger than the notion of orbital continuity:

Definition 1.4. *A self-mapping T of a metric space (X, d) is called $r-$continuous [24], $r = 1, 2, 3, ...$, if $T^r x_n \to Tz$, whenever $\{x_n\}$ is a sequence in X such that $T^{r-1}x_n \to z$.*

It is important to note that continuity of the mapping T implies $r-$continuity of T, but not conversely. Also, continuity of T^r and $r-$continuity of T are independent conditions when $r > 1$. In a metric space, the notion of $1-$continuity is equivalent to the notion of continuity. However, $r-$continuity of T implies $(r + 1)-$continuity of T, but the reverse implication is not true in general [24].

The following weaker form of continuity was given in [25].

Definition 1.5. *A self-mapping T of a metric space (X, d) is called weakly orbitally continuous if the set $\{y \in X : \lim_{i\to\infty} T^{m_i}y = u\}$ implies $\lim_{i\to\infty} TT^{m_i}y = Tu$ for some $u \in X$ is nonempty, whenever the set $\{x \in X : \lim_{i\to\infty} T^{m_i}x = u\}$ is nonempty.*

Orbital continuity and $r-$continuity imply weak orbital continuity, but the converse need not be true.

Example 1.6. *Let $X = [0, 2]$ equipped with the usual metric. Define $T : X \to X$ by*

$$Tx = \frac{(1+3x)}{4} \quad \text{if } 0 \leq x < 1, \quad Tx = 0 \quad \text{if } 1 \leq x < 2, \quad T2 = 2.$$

Then $T^n 0 \to 1$ and $T(T^n 0) \to 1 \neq T1$. Therefore, T is not orbitally continuous. However, T is weakly orbitally continuous. If we consider $x = 2$, then $T^n 2 \to 2$ and $T(T^n 2) \to 2 = T2$ and, hence, T is weakly orbitally continuous. If we take the sequence $\{T^n 0\}$, then for any integer $r \geq 1$, we have $T^{r-1}(T^n 0) \to 1$ and $T^r(T^n 0) \to 1 \neq T1$. This shows that T is not $r-$continuous [25].

The notion of T-orbital lower semi-continuity was given by Hicks and Rhoades [16].

Definition 1.7. *Let (X, d) be a metric space and $T : X \to X$. A mapping $\gamma : X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous at a point $z \in X$ if $\{x_n\}$ is a sequence in $O(x, T)$ for some $x \in X$, $\lim_{n\to\infty} x_n = z$ implies $\liminf_{n\to\infty} \gamma(x_n) \geq \gamma(z)$.*

In this paper, we extend a variant of the Caristi type fixed point theorem considered in [17] by using some weaker notions of continuity on a complete metric space. We also provide new solutions to Rhoades's open question regarding the continuity of contractive mappings at the fixed point in the form of the Caristi type fixed point theorem.

2. Main results

Our first theorem improves Theorem 5 given in [17]. We refer this theorem as Caristi-Jachymski-SteinJr (CJS) type fixed point theorem.

Theorem 2.1. *Let T be a self-mapping of a complete metric space (X, d). Suppose that $\alpha : X \to \mathbb{R}^+ := [0, \infty)$ is such that for each $x \in X$ and $m = 1, 2$; either*

$$d(x, T^m x) \leq \alpha(x) - \alpha(T^m x), \tag{1}$$

or

$$d(Tx, T^{m+1}x) \leq \alpha(Tx) - \alpha(T^{m+1}x). \tag{2}$$

If T is weakly orbitally continuous, then T possesses a fixed point in X.

