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A novel surface fuzzy analytic hierarchy process

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Abstract. For almost fifty years, the concept of fuzzy analytic hierarchy process based on triangular fuzzy numbers, supplied with various types of membership functions has experienced a wide variety of Intuitionistic fuzzy numbers, z-Numbers, and Spherical fuzzy numbers. This paper aims to upgrade the concept of Spherical Fuzzy Sets considering the hesitancy, its third coordinate, as a function instead of constant. This broads the decision-makers "spectrum" allowing them to describe the fuzzy part of data more comprehensively. As a path distance between two fuzzy sets, we used geodesic lines in place of great circles, connecting Fuzzy logic with Differential geometry. Also, we proved some algebraic properties concerning associativity and neutral elements for operations $\oplus$ and $\otimes$, for the defined Surface Fuzzy Sets. Additionally, the application of Surface Fuzzy Sets to within AHP is used for ranking Security, Privacy, and Authority sub-criteria affecting B2C e-commerce websites.

1. Introduction

Differential geometry has been applied in many scientific disciplines. The purpose of this paper is to apply differential geometry in the theory of Fuzzy Logic.

1.1. Previous research

Often, when making decisions, both on a personal and professional level, people are prevented from choosing the best of all the options offered due to insecurity or inaccurate information. Also, due to partial data on the observed occurrence, it is often impossible to properly determine the importance of one element over another. Zadeh [29], introducing the theory of fuzzy sets (type-1 fuzzy sets), defined useful tool for representation of uncertain and imprecise information applying mathematical tool. Using membership function, mapping universal set into interval $[0, 1]$, $\mu : \mathbb{R} \to [0, 1]$, an element from the universal set belongs to a defined set with some probability. Type-2 fuzzy sets [30] were introduced to ease determination of membership function since its membership function's grade is a type-1 fuzzy set. Combining both the membership and non-membership functions (μ and ν), Atanasov [1] introduced Intuitionistic Fuzzy Sets

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(IFS) and Intuitionistic Fuzzy Sets of second type [2]. To expand the decision makers range of possibilities, Yager [28] presented the condition $\mu^2+\nu^2 \in [0,1]$. Pythagorean numbers are defined in this way. Introducing the Neutrosophic fuzzy sets, Smarandache [23, 24] reduced the hesitancy of inconsistent information. About the generalization of Intuitionistic Fuzzy Sets based on Boolean Algebra and MARCOS technique may be read in [4, 16]. Some of the newest generalizations, comparisons of the approaches to the ranking and applications of Neutrosophic Fuzzy Sets in the e-commerce development are studied in [10, 26]. The recent applications of Fuzzy Sets Theory in machine learning regarding risk assessment of excavation system with the use of Pythagorean and Spherical Fuzzy TOPSIS and Artificial neural network, as well as operations on interpolative Boolean algebra with preserving idea of intuitionism and developing improved Einstein aggregation operators are presented in [12, 17, 25] and in many other articles.

Since its creation by Mahmood et al. [13] and Gündogdu and Kahraman [6], as a compound of Neutrosophic and Pythagorean fuzzy sets, Spherical fuzzy sets, being a part of Spherical AHP, VIKOR, TOPSIS, MULTIMOORA or WASPAS method, have been applied in many areas. Some of them are warehouse site selection [7], renewable energy application [8], manufacturing system selection [14], supply chain assessment and supplier selection [27], distribution center location selection [11], medical diagnostic [13], governmental strategies against the COVID-19 pandemic [19], and many other.

Gong et al. [5] measure the arc distances on the sphere, not using the Euclidean distance, and in [6] Gündogdu and Kahraman deal with a solid spherical volume, not a sphere, obtaining that for any two points on the spherical volume exists a sphere which they belong to. In this case, using a large circle of a sphere, Euclidean distance can be equal to a distance between spherical fuzzy sets.

