Filomat 37:11 (2023), 3591–3599
https://doi.org/10.2298/FIL2311591A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some remarks on $\mathcal{K}$-starcompact and related spaces

Nur Alam[a]

[a]*Department of Mathematics, University of Gour Banga, Malda-732103, West Bengal, India*

Abstract. This article is a continuation of the study on $\mathcal{K}$-starcompact and related spaces done in (Song, Bull. Malays. Math. Sci. Soc., 30(1) (2007)). We also introduce and study nearly $1\frac{1}{2}$-starcompact spaces as a generalization of $1\frac{1}{2}$-starcompact spaces.

1. Introduction

This article is a continuation of the study on $\mathcal{K}$-starcompact and related spaces done in [11]. The relationship between the variation of compactness using star operations is represented in the following diagram, where an arrow denotes the implication.

$$\text{starcompact} \longrightarrow \mathcal{K}\text{-starcompact} \longrightarrow 1\tfrac{1}{2}\text{-starcompact} \longrightarrow \text{star-Menger}$$

The existence of a $1\frac{1}{2}$-starcompact space which is not $\mathcal{K}$-starcompact is still unknown. Though an attempt was made in [11, Example 2.2] to produce an example of such a space, during our investigation it has been observed that the considered space is indeed $\mathcal{K}$-starcompact (see Example 3.3). So the following problem remains open.

Problem 1.1. *Does there exist a $1\frac{1}{2}$-starcompact space which is not $\mathcal{K}$-starcompact?*

We introduce nearly $1\frac{1}{2}$-starcompact spaces as a generalization of $1\frac{1}{2}$-starcompact spaces and observe that this class of spaces is distinct from both the class of star-Menger and $1\frac{1}{2}$-starcompact spaces as well. Accordingly the class of nearly $1\frac{1}{2}$-starcompact spaces can be distinguished from the class of $\mathcal{K}$-starcompact spaces. Few illustrative examples have been presented to study the behaviour of the extent of the spaces considered here. Certain observations on the Alexandroff duplicates are obtained. We also discuss preservation like properties of these spaces under certain topological operations.

2020 *Mathematics Subject Classification*. Primary 54D20; Secondary 54B05, 54C10, 54D99.
Keywords. $\mathcal{K}$-starcompact; $1\frac{1}{2}$-starcompact; nearly $1\frac{1}{2}$-starcompact.
Received: 31 May 2022; Revised: 25 December 2022; Accepted: 30 December 2022
Communicated by Ljubiša D.R. Kočinac
The author is thankful to University Grants Commission (UGC), New Delhi-110002, India for granting UGC-NET Junior Research Fellowship (1173/(CSIR-UGC NET JUNE 2017)) during the tenure of which this work was done.
Email address: nurrejwana@gmail.com (Nur Alam)

2. Preliminaries

Throughout the paper (X, τ) stands for a topological space. For undefined notions and terminologies see [5].

For a subset A of a space X and a collection $\mathcal{P}$ of subsets of X, $St(A, \mathcal{P})$ denotes the star of A with respect to $\mathcal{P}$, that is the set $\cup\{B \in \mathcal{P} : A \cap B \neq \emptyset\}$. For $A = \{x\}$, $x \in X$, we write $St(x, \mathcal{P})$ instead of $St(\{x\}, \mathcal{P})$ [5].

A space X is said to be $1\frac{1}{2}$-starcompact [11] if for every open cover $\mathcal{U}$ of X there exists a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $St(\cup\mathcal{V}, \mathcal{U}) = X$. A space X is said to be $\mathcal{K}$-starcompact [11] (resp. starcompact [11, 15]) if for every open cover $\mathcal{U}$ of X there exists a compact (resp. finite) $A \subseteq X$ such that $St(A, \mathcal{U}) = X$. A space X is said to be star-Menger [4, 6] (see also [7]) if for each sequence $(\mathcal{U}_n)$ of open covers of X there exists a sequence $(\mathcal{V}_n)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $\cup_{n\in\omega}\{St(V, \mathcal{U}_n) : V \in \mathcal{V}_n\}$ is an open cover of X. A space X is said to be star countable [15] if for every open cover $\mathcal{U}$ of X there exists a countable $A \subseteq X$ such that $X = St(A, \mathcal{U})$.

A family $\mathcal{A} \subseteq P(\omega)$ is said to be an almost disjoint family if each $A \in \mathcal{A}$ is infinite and for any two distinct elements $B, C \in \mathcal{A}$, $|B \cap C| < \omega$. For an almost disjoint family $\mathcal{A}$, let $\Psi(\mathcal{A}) = \mathcal{A} \cup \omega$ be the Isbell-Mrówka space (or, Ψ-space) (see [9]). It is well known that $\Psi(\mathcal{A})$ is pseudocompact if and only if $\mathcal{A}$ is a maximal almost disjoint family. In general, when talking about Isbell-Mrówka space we do not require the almost disjoint family to be maximal or the space to be pseudocompact.

A subset A of a space X is said to be regular-closed in X if $\mathrm{Cl}(\mathrm{Int}\, A) = A$. For a space X, $e(X) = \sup\{|Y| : Y \text{ is a closed and discrete subspace of } X\}$ is said to be the extent of X.

We use $|A|$ to denote the cardinality of a set A. For any cardinal κ, κ^+ denotes the smallest cardinal greater than κ. Let ω be the first infinite cardinal, ω_1 be the first uncountable cardinal and $\mathfrak{c}$ be the cardinality of the continuum. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. For each pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$ and $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$.

3. Main results

3.1. Certain observations on $\mathcal{K}$-starcompact and related spaces

We first introduce the following definition.

Definition 3.1. A space X is said to be nearly $1\frac{1}{2}$-starcompact if for every open cover $\mathcal{U}$ of X there exists a countable $A \subseteq X$ and a finite $\mathcal{V} \subseteq \mathcal{U}$ such that $X \setminus A \subseteq St(\cup\mathcal{V}, \mathcal{U})$.