Proof. Take $x_0 \in X$, and keeping in view of repeated operation of (1) or (2) with $m = 1$ paves an increasing sequence $\{i_n\}$ such that $i_{n+1} \leq i_n + 2$, and

$$d(T^{i_n}x_0, T^{i_{n+1}}x_0) \leq \alpha(T^{i_n}x_0) - \alpha(T^{i_{n+1}}x_0).$$

Similarly, repeated operation of (1) or (2) with $m = 2$ provides an increasing sequence $\{j_n\}$ such that $j_{n+1} \leq j_n + 2$, and

$$d(T^{j_n}x_0, T^{j_{n+2}}x_0) \leq \alpha(T^{j_n}x_0) - \alpha(T^{j_{n+2}}x_0).$$

Let $g_n = \alpha(T^n x_0) - \alpha(T^{n+1}x_0)$ and $h_n = \alpha(T^n x_0) - \alpha(T^{n+2}x_0)$. Notice that $h_n = g_n + g_{n+1}$. Following [17], we estimate $d(T^k x_0, T^{k+1}x_0)$. If $k = i_n$ for some n, then $d(T^k x_0, T^{k+1}x_0) \leq g_k$.

If $k \notin \{i_n : n \in \mathbb{N}\}$, then both $k-1$ and $k+1$ belong to $\{i_n : n \in \mathbb{N}\}$, in which case both $d(T^{k-1}x_0, T^k x_0) \leq g_{k-1}$ and $d(T^{k+1}x_0, T^{k+2}x_0) \leq g_{k+1}$.

Either k or $k - 1 \in \{j_n : n \in \mathbb{N}\}$. If $k \in \{j_n : n \in \mathbb{N}\}$, then

$$d(T^k x_0, T^{k+1}x_0) \leq d(T^k x_0, T^{k+2}x_0) + d(T^{k+2}x_0, T^{k+1}x_0) \leq g_k + 2g_{k+1}.$$

If $k - 1 \in \{j_n : n \in \mathbb{N}\}$, then

$$d(T^k x_0, T^{k+1}x_0) \leq d(T^k x_0, T^{k-1}x_0) + d(T^{k-1}x_0, T^{k+1}x_0) \leq 2g_{k-1} + g_k.$$

Therefore in all cases we always have,

$$d(T^k x_0, T^{k+1} x_0) \le 2g_{k-1} + g_k + 2g_{k+1}.$$

Since $g_1 + g_2 + ... + g_n = \alpha(Tx) - \alpha(T^{n+1}x) \le \alpha(Tx)$ and the sequence of partial sums of the series $\sum_{n=0}^{\infty} d(T^n x_0, T^{n+1} x_0)$ form a bounded monotone sequence. So the sequence $\{T^n x_o\}$ converges to some point $z \in X$. Suppose that T is weakly orbitally continuous. Since $T^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of T, we get $T^n y_0 \to z$ and $T^{n+1} y_0 \to Tz$ for some $y_0 \in X$. This implies that $z = Tz$, i.e., z is a fixed point of T. □

Example 2.2. *Let $X = [0, 2]$ equipped with the usual metric. Define $T : X \to X$ by*

$$Tx = \frac{(1+x)}{2} \quad \text{if } 0 \le x < 1, \quad Tx = 0 \quad \text{if } 1 \le x < 2, \quad T2 = 2.$$

Then T has a fixed point $x = 2$. Further, T satisfies the following conditions of Theorem 2.1 [25]:

(i) *T is weakly orbitally continuous.*

(ii) *For*

$$\alpha(x) = \begin{cases} 1 - x & \text{if} \quad x < 1 \\ 1 + x & \text{if} \quad x \ge 1 \end{cases}$$

T satisfies the condition

$$d(x, Tx) \le \alpha(x) - \alpha(Tx).$$

Remark 2.3. *It is important to note that in the above example $\lim_{x\to 1} \inf \alpha(x) = 0 < \alpha(1)$, i.e., α is not lower semi-continuous. This infers that T does not satisfy the conditions of Caristi's theorem. Therefore, Theorem 2.1 can be seen as a weaker variant of Caristi's result.*

Corollary 2.4. *Let T be a self-mapping of a complete metric space (X, d). Suppose that $\alpha : X \to \mathbb{R}^+ := [0, \infty)$ is such that for each $x \in X$ and $m = 1, 2$; either (1) or (2). If T is orbitally continuous or T is $r-$continuous for $r \ge 1$ or T^r is continuous, then T possesses a fixed point in X.*

The next result provides a new answer to the Rhoades open question of discontinuity of contractive mappings at the fixed point in the form of a variant of the Caristi type fixed point theorem.