1.2. *Definition and basic properties of spherical fuzzy sets*

A spherical fuzzy set $\tilde{S}$ of the non-empty universe of discourse $\mathcal{X}$ is defined as:

$$\tilde{S} = \Big\{\Big(\mu_{\tilde{S}}(x), \nu_{\tilde{S}}(x), \pi_{\tilde{S}}(x)\Big) | x \in \mathcal{X}\Big\}, \tag{1}$$

where $\mu_{\tilde{S}}(x) : \mathcal{X} \to [0,1]$, $\nu_{\tilde{S}}(x) : \mathcal{X} \to [0,1]$ and $\pi_{\tilde{S}}(x) : \mathcal{X} \to [0,1]$ are the membership, non-membership, and hesitancy functions, satisfying the condition

$$0 \leq \mu_{\tilde{S}}^2 + \nu_{\tilde{S}}^2 + \pi_{\tilde{S}}^2 \leq 1. \tag{2}$$

For two spherical fuzzy sets $\tilde{S}_1 = \Big(\mu_{\tilde{S}_1}, \nu_{\tilde{S}_1}, \pi_{\tilde{S}_1}\Big)$ and $\tilde{S}_2 = \Big(\mu_{\tilde{S}_2}, \nu_{\tilde{S}_2}, \pi_{\tilde{S}_2}\Big)$ defined on the non-empty universal set $\mathcal{X}$, the basic operations (intersection, union, addition, multiplication, multiplication by a scalar k, power) are defined as follows:

$$\begin{aligned}\tilde{S}_1 \cap \tilde{S}_2 =&\Big\{ \min\Big\{\mu_{\tilde{S}_1}, \mu_{\tilde{S}_2}\Big\}, \max\Big\{\nu_{\tilde{S}_1}, \nu_{\tilde{S}_2}\Big\},\\ &\max\Big\{ \sqrt{1-(\min\{\mu_{\tilde{S}_1}, \mu_{\tilde{S}_2}\})^2-(\max\{\nu_{\tilde{S}_1}, \nu_{\tilde{S}_2}\})^2}, \min\{\pi_{\tilde{S}_1}\pi_{\tilde{S}_2}\}\Big\}\Big\},\end{aligned} \tag{3}$$

$$\begin{aligned}\tilde{S}_1 \cup \tilde{S}_2 =&\Big\{ \max\Big\{\mu_{\tilde{S}_1}, \mu_{\tilde{S}_2}\Big\}, \min\Big\{\nu_{\tilde{S}_1}, \nu_{\tilde{S}_2}\Big\},\\ &\min\{ \sqrt{1-(\max\{\mu_{\tilde{S}_1}, \mu_{\tilde{S}_2}\})^2-(\min\{\nu_{\tilde{S}_1}, \nu_{\tilde{S}_2}\})^2}, \max\{\pi_{\tilde{S}_1}\pi_{\tilde{S}_2}\}\}\Big\},\end{aligned} \tag{4}$$

$$\begin{aligned}\tilde{S}_1 \oplus \tilde{S}_2 =&\Big\{ \sqrt{\mu_{\tilde{S}_1}^2+\mu_{\tilde{S}_2}^2-\mu_{\tilde{S}_1}^2\mu_{\tilde{S}_2}^2}, \nu_{\tilde{S}_1}\nu_{\tilde{S}_2},\\ &\sqrt{\pi_{\tilde{S}_1}^2(1-\mu_{\tilde{S}_2}^2)+\pi_{\tilde{S}_2}^2(1-\nu_{\tilde{S}_1}^2)-\pi_{\tilde{S}_1}^2\pi_{\tilde{S}_2}^2}\Big\},\end{aligned} \tag{5}$$

$$\begin{aligned}\tilde{S}_1 \odot \tilde{S}_2 =&\Big\{\mu_{\tilde{S}_1}\mu_{\tilde{S}_2}, \sqrt{\nu_{\tilde{S}_1}^2+\nu_{\tilde{S}_2}^2-\nu_{\tilde{S}_1}^2\nu_{\tilde{S}_2}^2},\\ &\sqrt{\pi_{\tilde{S}_1}^2(1-\nu_{\tilde{S}_2}^2)+\pi_{\tilde{S}_2}^2(1-\mu_{\tilde{S}_1}^2)-\pi_{\tilde{S}_1}^2\pi_{\tilde{S}_2}^2}\Big\},\end{aligned} \tag{6}$$