Clearly, every countable space (and also every $1\frac{1}{2}$-starcompact space) is nearly $1\frac{1}{2}$-starcompact.

In the following example, we observe that there exists a nearly $1\frac{1}{2}$-starcompact space which is not $1\frac{1}{2}$-starcompact (hence not $\mathcal{K}$-starcompact).

Example 3.2. There exists a Hausdorff nearly $1\frac{1}{2}$-starcompact space which is not $1\frac{1}{2}$-starcompact.

Proof. Let $P = \{x_\alpha : \alpha < \mathfrak{c}\}$, $Q = \{y_n : n \in \omega\}$ and $Y = \{\langle x_\alpha, y_n\rangle : \alpha < \mathfrak{c}, n \in \omega\}$, and let $X = Y \cup P \cup \{p\}$ where $p \notin Y \cup P$. We define a topology on X as follows: every point of Y is isolated, a basic neighbourhood of a point $x_\alpha \in P$ for each $\alpha < \mathfrak{c}$ is of the form

$$U_{x_\alpha}(n) = \{x_\alpha\} \cup \{\langle x_\alpha, y_m\rangle : m > n\}$$

for $n \in \omega$ and a basic neighbourhood of p is of the form

$$U_p(A) = \{p\} \cup \{\langle x_\alpha, y_n\rangle : x_\alpha \in P \setminus A, n \in \omega\}$$

for a countable subset A of P. It is clear that X is a Hausdorff space from the definition of the topology on X. We now show that X is a nearly $1\frac{1}{2}$-starcompact space. Let $\mathcal{U}$ be an open cover of X. Then we can find a countable subset A of P and a member U of $\mathcal{U}$ such that $U_p(A) \subseteq U$. By the construction of the topology on X, we have $(P \setminus A) \cup U_p(A) \subseteq St(U, \mathcal{U})$. For each $x_\alpha \in A$, $C_{x_\alpha} = \{x_\alpha\} \cup \{\langle x_\alpha, y_n\rangle : n \in \omega\}$ is countable. Choose $C = \cup_{x_\alpha \in A} C_{x_\alpha}$. Thus C is a countable subset of X with $X = C \cup (P \setminus A) \cup U_p(A)$. Therefore X is nearly $1\frac{1}{2}$-starcompact. But X is not $1\frac{1}{2}$-starcompact. Indeed, consider the open cover

$$\mathcal{U} = \{U_p(A)\} \cup \{U_{x_\alpha}(0) : x_\alpha \in A\} \cup \{\{\langle x_\alpha, y_0\rangle\} : x_\alpha \in A\}$$

of X, where A is a countably infinite subset of P. □

Also note that $(\omega + 1) \times (\omega + 1) \setminus \{\langle \omega, \omega\rangle\}$ is another example of a nearly $1\frac{1}{2}$-starcompact space which is not $1\frac{1}{2}$-starcompact.

In the next example, we observe that the conclusion of [11, Example 2.2] is not correct, where it was shown that the considered space is not $\mathcal{K}$-starcompact.

Example 3.3. The space X as in [11, Example 2.2] is $\mathcal{K}$-starcompact.

Proof. Let X be the space as in [11, Example 2.2]. Then $X = \omega_1 \cup A$, where $A = \{a_\alpha : \alpha < \omega_1\}$ is a set of cardinality ω_1 and the topology on X is defined as follows. ω_1 has the usual order topology and is an open subspace of X, a basic neighbourhood of a point $a_\alpha \in A$ is of the form

$$O_\beta(a_\alpha) = \{a_\alpha\} \cup (\beta, \omega_1), \text{ where } \beta < \omega_1.$$

We claim that for each $\alpha \in \omega_1$, $\omega_1 \cup \{a_\alpha\}$ is a compact subspace of X. Let $\alpha \in \omega_1$ be fixed and $\mathcal{U}$ be a cover of $\omega_1 \cup \{a_\alpha\}$ by open sets in X. Then there exists a $U \in \mathcal{U}$ such that $O_\alpha(a_\alpha) \subseteq U$. Since $[0, \alpha]$ is compact, we get a finite $\mathcal{V} \subseteq \mathcal{U}$ with $[0, \alpha] \subseteq \cup\mathcal{V}$. It follows that $\omega_1 \cup \{a_\alpha\} \subseteq \cup(\{U\} \cup \mathcal{V})$ and hence $\omega_1 \cup \{a_\alpha\}$ is compact. Thus for each $\alpha \in \omega_1$, $\omega_1 \cup \{a_\alpha\}$ is a compact subspace of X. We pick an open cover $\mathcal{W}$ of X to show that X is $\mathcal{K}$-starcompact. Let $\beta \in \omega_1$ be fixed and choose $K = \omega_1 \cup \{a_\beta\}$. Since K intersects every member of $\mathcal{W}$, thus $X = St(K, \mathcal{W})$. Consequently X is $\mathcal{K}$-starcompact. □

Theorem 3.4. *Every nearly $1\frac{1}{2}$-starcompact space is star-Menger.*

Proof. Let X be a nearly $1\frac{1}{2}$-starcompact space. We pick a sequence $(\mathcal{U}_n)$ of open covers of X to show that X is star-Menger. Since X is nearly $1\frac{1}{2}$-starcompact, for each n there exists a finite $\mathcal{V}_n \subseteq \mathcal{U}_n$ and a countable $A_n \subseteq X$ such that $X \setminus A_n \subseteq St(\cup\mathcal{V}_n, \mathcal{U}_n)$. Since $A = \cup_{n\in\omega} A_n$ is countable, we enumerate it as $\{a_n : n \in \omega\}$. For each n let $U_n \in \mathcal{U}_n$ with $a_n \in U_n$. Then for each $n \in \omega$, $\mathcal{H}_n = \mathcal{V}_n \cup \{U_n\}$ is a finite subset of $\mathcal{U}_n$ and $\cup_{n\in\omega}\{St(V, \mathcal{U}_n) : V \in \mathcal{H}_n\}$ is an open cover of X. Hence the result. □

It is easy to observe that the set of all reals $\mathbb{R}$ is star-Menger but not nearly $1\frac{1}{2}$-starcompact.