Theorem 2.5. *Let T be a self-mapping of a complete metric space (X, d). Suppose that $\alpha : X \to \mathbb{R}^+ := [0, \infty)$ is such that for each $x \in X$ and $m = 1, 2$; either (1) or (2) and T satisfies*

$$d(Tx, Ty) \le \beta(\max\{d(x, y), d(x, Tx), d(y, Ty), \frac{[d(x, Ty) + d(y, Tx)]}{2}\}), \tag{3}$$

where $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\beta(t) < t$ for $t > 0$. Then T possesses a unique fixed point in X.

Proof. Following the same arguments as in Theorem 2.1, we can establish that the sequence $\{T^n x_0\}$ in X is Cauchy. Since X is complete, there exists a point $z \in X$ such that $T^n x_0 \to z$ as $n \to \infty$. We claim that $Tz = z$. For if $Tz \ne z$, using (3), we get a contradiction. Thus z is a fixed point of T. Uniqueness of the fixed point follows easily. □

Corollary 2.6. *Let T be a contractive type self-mapping of a complete metric space (X, d). Suppose that $\alpha : X \to \mathbb{R}^+ := [0, \infty)$ is such that for each $x \in X$ and $m = 1, 2$; either (1) or (2). If T is weakly orbitally continuous or T is orbitally continuous or T is $r-$continuous for $r \ge 1$ or T^r is continuous, then T possesses a unique fixed point in X.*

The following example illustrates Corollary 2.6.

Example 2.7. *Let $X = (-\infty, \infty)$ equipped with the usual metric. Define $T : X \to X$ by*

$$Tx = 1 \quad \text{if } x \le 1, \quad Tx = 0 \quad \text{if } x > 1.$$

Then T satisfies all the conditions of Corollary 2.6 and has a unique fixed point $x = 1$ at which T is discontinuous [25]. The mapping T satisfies the contractive condition

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{[d(x, Ty) + d(y, Tx)]}{2}\}. \tag{4}$$

The mapping T also satisfies condition (1) (for $m = 1$) with $\alpha : X \to \mathbb{R}^+$ defined by

$$\alpha(x) = \begin{cases} 1 - x & \text{if} \quad x \le 1 \\ 1 + x & \text{if} \quad x > 1 \end{cases}$$

In Theorem 2.12 [1], the authors used the following condition which is equivalent to the notion of orbital continuity (see Pant et al. [25]).

(A01): If $x_n \to x$, then $d(x, Tx) = 0$; here $x_n = T^n x_0$ for $n \in \mathbb{N}$.

Corollary 2.8. *Let T be a self-mapping of a complete metric space (X, d). Suppose that $\alpha : X \to \mathbb{R}^+ := [0, \infty)$ is such that for each $x \in X$ and $m = 1, 2$; either (1) or (2) and there exists $x_0 \in X$ satisfying **(A01)**. Then T possesses a fixed point in X.*

The next theorem is also a weaker version of [17]. It is important to note that weak orbital continuity and $T-$orbital lower semi-continuity are independent to each other.

Theorem 2.9. *Let T be a self-mapping of a $T-$orbitally complete metric space (X, d). Suppose that $\alpha : X \to \mathbb{R}^+ := [0, \infty)$ is such that for each $x \in X$ and $m = 1, 2$; either*

$$d(x, T^m x) \le \alpha(x) - \alpha(T^m x),$$

or

$$d(Tx, T^{m+1} x) \le \alpha(Tx) - \alpha(T^{m+1} x).$$

If $x \to d(x, Tx)$ is $T-$orbitally lower semi-continuous, then T has a unique fixed point z and $T^n x \to z$ as $n \in \infty$ for all $x \in O(x, T)$.