$$k \times \tilde{S} = \left\{ \sqrt{1 - \left(1 - \mu_{\tilde{S}}^2\right)^k}, \nu_{\tilde{S}}^k, \sqrt{\left(1 - \mu_{\tilde{S}}^2\right)^k - (1 - \mu_{\tilde{S}}^2 - \pi_{\tilde{S}}^2)^k} \right\}, \tag{7}$$

$$\tilde{S}^k = \left\{ \mu_{\tilde{S}}^k, \sqrt{1 - \left(1 - \nu_{\tilde{S}}^2\right)^k}, \sqrt{\left(1 - \nu_{\tilde{S}}^2\right)^k - \left(1 - \nu_{\tilde{S}}^2 - \pi_{\tilde{S}}^2\right)^k} \right\}. \tag{8}$$

For spherical fuzzy sets $\tilde{S}_1 = (\mu_{\tilde{S}_1}, \nu_{\tilde{S}_1}, \pi_{\tilde{S}_1})$ and $\tilde{S}_2 = (\mu_{\tilde{S}_2}, \nu_{\tilde{S}_2}, \pi_{\tilde{S}_2})$, and scalars k, k_1 and $k_2 \geq 0$, the commutative, distributive and power laws of operations $\oplus$, $\otimes$ are proved in [6].

1.3. *More definitions related to spherical fuzzy sets*

Suppose that $\tilde{S}_1$ and $\tilde{S}_2$ be two spherical fuzzy sets. Then to compare these SFSs, the score function and accuracy function are defined as follows:

$$s(\tilde{S}_1) = \frac{\mu_{\tilde{S}_1} + 2\left(1 - \nu_{\tilde{S}_1}\right) - \pi_{\tilde{S}_1}}{3}, \tag{9}$$

$$a(\tilde{S}_1) = \mu_{\tilde{S}_1}^2 + \nu_{\tilde{S}_1}^2 + \pi_{\tilde{S}_1}^2. \tag{10}$$

For a spherical fuzzy set $\tilde{S} = (\mu_{\tilde{S}}, \nu_{\tilde{S}}, \pi_{\tilde{S}})$, the Score Indices (SI) are

$$SI = \begin{cases} 10\sqrt{\left|\mu_{\tilde{S}}^2 - 2\pi_{\tilde{S}}(\mu_{\tilde{S}} - \nu_{\tilde{S}}) - \nu_{\tilde{S}}^2\right|}, & \text{for E, AS, ES, VS, FS,} \\ \dfrac{1}{10\sqrt{\left|\mu_{\tilde{S}}^2 - 2\pi_{\tilde{S}}(\mu_{\tilde{S}} - \nu_{\tilde{S}}) - \nu_{\tilde{S}}^2\right|}}, & \text{for AW, EW, VW, FW,} \end{cases} \tag{11}$$

where E, AS, ES, VS, FS, AW, EW, VW, FW, are explained in Table 1 (see [8, 9, 20]).

SI	LM	Meaning of LM	SFNs
9	AS	Absolutely strong dominance	$(0.9, 0.1, 0.0)$
7	ES	Extremely strong dominance	$(0.8, 0.2, 0.1)$
5	VS	Very strong dominance	$(0.7, 0.3, 0.2)$
3	FS	Fairly strong dominance	$(0.6, 0.4, 0.3)$
1	E	Equal importance	$(0.5, 0.4, 0.4)$
1/3	FW	Fairly weak dominance	$(0.4, 0.6, 0.3)$
1/5	VW	Very weak dominance	$(0.3, 0.7, 0.2)$
1/7	EW	Extremely weak dominance	$(0.2, 0.8, 0.1)$
1/9	AW	Absolutely weak dominance	$(0.1, 0.9, 0.0)$