We now present a few illustrative examples to study the behaviour of the extent of $\mathcal{K}$-starcompact ($1\frac{1}{2}$-starcompact, nearly $1\frac{1}{2}$-starcompact) spaces.

Example 3.5. For any infinite cardinal κ, there exists a Tychonoff $\mathcal{K}$-starcompact (and hence $1\frac{1}{2}$-starcompact, nearly $1\frac{1}{2}$-starcompact) star countable space $X(\kappa)$ with $e(X(\kappa)) \geq \kappa$.

Proof. For each $\alpha < \kappa$, choose a point $f_\alpha \in \{0, 1\}^\kappa$ which is defined by $f_\alpha(\alpha) = 1$ and $f_\alpha(\beta) = 0$ if $\beta \neq \alpha$. Let $D = \{f_\alpha : \alpha < \kappa\}$. Consider

$$X(\kappa) = (\{0, 1\}^\kappa \times (\kappa^+ + 1)) \setminus ((\{0, 1\}^\kappa \setminus D) \times \{\kappa^+\})$$

as a subspace of the product space $\{0, 1\}^\kappa \times (\kappa^+ + 1)$. In [8, Theorem 1], Matveev proved that $X(\kappa)$ is a Tychonoff star countable space and $D \times \{\kappa^+\}$ is a closed and discrete subset of it i.e. $e(X(\kappa)) \geq \kappa$. We now show that $X(\kappa)$ is $\mathcal{K}$-starcompact. Let $\mathcal{U}$ be an open cover of $X(\kappa)$. Then for each $\alpha < \kappa$, we can say that $(U_\alpha \times (\gamma, \kappa^+]) \cap X(\kappa)$ is contained in some element of $\mathcal{U}$, where U_α is an open set in $\{0, 1\}^\kappa$ containing $f_\alpha \in D$ and $\gamma < \kappa^+$ is fixed. We can easily obtain a $\gamma < \beta < \kappa^+$ such that $K_1 = \{0, 1\}^\kappa \times (\beta + 1)$ is compact and $D \times \{\kappa^+\} \subseteq St(K_1, \mathcal{U})$. Now $\{0, 1\}^\kappa \times \{\kappa^+\}$ is $\mathcal{K}$-starcompact since it is countably compact. It follows that there exists a compact subspace K_2 of $X(\kappa)$ such that $\{0, 1\}^\kappa \times \{\kappa^+\} \subseteq St(K_2, \mathcal{U})$. Then the set $K_1 \cup K_2$ witnesses for $\mathcal{U}$ that $X(\kappa)$ is $\mathcal{K}$-starcompact (and hence $1\frac{1}{2}$-starcompact, nearly $1\frac{1}{2}$-starcompact). □

For a Tychonoff space X, βX denotes the Čech-Stone compactification of X.

Example 3.6. For any infinite cardinal $\kappa > \omega$, there exists a Tychonoff $\mathcal{K}$-starcompact (and hence $1\frac{1}{2}$-starcompact, nearly $1\frac{1}{2}$-starcompact) space $Y(\kappa)$ with $e(Y(\kappa)) \geq \kappa$ which is not star countable.

Proof. Let $D = \{d_\alpha : \alpha < \kappa\}$ be the discrete space of cardinality κ. Consider

$$Y(\kappa) = (\beta D \times \kappa^+) \cup (D \times \{\kappa^+\})$$

as a subspace of $\beta D \times (\kappa^+ + 1)$. By [14, Lemma 2.3], $Y(\kappa)$ is a Tychonoff $\mathcal{K}$-starcompact (and hence $1\frac{1}{2}$-starcompact, nearly $1\frac{1}{2}$-starcompact) space. It can be easily concluded that $D \times \{\kappa^+\}$ is a discrete closed set in $Y(\kappa)$. Thus $e(Y(\kappa)) \geq \kappa$. Now $Y(\kappa)$ is not star countable since for the open cover

$$\mathcal{U} = \{\beta D \times \kappa^+\} \cup \{\{d_\alpha\} \times (\kappa^+ + 1) : \alpha < \kappa\},$$

$Y(\kappa)$ has no countable subset A such that $Y(\kappa) = St(A, \mathcal{U})$. □

Recall that a space X is said to be metacompact (resp. subparacompact) [2] if every open cover of it has a point-finite open refinement (resp. σ-discrete closed refinement). Since the spaces $X(\kappa)$ and $Y(\kappa)$ contain a non-compact countably compact closed subspace which is homeomorphic to κ^+, they are neither metacompact nor subparacompact. Interestingly the following question can be made.

Problem 3.7. *Can the extent of a metacompact (or, subparacompact) $\mathcal{K}$-starcompact ($1\frac{1}{2}$-starcompact, nearly $1\frac{1}{2}$-starcompact) space be arbitrarily large?*

In the next example, we answer the above question positively.

Example 3.8. For any infinite cardinal κ, there exists a Hausdorff (non-regular) metacompact subparacompact $\mathcal{K}$-starcompact (and hence $1\frac{1}{2}$-starcompact, nearly $1\frac{1}{2}$-starcompact) space $Z(\kappa)$ with $e(Z(\kappa)) \geq \kappa$ (which is not star countable when κ is uncountable).