The following fixed point theorem extends Theorem 3 of Bollenbacher and Hicks [9]:

Theorem 2.10. *Let T be a self-mapping of a $T-$orbitally complete metric space (X, d) and $\alpha : X \to \mathbb{R}^+ := [0, \infty)$. Suppose there exist an $x \in X$ and $m = 1, 2$; either*

$$d(y, T^m y) \le \alpha(y) - \alpha(T^m y),$$

or

$$d(Ty, T^{m+1} y) \le \alpha(Ty) - \alpha(T^{m+1} y),$$

for every $y \in O(x, T)$ and any Cauchy sequence in $O(x, T)$ converges to a point in X. Then $Tz = z$ iff $G(x) = d(x, T^m x)$ or $d(Tx, T^{m+1} x)$ is $T-$orbitally lower semicontinuous at x.

Proof. The proof follows from the arguments given in Theorem 2.1 and [9]. □

3. Characterization of metric completeness

Various authors have studied fixed point theorems that characterize metric completeness ([19, 20, 27, 29, 30, 34]). In the next result, we show that Theorem 2.1 characterizes the metric completeness of X. Also, we prove that completeness of the space is equivalent to fixed point property for a larger class of mappings including both continuous and discontinuous mappings (see [25]). It may be observed that there is a substantive difference between the next theorem and similar theorems (e. g., Subrahmanyam [29], Suzuki [30]) giving a characterization of completeness in terms of fixed point property for contractive type mappings (see [25]). In [19], Kirk showed that Caristi's fixed theorem characterizes metric completeness.

Theorem 3.1. *If every weak orbitally continuous self-mapping T of a metric space (X, d) satisfying (1) or (2) for $m = 1, 2$ of Theorem 2.1 possesses a fixed point, then (X, d) is complete.*

Proof. Suppose that every weak orbitally continuous self-mapping of a metric space (X, d) satisfying (1) or (2) for $m = 1, 2$ of Theorem 2.1 possesses a fixed point. We will show that (X, d) is complete. Arguing by contradiction, suppose that (X, d) is not complete. Then there exists a Cauchy sequence in (X, d), say $P = \{v_n\}_{n\in\mathbb{N}}$ having distinct points which does not converge in (X, d). Let $x \in X$ be any arbitrary point. Then, since x is not a limit point of the Cauchy sequence P, and we have $d(x, P - \{x\}) > 0$ and there exists an integer $n_x \in \mathbb{N}$ such that $x \neq v_{n_x}$ and for each $l \geq n_x$

$$d(v_{n_x}, v_l) < \frac{1}{2}d(x, v_{n_x}). \tag{5}$$

Consider a mapping $T : X \mapsto X$ by $T(x) = v_{n_x}$. Then $T(x) \neq x$ for each x and using (5), for any $x, y \in X$, we get

$$d(Tx, Ty) = d(v_{n_x}, v_{n_y}) < \frac{1}{2}d(x, v_{n_x}) = d(x, Tx), \text{if } \; n_x \leq n_y. \tag{6}$$

Taking $y = Tx$ in (6), we get $n_x \leq n_{Tx}$ and

$$d(Tx, T^2x) = d(v_{n_x}, v_{n_{Tx}}) < \frac{1}{2}d(x, v_{n_x}) = d(x, Tx). \tag{7}$$

Now let us define a function $\alpha : X \to [0, \infty)$ by $\alpha(x) = 2d(x, Tx)$. Then using (7), we get

$$\alpha(x) - \alpha(Tx) = 2d(x, Tx) - 2d(Tx, T^2x) \geq 2d(x, Tx) - d(x, Tx) = d(x, Tx). \tag{8}$$

It is clear from (8) that the mapping T satisfies (1) for m=1 of Theorem 2.1. Similarly, one can show that the mapping T satisfies (1) for m=2 or (2) for m=1, 2 of Theorem 2.1. Moreover, T is a fixed point free mapping whose range is contained in the non-convergent Cauchy sequence. Hence, there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which $\{Tx_n\}_{n\in\mathbb{N}}$ converges, i.e., there exists no sequence $\{x_n\}_{n\in\mathbb{N}}$ in X for which the condition $T^{r-1}x_n \to t \Rightarrow T^r x_n \to Tt$ for $r > 1$ is violated. Therefore, T is r-continuous mapping. Hence, T is weak orbitally continuous. Thus, we have a self-mapping T of X which satisfies conditions of Theorem 2.1 but does not possess a fixed point. This contradicts the hypothesis of the theorem. Hence (X, d) is complete. □