Table 1: Score indices, Linguistic measures of criteria and SFNs

Spherical Fuzzy Weighted Arithmetic Mean ($SFWAM$) with respect to $k = (k_1, k_2, \ldots, k_n)$; $k_i \in [0, 1]$; $\sum_{i=1}^n k_i = 1$, the value of $SFWAM$ is

$$\begin{aligned} SFWAM_w(\tilde{A}_{S1}, \tilde{A}_{S2}, \ldots, \tilde{A}_{Sn}) &= k_1\tilde{A}_{S1} + k_2\tilde{A}_{S2} + \ldots + k_n\tilde{A}_{Sn} \\ &= \left\{ \sqrt{1 - \prod_{i=1}^n (1 - \mu_{\tilde{A}_S}^2)^{k_i}}, \prod_{i=1}^n \nu_{A_s}^{k_i}, \sqrt{\prod_{i=1}^n (1 - \mu_{\tilde{A}_S}^2)^{k_i} - \prod_{i=1}^n (1 - \mu_{\tilde{A}_S}^2 - \pi_{\tilde{A}_s}^2)^{k_i}} \right\}. \end{aligned} \tag{12}$$

Spherical Fuzzy Weighted Geometric Mean ($SFWGM$) with respect to $k = (k_1, k_2, \ldots, k_n)$; $k_i \in [0, 1]$; $\sum_{i=1}^n k_i = 1$, the value of $SFWGM$ is

$$\begin{aligned} SFWGM_w(\tilde{A}_{S1}, \tilde{A}_{S2}, \ldots, \tilde{A}_{Sn}) &= \tilde{A}_{S1}^{k_1} + \tilde{A}_{S2}^{k_2} + \ldots + \tilde{A}_{Sn}^{k_n} \\ &= \left\{ \prod_{i=1}^n \mu_{A_s}^{k_i}, \sqrt{1 - \prod_{i=1}^n (1 - \nu_{\tilde{A}_S}^2)^{k_i}}, \sqrt{\prod_{i=1}^n (1 - \nu_{\tilde{A}_S}^2)^{k_i} - \prod_{i=1}^n (1 - \nu_{\tilde{A}_S}^2 - \pi_{\tilde{A}_s}^2)^{k_i}} \right\}. \end{aligned} \tag{13}$$

1.4. Motivation

If we rotate the curve $\ell : x = \varphi(u) > 0, y = 0, z = \psi(u)$ around the Oz-axis, the equation of resulting surface is [18]

$$r = r(u, v) = \Big(\varphi(u)\cos v, \varphi(u)\sin v, \psi(u)\Big), \tag{14}$$

$u \in [a, b]$, $v \in [0, 2\pi)$.

The volume of this surface is

$$V = \pi \int_{u_1}^{u_2} \varphi^2(u) \frac{\psi(u)}{du} du. \tag{15}$$

The partial derivatives of surface r by u and v are $r_u = \partial r/\partial u$ and $r_v = \partial r/\partial v$. The coefficients of first square form of the surface r are

$$g_{\underline{11}} = E = r_u \cdot r_u, \quad g_{\underline{12}} = g_{\underline{21}} = F = r_u \cdot r_v, \quad g_{\underline{22}} = G = r_v \cdot r_v. \tag{16}$$

If the points $A = r(u_1, v_1)$ and $B = r(u_2, v_2)$ are two different points, the shortest line which connects these points on the surface r is the geodesic line of this surface [15]. The system of differential equations which generates the geodesic line $\gamma = \gamma(t) = (\gamma^1, \gamma^2, \gamma^3)$ of the surface r is [15]

$$\frac{d^2\gamma^i}{dt^2} + \frac{1}{2} g^{\underline{ip}} \Big(g_{\underline{jp},k} - g_{\underline{jk},p} + g_{\underline{pk},j}\Big) \frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt} = \rho\gamma^i, \tag{17}$$

for $x = x^1, y = x^2, z = x^3$ and partial derivative $\partial g_{\underline{ij}}/\partial x^k$ signed as $g_{\underline{ij},k}$, $i, j, k = 1, 2, 3$. In the previous equation, we used the Einstein Summation Convention, $g^{\underline{ip}} g_{\underline{jk},p} = \sum_{a=1}^{3} g^{\underline{ia}} g_{\underline{jk},a}$, and analogously for $g^{\underline{ip}} g_{\underline{jp},k}$ and $g^{\underline{ip}} g_{\underline{pk},j}$. In this equation, we used the structure $g^{\underline{ij}}$ determined by $\Big[g^{\underline{ij}}\Big] = \Big[g_{\underline{ij}}\Big]^{-1}$.

Geodesic lines on a sphere are great circles of this sphere. That fact motivated us to write this paper.