Proof. Let $D = \{d_\alpha : \alpha < \kappa\}$ be the discrete space of cardinality κ and $aD = D \cup \{\infty\}$ be the one point compactification of D. In the product space $aD \times (\omega + 1)$, replace the local base of the point $\langle \infty, \omega \rangle$ by the family

$$\mathcal{B} = \{U \setminus (D \times \{\omega\}) : \langle \infty, \omega \rangle \in U \text{ and } U \text{ is an open set in } aD \times (\omega + 1)\}.$$

Let $Z(\kappa)$ be the space obtained by such a replacement. By [10, Example 3.4], $Z(\kappa)$ is a Hausdorff (non-regular) metacompact subparacompact space and if κ is uncountable, then it is not star countable. To show that $Z(\kappa)$ is $\mathcal{K}$-starcompact we pick an open cover $\mathcal{U}$ of it. We choose a $V \in \mathcal{U}$ such that $\langle \infty, \omega \rangle \in V$. Then there exists a $U \setminus (D \times \{\omega\}) \in \mathcal{B}$ with $U \setminus (D \times \{\omega\}) \subseteq V$. We now show that $U \setminus (D \times \{\omega\})$ is a compact subspace of $Z(\kappa)$. Let $\mathcal{W}$ be a cover of $U \setminus (D \times \{\omega\})$ by open sets in $Z(\kappa)$. Then we get a $W \in \mathcal{W}$ containing $\langle \infty, \omega \rangle$ and a $H \setminus (D \times \{\omega\}) \in \mathcal{B}$ with $H \setminus (D \times \{\omega\}) \subseteq W$. Since $H \setminus (D \times \{\omega\})$ contains all but finitely many elements of $U \setminus (D \times \{\omega\})$, a finite subset of $\mathcal{W}$ covers $U \setminus (D \times \{\omega\})$. It follows that $U \setminus (D \times \{\omega\})$ is a compact subspace of $Z(\kappa)$. It is easy to see that $U \setminus (D \times \{\omega\})$ does not contain only the points of $D \times \{\omega\}$ and the points of $\{\langle d_\alpha, m \rangle : \alpha < \alpha_0 \text{ and } m < m_0\}$ for some finite $\alpha_0 < \kappa$ and for some $m_0 \in \omega$. Then

$$K = (U \setminus (D \times \{\omega\})) \cup \{\langle d_\alpha, m \rangle : \alpha < \alpha_0 \text{ and } m < m_0\}$$

witnesses for $\mathcal{U}$ that $Z(\kappa)$ is $\mathcal{K}$-starcompact because K intersects every member of $\mathcal{U}$. □

Note that the $\mathcal{K}$-starcompact property is not preserved under regular-closed subsets (see [11, Example 3.1]). Again if we consider the example [11, Example 3.1] and go through the proof of it, then we can say that there exists a Tychonoff $1\frac{1}{2}$-starcompact space having a regular-closed subset which is not $1\frac{1}{2}$-starcompact. We now give a counterexample in the context of nearly $1\frac{1}{2}$-starcompact property. We first need the following result from [1].

Lemma 3.9. ([1, Corollary 11]) *If* $|\mathcal{A}| = \mathfrak{c}$, *then* $\Psi(\mathcal{A})$ *is not star-Menger (hence not nearly* $1\frac{1}{2}$*-starcompact).*

Example 3.10. There exists a Tychonoff pseudocompact nearly $1\frac{1}{2}$-starcompact space having a regular-closed subset which is not nearly $1\frac{1}{2}$-starcompact.

Proof. Consider $X = \Psi(\mathcal{A})$. Suppose that $\mathcal{A}$ is a maximal almost disjoint family with $|\mathcal{A}| = \mathfrak{c}$. Then X is a Tychonoff pseudocompact space which is not nearly $1\frac{1}{2}$-starcompact by Lemma 3.9. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be the discrete space with cardinality $\mathfrak{c}$ and $aD = D \cup \{d\}$ be the one point compactification of D. Choose $Y = (aD \times [0, \mathfrak{c}^+]) \setminus \{\langle d, \mathfrak{c}^+\rangle\}$ as a subspace of $aD \times [0, \mathfrak{c}^+]$. Then Y is a Tychonoff pseudocompact nearly $1\frac{1}{2}$-starcompact space (see [12, Example 2.2]). Let $f : \mathcal{A} \to D \times \{\mathfrak{c}^+\}$ be a bijection and Z be the quotient image of the topological sum $X \oplus Y$ obtained by identifying A of X with $f(A)$ of Y for every $A \in \mathcal{A}$. Let $q : X \oplus Y \to Z$ be the quotient map. It is clear that $q(X)$ is a regular-closed subset of Z, and $q(X)$ is not nearly $1\frac{1}{2}$-starcompact as $q(X)$ is homeomorphic to X. Again by [12, Example 2.2], Z is a Tychonoff pseudocompact nearly $1\frac{1}{2}$-starcompact space. □

In the following result, we observe that, like the $\mathcal{K}$-starcompact and $1\frac{1}{2}$-starcompact property, the nearly $1\frac{1}{2}$-starcompact property is also preserved under clopen subsets.

Theorem 3.11. *A clopen subset of a nearly* $1\frac{1}{2}$*-starcompact space is nearly* $1\frac{1}{2}$*-starcompact.*

Proof. Let Y be a clopen subset of a nearly $1\frac{1}{2}$-starcompact space X. Let $\mathcal{U}$ be an open cover of Y. Then $\mathcal{W} = \mathcal{U} \cup \{X \setminus Y\}$ is an open cover of X. Applying the nearly $1\frac{1}{2}$-starcompact property of X we obtain a countable subset A of X and a finite subset $\mathcal{H}$ of $\mathcal{W}$ such that $X \setminus A \subseteq St(\cup\mathcal{H}, \mathcal{W})$. Choose $B = A \cap Y$ and $\mathcal{V} = \{U \in \mathcal{U} : U \in \mathcal{H}\}$. We now show that B and $\mathcal{V}$ guarantee for $\mathcal{U}$ that Y is nearly $1\frac{1}{2}$-starcompact. Let $x \in Y \setminus B$. This gives us $x \in X \setminus A$ and hence there exist $U \in \mathcal{W}$ and $V \in \mathcal{H}$ such that $x \in U$ and $U \cap V \neq \emptyset$. It is easy to observe that $U \neq X \setminus Y$ i.e. $U \in \mathcal{U}$ and so $V \in \mathcal{V}$. Thus we can say that $x \in St(\cup\mathcal{V}, \mathcal{U})$ and hence $Y \setminus B \subseteq St(\cup\mathcal{V}, \mathcal{U})$. This completes the proof. □