The following variant of Cantor's intersection theorem holds in a complete metric space:

Theorem 3.2. *Suppose that (X, d) is a complete metric space and C_k is a sequence of non-empty closed nested subsets of X whose diameters tend to zero, i.e., $\lim_{k\to\infty} diam(C_k) = 0$, where $diam(C_k)$ is defined by*

$$diam(C_k) = \sup\{d(x, y) | x, y \in C_k\}.$$

Then the intersection of the C_k contains exactly one point, that is, $\bigcap_{k=1}^{\infty} C_k = \{x\}$ for some $x \in X$.

Converse of the above theorem also holds good: If (X, d) is a metric space with the property that the intersection of any nested family of non-empty closed subsets whose diameters tend to zero is non-empty, then X is a complete metric space.

We now show that Theorem 2.1 characterizes Cantor's intersection property.

Theorem 3.3. *Let (X,d) be a metric space and T a self-mapping of X satisfying (1) or (2) for $m = 1,2$ of Theorem 2.1. Suppose X satisfies Cantor's intersection property and T is weak orbitally continuous mapping. Then T has a fixed point.*

Proof. Let x be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = T^n x, n = 0,1,2,3....$ Then following the proof of Theorem 2.1, we have $\{x_n\}$ is a Cauchy sequence. Define a sequence $\{f_n\}_{n\in\mathbb{N}}$ of nonempty subsets of X by $f_n = \{x_i : i \geq n\}$. Let C_n denote the closure of $f_n \in X$. Then for each n, it is obvious that C_n is a nonempty closed subset of X, $C_{n+1} \subseteq C_n$ and $\lim_{n\to\infty} diam(C_n) = 0$. Since Cantor's intersection property holds in X, $\cap\{C_n\}$ consists of exactly one point, say z, which is nothing but the limit of the Cauchy sequence $\{x_n\}$. Rest of the proof follows from Theorem 2.1. □

Theorem 3.4. *Let (X,d) be a metric space. Every weak orbitally continuous self-mapping T of a metric space (X,d) satisfying (1) or (2) for $m = 1,2$ of Theorem 2.1 possesses a fixed point, then X satisfies Cantor's intersection property.*

Proof. Suppose that every weak orbitally continuous self-mapping T of a metric space (X,d) satisfying (1) or (2) for $m = 1,2$ of Theorem 2.1 possesses a fixed point. We assert that X satisfies Cantor's intersection property. If possible, suppose X does not satisfy Cantor's intersection property, then there exists a sequence $\{C_n\}$ of nonempty closed subsets of X satisfying $C_{n+1} \subseteq C_n$ and $\lim_{n\to\infty} diam(C_n) = 0$ and having empty intersection. Construct a sequence $f = \{x_n\} \in X$ such that $x_i \in C_i$. $\lim_{n\to\infty} diam(C_n) = 0$, given $\epsilon > 0$ there exists a positive integer N such that $n,m \geq N$ implies $d(x_n,x_m) < \epsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence. However, $f = \{x_n\}$ is a non-convergent Cauchy sequence since the sequence $\{C_n\}$ has empty intersection. As done in the proof of Theorem 3.1, we can now define weakly orbitally continuous self-mappin! g T on a metric space (X,d) satisfying (1) or (2) which does not possess a fixed point. This contradicts our hypothesis. Therefore, Cantor's intersection property holds in X. □

In view of Theorems 3.1 and 3.4, we get the following result:

Theorem 3.5. *For a metric space (X,d), the following are equivalent:*
(a) (X,d) is complete.
(b) (X,d) satisfies Cantor's intersection property.
(c) every weak orbitally continuous self-mapping T of (X,d) such that there exists a function $\alpha : X \to [0,\infty)$ satisfying for each $x \in X$ and $m = 1,2$; either

$$d(x,T^m x) \leq \alpha(x) - \alpha(T^m x),$$

or

$$d(Tx,T^{m+1}x) \leq \alpha(Tx) - \alpha(T^{m+1}x).$$

has a fixed point.