Next, we turn our attention to the Alexandroff duplicate of the spaces considered here. We first recall that the Alexandroff duplicate $AD(X)$ of a space X (see [3, 5]) is defined as follows. $AD(X) = X \times \{0, 1\}$; each point of $X \times \{1\}$ is isolated and a basic neighbourhood of $\langle x, 0\rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighbourhood of x in X.

It is to be noted that if X is a $\mathcal{K}$-starcompact (resp. $1\frac{1}{2}$-starcompact) space, then $AD(X)$ may not be $\mathcal{K}$-starcompact (resp. $1\frac{1}{2}$-starcompact). The same is true for nearly $1\frac{1}{2}$-starcompact spaces, the reason is as follows. Consider X as in [11, Example 3.1]. Then X is a Tychonoff nearly $1\frac{1}{2}$-starcompact space. Choose $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ and $A = \{\langle\langle d_\alpha, \mathfrak{c}^+\rangle, 1\rangle : \alpha < \mathfrak{c}\}$. Clearly, A is a discrete clopen subset of $AD(X)$ with $|A| = \mathfrak{c}$. By Theorem 3.11, $AD(X)$ is not nearly $1\frac{1}{2}$-starcompact. However we obtain the following result.

Theorem 3.12. *For a space X the following assertions hold.*

(1) *If* $AD(X)$ *is a nearly* $1\frac{1}{2}$*-starcompact space, then X is nearly* $1\frac{1}{2}$*-starcompact.*

(2) *If* $AD(X)$ *is a* $1\frac{1}{2}$*-starcompact space, then X is* $1\frac{1}{2}$*-starcompact.*

(3) *If* $AD(X)$ *is a* $\mathcal{K}$*-starcompact space, then X is* $\mathcal{K}$*-starcompact.*

Proof. (1) To show that X is nearly $1\frac{1}{2}$-starcompact we pick an open cover $\mathcal{U}$ of X. Then $\mathcal{W} = \{U \times \{0, 1\} : U \in \mathcal{U}\}$ is an open cover of $AD(X)$. Since $AD(X)$ is nearly $1\frac{1}{2}$-starcompact, there exists a finite $\mathcal{H} \subseteq \mathcal{W}$ and a countable $A \subseteq AD(X)$ such that $AD(X) \setminus A \subseteq St(\cup\mathcal{H}, \mathcal{W})$. Choose $\mathcal{V} = \{U \in \mathcal{U} : U \times \{0, 1\} \in \mathcal{H}\}$ and a countable $C \subseteq X$ such that $A \subseteq C \times \{0, 1\}$. Clearly, $\mathcal{V}$ and C witness for $\mathcal{U}$ that X is nearly $1\frac{1}{2}$-starcompact.

(3) Let $\mathcal{U}$ be an open cover of X. Then $\mathcal{W} = \{U \times \{0, 1\} : U \in \mathcal{U}\}$ is an open cover of $AD(X)$. Apply the $\mathcal{K}$-starcompact property of $AD(X)$ to $\mathcal{W}$ to obtain a compact subset K of $AD(X)$ such that $St(K, \mathcal{W}) = AD(X)$. Put $F = \{x \in X : \text{either } \langle x, 0\rangle \in K \text{ or } \langle x, 1\rangle \in K\}$. One can readily observe that F is a compact subset of X. Then F guarantees for $\mathcal{U}$ that X is $\mathcal{K}$-starcompact. □

Theorem 3.13. ([13, Theorem 2.5]) *If X is a T_1 star-Menger space with $e(X) < \omega_1$, then $AD(X)$ is star-Menger.*

Using Theorem 3.4 we obtain the following corollary.

Corollary 3.14. *If X is a T_1 nearly $1\frac{1}{2}$-starcompact space with $e(X) < \omega_1$, then $AD(X)$ is star-Menger.*

Theorem 3.15. ([13, Theorem 2.5]) *If X is a T_1 space and $AD(X)$ is a star-Menger space, then $e(X) < \omega_1$.*

Corollary 3.16. *If X is a T_1 space and $AD(X)$ is a nearly $1\frac{1}{2}$-starcompact space, then $e(X) < \omega_1$.*

3.2. Few more properties of nearly $1\frac{1}{2}$-starcompact spaces

In this section we discuss certain preservation likes properties of the nearly $1\frac{1}{2}$-starcompact property. We start with the following basic observation.

Theorem 3.17. *The union of finitely many nearly $1\frac{1}{2}$-starcompact spaces is also nearly $1\frac{1}{2}$-starcompact.*

Proof. Let $\{X_k : 1 \leq k \leq n\}$ be a finite family of nearly $1\frac{1}{2}$-starcompact spaces and $X = \cup_{k=1}^{n} X_k$. To show that X is nearly $1\frac{1}{2}$-starcompact we choose an open cover $\mathcal{U}$ of X. Since for each $1 \leq k \leq n$, X_k is nearly $1\frac{1}{2}$-starcompact, we get two finite families $\{A_k : 1 \leq k \leq n\}$ and $\{\mathcal{V}_k : 1 \leq k \leq n\}$ such that A_k is a countable subset of X_k and $\mathcal{V}_k$ is a finite subset of $\mathcal{U}$ and $X_k \setminus A_k \subseteq St(\cup\mathcal{V}_k, \mathcal{U})$. Choose $A = \cup_{k=1}^{n} A_k$ and $\mathcal{V} = \cup_{k=1}^{n} \mathcal{V}_k$. We claim that A and $\mathcal{V}$ witnessed for $\mathcal{U}$ that X is nearly $1\frac{1}{2}$-starcompact. Let $x \in X \setminus A$. Then $x \in X_{k_0}$ for some $1 \leq k_0 \leq n$ and $x \notin A_k$ for each $1 \leq k \leq n$. It follows that $x \in X_{k_0} \setminus A_{k_0}$ and hence $x \in St(\cup\mathcal{V}_{k_0}, \mathcal{U})$ i.e. $x \in St(\cup\mathcal{V}, \mathcal{U})$. Thus $X \setminus A \subseteq St(\cup\mathcal{V}, \mathcal{U})$. □

The above result need not be true if we consider countably many nearly $1\frac{1}{2}$-starcompact spaces instead of finitely many nearly $1\frac{1}{2}$-starcompact spaces. Indeed, the set of all reals $\mathbb{R}$ is not nearly $1\frac{1}{2}$-starcompact and $\mathbb{R} = \cup_{n\in\omega}[-n, n]$ with each $[-n, n]$ is nearly $1\frac{1}{2}$-starcompact.

Theorem 3.18. *Any continuous image of a nearly $1\frac{1}{2}$-starcompact space is nearly $1\frac{1}{2}$-starcompact.*

Proof. Let X be a nearly $1\frac{1}{2}$-starcompact space and $f : X \to Y$ be a continuous mapping from X onto Y. We pick an open cover $\mathcal{U}$ of Y to prove that Y is nearly $1\frac{1}{2}$-starcompact. Choose $\mathcal{W} = \{f^{-1}(U) : U \in \mathcal{U}\}$. Then $\mathcal{W}$ is an open cover of X and since X is nearly $1\frac{1}{2}$-starcompact, we get a countable subset A of X and a finite subset $\mathcal{H}$ of $\mathcal{W}$ such that $X \setminus A \subseteq St(\cup\mathcal{H}, \mathcal{W})$. Let $\mathcal{V} = \{U \in \mathcal{U} : f^{-1}(U) \in \mathcal{H}\}$ and $B = f(A)$. We claim that B and $\mathcal{V}$ witnessed for $\mathcal{U}$ that Y is nearly $1\frac{1}{2}$-starcompact. Let $y \in Y \setminus B$. Then there exists a $x \in X$ with $x \notin A$ such that $y = f(x)$. It follows that $x \in X \setminus A$ and so $x \in f^{-1}(U)$ for some $f^{-1}(U) \in \mathcal{W}$ satisfying $f^{-1}(U) \cap f^{-1}(V) \neq \emptyset$ for some $f^{-1}(V) \in \mathcal{H}$. Thus $y \in St(\cup\mathcal{V}, \mathcal{U})$ and consequently $Y \setminus B \subseteq St(\cup\mathcal{V}, \mathcal{U})$. Hence the result. □

The $\mathcal{K}$-starcompact property is an inverse invariant of open perfect continuous mappings (see [11, Theorem 3.2]) and hence the product of a $\mathcal{K}$-starcompact space and a compact space is $\mathcal{K}$-starcompact. A similar characterization for the $1\frac{1}{2}$-starcompact property has been observed in the following result.

Theorem 3.19 (Folklore). *If $f : X \to Y$ is an open perfect continuous mapping from a space X onto a $1\frac{1}{2}$-starcompact space Y, then X is also $1\frac{1}{2}$-starcompact.*

Proof. Let $\mathcal{U}$ be an open cover of X and $y \in Y$. Since $f^{-1}(y)$ is compact, there exists a finite subset $\mathcal{V}_y$ of $\mathcal{U}$ such that $f^{-1}(y) \subseteq \cup\mathcal{V}_y$ and $f^{-1}(y) \cap U \neq \emptyset$ for each $U \in \mathcal{V}_y$. Since f is closed, there exists an open set U_y in Y containing y such that $f^{-1}(U_y) \subseteq \cup\mathcal{V}_y$. Then by the openness of f, we can find an open set V_y in Y containing y such that $V_y \subseteq \cap\{f(U) : U \in \mathcal{V}_y\}$ and $f^{-1}(V_y) \subseteq f^{-1}(U_y)$. Thus we obtain an open cover $\mathcal{V} = \{V_y : y \in Y\}$ of Y. Since Y is $1\frac{1}{2}$-starcompact, we get a finite $\mathcal{H} \subseteq \mathcal{V}$ such that $St(\cup\mathcal{H}, \mathcal{V}) = Y$. Choose $\mathcal{H} = \{V_{y_i} : 1 \leq i \leq k\}$ and $\mathcal{W} = \cup_{1\leq i\leq k}\mathcal{V}_{y_i}$. Then $\mathcal{W}$ is a finite subset of $\mathcal{U}$. Since for each $1 \leq i \leq k$,

$f^{-1}(V_{y_i}) \subseteq \cup\mathcal{V}_{y_i}$, we have $f^{-1}(\cup\mathcal{H}) \subseteq \cup\mathcal{W}$. We claim that $\mathcal{W}$ guarantees for $\mathcal{U}$ that X is $1\frac{1}{2}$-starcompact. Let $x \in X$. Then $f(x) \in St(\cup\mathcal{H}, \mathcal{V})$ and subsequently we have a $V_{y_0} \in \mathcal{V}$ such that $f(x) \in V_{y_0}$ and $V_{y_0} \cap (\cup\mathcal{H}) \neq \emptyset$. Since $f^{-1}(U_{y_0}) \subseteq \cup\mathcal{V}_{y_0}$, we can find a $U_0 \in \mathcal{V}_{y_0}$ such that $x \in U_0$. On the other hand, $V_{y_0} \subseteq \cap\{f(U) : U \in \mathcal{V}_{y_0}\}$ gives $V_{y_0} \subseteq f(U_0)$. It follows that $U_0 \cap f^{-1}(\cup\mathcal{H}) \neq \emptyset$ i.e. $U_0 \cap (\cup\mathcal{W}) \neq \emptyset$ and hence $x \in St(\cup\mathcal{W}, \mathcal{U})$. Thus X is $1\frac{1}{2}$-starcompact. □