4. Common fixed points of Banach-Caristi contractive condition

In 2013, Turinici [32] proved an interesting common fixed point theorem which includes a wide class of well-known fixed point theorems in the existing literature (see, [2, 13, 21, 33]). Before positing the main result of Turinici, we recall here some related definitions and notations from [32].

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. φ is called regressive provided $\varphi(0) = 0$ and $\varphi(t) < t$, $t \in \mathbb{R}^0_+ = (0,\infty)$. For some fix φ. Let us define, the complement of φ, i.e., $\psi = J - \varphi$, here J is the identity function, i.e., $J(t) = t, t \in \mathbb{R}_+$. Clearly, $\psi : \mathbb{R}_+ \to \mathbb{R}_+$; precisely,

$$\psi(0) = 0, 0 < \psi(t) \leq t, t \in \mathbb{R}^0_+.$$

Now following Turinici [32], we define:

(A) φ is super-additive: $\varphi(t+s) \geq \varphi(t) + \varphi(s)$ for all $t, s \geq 0$. In this case φ must be increasing.

(B) $\psi := J - \varphi$ is coercive: $\psi(t) \to \infty$ as $t \to \infty$. φ can be termed as complementary coercive. By this definition

$$g(r) := sup\{t \geq 0 : \psi(t) \leq r\} < \infty, \text{for each } \ r \in \mathbb{R}_+,$$

wherefrom, $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $g(0) = 0,\ g(r) \geq r,\ r \in \mathbb{R}_+$.

Lemma 4.1. *[32] Let φ be regressive, super-additive complementary coercive. Further, let the sequence $(\theta_n; n \geq 0)$ in $\mathbb{R}_+$ be such that*

$$\theta_{m+n} \leq \varphi(\theta_m) - \delta_m - \delta_{m+1}, \tag{9}$$

where $(\delta_n; n \geq 0)$ is a sequence in $\mathbb{R}_+$. Then, the series $\Sigma_n \theta_n$ converges.

Now, we show that the Banach-Caristi contractive condition studied by Turinici still guarantees the existence of common fixed points under various weaker form of continuity.

Theorem 4.2. *Suppose A and B are weakly orbitially continuous selfmappings of a complete metric space (X, d). Let the function φ be regressive, super-additive, complementary coercive and $\gamma : X^2 \to \mathbb{R}_+$ such that*

$$d\,(Ax, By) \leq \varphi(d\,(x, y)) + \gamma(x, y) - \gamma(Ax, By);$$

for all $x, y \in X$ and all $n \geq 1$. Then

(a) A and B have a unique common fixed point $z \in X$;

(b) $A^n x \to z$ and $B^n x \to z$ as $n \to \infty$, for each $x \in X$.

Proof. Given $x_0, y_0 \in X$, put $x_n = A^n x_0 : n \geq 0$ and $y_n = B^n y_0 : n \geq 0$. By applying Lemma 4.1 and following the proof of Turinici [32], we can show that $\{A^n x_0\}$ and $\{B^n y_0\}$ are Cauchy sequences. In view of weak orbital continuity of A and B, we get z is a common fixed point of A and B. □

An extension of Theorem 4.2 is the following:

Theorem 4.3. *Suppose A and B are weakly orbitially continuous selfmappings of a complete metric space (X, d). Let the function φ be regressive, complementary coercive and $\gamma : X^2 \to \mathbb{R}_+$ such that*

$$\sum_{j=1}^{n} d\left(A^j x, B^j y\right) \leq \varphi(\sum_{j=0}^{n-1} d\left(A^j x, B^j y\right)) + \gamma(x, y) - \gamma(A^n x, B^n y);$$

for all $x, y \in X$ and all $n \geq 1$. Then conclusions of Theorem 4.2 are retainable.

Proof. The proof follows on the same lines as given in Theorem 5 [32]. □

Remark 4.4. *Theorem 4.3 extends various common fixed point theorems in the existing literature (see, [2, 13, 21, 26, 33]).*

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