Corollary 3.20. *If X is a $1\frac{1}{2}$-starcompact space and Y is a compact space, then $X \times Y$ is $1\frac{1}{2}$-starcompact.*

Next, we observe that the nearly $1\frac{1}{2}$-starcompact property is not an inverse invariant of open perfect continuous mappings. Let $D_1 = \{a_\alpha : \alpha < \omega\}$ and $D_2 = \{d_\alpha : \alpha < \omega_1\}$ be two discrete spaces, and $aD_2 = D_2 \cup \{d\}$ be the one point compactification of D_2. Then D_1 is a nearly $1\frac{1}{2}$-starcompact space. Now the open cover $\{\{a_\alpha\} \times aD_2 : \alpha < \omega\}$ witnesses that $D_1 \times aD_2$ is not nearly $1\frac{1}{2}$-starcompact, but the projection mapping $p : D_1 \times aD_2 \to D_1$ is open perfect continuous. From this example, we can also conclude that the product of a nearly $1\frac{1}{2}$-starcompact and a compact space need not be nearly $1\frac{1}{2}$-starcompact. It follows that the product of two nearly $1\frac{1}{2}$-starcompact spaces need not be nearly $1\frac{1}{2}$-starcompact. The following well-known example (see [11, Example 3.3]) shows that the product of two countably compact (hence nearly $1\frac{1}{2}$-starcompact) spaces need not be nearly $1\frac{1}{2}$-starcompact. For the sake of completeness, we give a sketch of the proof.

Example 3.21. There exists two countably compact spaces X and Y such that $X \times Y$ is not nearly $1\frac{1}{2}$-starcompact.

Proof. Let D be the discrete space with cardinality $\mathfrak{c}$. Let $X = \cup_{\alpha<\omega_1} E_\alpha$ and $Y = \cup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are subsets of βD such that

(1) $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;

(2) $|E_\alpha| \leq \mathfrak{c}$ and $|F_\beta| \leq \mathfrak{c}$;

(3) every infinite subset of E_α (resp. F_α) has an accumulation point in $E_{\alpha+1}$ (resp. $F_{\alpha+1}$).

These sets E_α and F_α are well-defined as every infinite closed subset of βD has cardinality $2^{\mathfrak{c}}$ (see [16]). Then X and Y are countably compact. Since the diagonal $\{\langle d, d\rangle : d \in D\}$ is a discrete clopen subset of $X \times Y$ with cardinality $\mathfrak{c}$, by Theorem 3.11, $\{\langle d, d\rangle : d \in D\}$ is nearly $1\frac{1}{2}$-starcompact, which is absurd. Hence $X \times Y$ is not nearly $1\frac{1}{2}$-starcompact. □

In [15, Example 3.3.3], van Douwen et al. gave an example showing that there exists a countably compact (hence nearly $1\frac{1}{2}$-starcompact) space and a Lindelöf space Y such that $X \times Y$ is not star countable. In the next example, we observe that $X \times Y$ is not nearly $1\frac{1}{2}$-starcompact.

Example 3.22. There exist a countably compact (hence nearly $1\frac{1}{2}$-starcompact) space X and a Lindelöf space Y such that $X \times Y$ is not nearly $1\frac{1}{2}$-starcompact.

Proof. Let $X = [0, \omega_1)$ with the usual order topology. Then X is a countably compact space. Let us define a topology on $Y = \omega_1 + 1$ as follows: each point $\alpha < \omega_1$ is isolated and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Clearly, Y is Lindelöf. We claim that $X \times Y$ is not nearly $1\frac{1}{2}$-starcompact. Suppose that $X \times Y$ is nearly $1\frac{1}{2}$-starcompact. For each $\alpha < \omega_1$, let $U_\alpha = [0, \alpha] \times [\alpha, \omega_1]$ and $V_\alpha = (\alpha, \omega_1) \times \{\alpha\}$. It is immediate that $U_\alpha \cap V_\beta = \emptyset$ for any $\alpha, \beta < \omega_1$ and $V_\alpha \cap V_\beta = \emptyset$ if $\alpha \neq \beta$. Choose $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}$ and then $\mathcal{U}$ is an open cover of $X \times Y$. Applying the nearly $1\frac{1}{2}$-starcompact property of $X \times Y$ we get a countable subset C of $X \times Y$ and a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $(X \times Y) \setminus C \subseteq St(\cup\mathcal{V}, \mathcal{U})$. Since $\mathcal{V}$ is a finite subset of $\mathcal{U}$, there exists a finite $\alpha_0 < \omega_1$ such that $V_\alpha \notin \mathcal{V}$ for each $\alpha > \alpha_0$. It follows that $\langle \beta + 1, \beta\rangle \notin St(\cup\mathcal{V}, \mathcal{U})$ for each $\beta > \alpha_0$ as V_β is the only member of $\mathcal{U}$ containing the point $\langle \beta + 1, \beta\rangle$ and $V_\beta \cap \mathcal{V} = \emptyset$. Also since C is a countable subset of $X \times Y$, there exists a $\beta_0 \in (\alpha_0, \omega_1)$ such that $\langle \beta_0 + 1, \beta_0\rangle \in (X \times Y) \setminus C$ and $\langle \beta_0 + 1, \beta_0\rangle \notin St(\cup\mathcal{V}, \mathcal{U})$, which is a contradiction. Thus $X \times Y$ is not nearly $1\frac{1}{2}$-starcompact. □

We end this section with the following theorem in which we observe that the nearly $1\frac{1}{2}$-starcompact property is an inverse invariant of certain mappings. We say that a mapping $f : X \to Y$ is countable-to-one if for each $y \in Y$, $f^{-1}(y)$ is countable.

Theorem 3.23. *If f is an open, closed, and countable-to-one continuous mapping from a space X onto a nearly $1\frac{1}{2}$-starcompact space Y, then X is nearly $1\frac{1}{2}$-starcompact.*

Proof. To show that X is nearly $1\frac{1}{2}$-starcompact we pick an open cover $\mathcal{U}$ of X. Let $y \in Y$. Then we get a finite subset $\mathcal{V}_y$ of $\mathcal{U}$ such that $f^{-1}(y) \subseteq \cup\mathcal{V}_y$ and $f^{-1}(y) \cap U \neq \emptyset$ for each $U \in \mathcal{V}_y$ because $f^{-1}(y)$ is compact. We can obtain an open set U_y in Y containing y such that $f^{-1}(U_y) \subseteq \cup\mathcal{V}_y$ since f is closed. Again since f is open, there is an open set V_y in Y containing y such that $V_y \subseteq \cap\{f(U) : U \in \mathcal{V}_y\}$ and $f^{-1}(V_y) \subseteq f^{-1}(U_y)$. Thus we obtain an open cover $\mathcal{V} = \{V_y : y \in Y\}$ of Y. Applying the nearly $1\frac{1}{2}$-starcompact property of Y we get a finite $\mathcal{H} \subseteq \mathcal{V}$ and a countable $B \subseteq Y$ such that $Y \setminus B \subseteq St(\cup\mathcal{H}, \mathcal{V})$. Let $\mathcal{H} = \{V_{y_i} : 1 \leq i \leq k\}$, $\mathcal{W} = \cup_{1\leq i\leq k}\mathcal{V}_{y_i}$ and $A = f^{-1}(B)$. It follows that $\mathcal{W}$ is a finite subset of $\mathcal{U}$ and since f is countable-to-one, A is a countable subset of X. Proceeding similarly as in the proof of Theorem 3.19, one can readily observe that $\mathcal{W}$ and A witness for $\mathcal{U}$ that X is nearly $1\frac{1}{2}$-starcompact. □

4. Concluding remarks and open problems

In this article, we only consider the generalization (nearly $1\frac{1}{2}$-starcompact property) of the $1\frac{1}{2}$-starcompact property. Similar types of investigations on the generalizations (nearly starcompact and nearly $\mathcal{K}$-starcompact property) of the starcompact and $\mathcal{K}$-starcompact property can be carried out. Next, we give definitions of these generalized properties.

Definition 4.1. A space X is said to be

(1) nearly starcompact if for every open cover $\mathcal{U}$ of X there exists a countable $A \subseteq X$ and a finite $F \subseteq X$ such that $X \setminus A \subseteq St(F, \mathcal{U})$.

(2) nearly $\mathcal{K}$-starcompact if for every open cover $\mathcal{U}$ of X there exists a countable $A \subseteq X$ and a compact $K \subseteq X$ such that $X \setminus A \subseteq St(K, \mathcal{U})$.

In addition to Problem 1.1, we are not able to find answers to the following problems during the preparation of this article.

Problem 4.2. *Find conditions under which the $1\frac{1}{2}$-starcompact and nearly $1\frac{1}{2}$-starcompact property are equivalent.*

Problem 4.3. *Find conditions under which the nearly $1\frac{1}{2}$-starcompact and star-Menger property are equivalent.*

Problem 4.4. *Is the space $AD(X)$ of a nearly $1\frac{1}{2}$-starcompact space X with $e(X) < \omega_1$ also nearly $1\frac{1}{2}$-starcompact?*

Acknowledgement

The author thanks the referee(s) for making numerous useful comments that considerably enhanced the paper's presentation.

References

[1] M. Bonanzinga, M. Matveev, Some covering properties for Ψ-spaces, Mat. Vesnik 61 (2009) 3–11.
[2] D.K. Burke, Covering properties, in: K. Kunen, J.E. Vaughan (Eds.), Handbook of Set-Theoretic Topology, Elsevier Science Publishers B.V., 1984, pp. 347–422.
[3] A. Caserta, S. Watson, The Alexandroff duplicate and its subspaces, Appl. Gen. Topol. 8 (2007) 187–205.
[4] D. Chandra, N. Alam, Some remarks on star-Menger spaces using box products, Filomat 36 (2022) 1769–1774.
[5] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[6] Lj.D.R. Kočinac, Star-Menger and related spaces, Publ. Math. Debrecen 55 (1999) 421–431.

[7] Lj.D.R. Kočinac, Star selection principles: A survey, Khayyam J. Math. 1 (2015) 82–106.
[8] M.V. Matveev, How weak is weak extent?, Topology Appl. 119 (2002) 229–232.
[9] S. Mrówka, On completely regular spaces, Fund. Math. 41 (1954) 105–106.
[10] M. Sakai, Star versions of the Menger property, Topology Appl. 176 (2014) 22–34.
[11] Y.-K. Song, On $\mathcal{K}$-starcompact spaces, Bull. Malays. Math. Sci. Soc. 30(1) (2007) 59–64.
[12] Y.-K. Song, R. Li, A note on star-Hurewicz spaces, Filomat 27(6) (2013) 1091–1095.
[13] Y.-K. Song, Remarks on star-Menger spaces II, Houston J. Math. 41 (2015) 357–366.
[14] Y.-K. Song, On star-K-Hurewicz spaces, Filomat 31 (2017) 1279–1285.
[15] E.K. van Douwen, G.M. Reed, A.W. Roscoe, I.J. Tree, Star covering properties, Topology Appl. 39 (1991) 71–103.
[16] R.C. Walker, The Stone-Čech compactification, Springer, New York-Berlin, 1974.