In [3, 21], authors defined different kinds of fuzzy sets and examined algebraic properties (associativity, commutativity, distributivity, neutral, and inverse elements) of operations specified on them.

The main aims of this research are

1. To generalize concept of Spherical Fuzzy Sets to the concept of Surface Fuzzy Sets.
2. To examine associativity, neutral and inverse Surface Fuzzy Sets for operations $\oplus$ and $\otimes$ and distributivity of these operations to each other.
3. To analyze the component π as a function, not as a concrete number.

2. Necessary mathematics

Let us consider the plane curve $\ell = \ell(u)$ and the surface

$$r = r(u, v) = \Big(\ell(u)\cos v, \ell(u)\sin v, h(v)\Big), \tag{18}$$

for $\ell : \mathcal{U} \to [0, 1]$, $h : \mathcal{V} \to [0, 1]$, $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}$.

The partial derivatives of surface r, $r_u = \partial r/\partial u$ and $r_v = \partial r/\partial v$, are

$$\begin{cases} r_u = \Big(\ell_u(u)\cos v, \ell_u(u)\sin v, 0\Big), \\ r_v = \Big(-\ell(u)\sin v, \ell(u)\cos v, h_v(v)\Big), \end{cases} \tag{19}$$

for $h_v(v) = d\Big\{h(v)\Big\}/dv$.

The normal vector N of the surface r is

$$N = r_u \times r_v = \Big(\ell_u(u)h_v(v)\sin v, -\ell_u(u)h_v(v)\cos v, \ell(u)\ell_u(u)\Big). \tag{20}$$

The normal vector N at the point $A = r(u_0, v_0)$ is

$$N_0 = \Big(\ell_u(u_0)h_v(v_0)\sin v_0, -\ell_u(u_0)h_v(v_0)\cos v_0, \ell(u_0)\ell_u(u_0)\Big). \tag{21}$$

The normal of surface r at the point $r(u_0, v_0)$ is

$$(n) : \frac{x - \ell(u_0)\cos v_0}{\ell_u(u_0)h_v(v_0)\sin v_0} = \frac{y - \ell(u_0)\sin v_0}{-\ell_u(u_0)h_v(v_0)\cos v_0} = \frac{z - h(v_0)}{\ell(u_0)\ell_u(u_0)} = t. \tag{22}$$

Parametrically, the normal (n) is expressed as

$$(n) : \begin{cases} x = t\ell_u(u_0)h_v(v_0)\sin v_0 + \ell(u_0)\cos v_0 \\ y = -t\ell_u(u_0)h_v(v_0)\cos v_0 + \ell(u_0)\sin v_0 \\ z = t\ell(u_0)\ell_u(u_0) + h(v_0). \end{cases} \tag{23}$$

Let us consider the point $B = r(u_1, v_1)$. The equation of plane which contains the normal n and the point B is determined by the corresponding normal vector $d = (d_1, d_2, d_3) = N_0 \times \vec{AB}$ for

$$d_1 = \ell(u_0)\Big[\Big(h(v_0) - h(v_1)\Big)h_v(v_0)\cos v_0 + \ell(u_0)\Big(\ell(u_1)\sin v_1 - \ell(u_0)\sin v_0\Big)\Big], \tag{24}$$

$$d_2 = \ell_u(u_0)\Big[\ell^2(u_0)\cos v_0 - \ell(u_0)\ell(u_1)\cos v_1 + \Big(h(v_1) - h(v_0)\Big)h_v(v_0)\sin v_0\Big], \tag{25}$$

$$d_3 = \ell_u(u_0)h_v(v_0)\Big[\ell(u_1)\cos(v_0 + v_1) - \ell(u_0)\cos 2v_0\Big]. \tag{26}$$

Hence, the equation of plane (α) which contains the line n and the point B is

$$(\alpha) : d_1 \cdot \Big(x - \ell(u_0)\cos v_0\Big) + d_2 \cdot \Big(y - \ell(u_0)\sin v_0\Big) + d_3 \cdot \Big(z - h(v_0)\Big) = 0. \tag{27}$$

After comparing the equation (27) with the equation (18), we obtain that the equation of intersection of surface r and plane α is curve $\gamma = \gamma(v)$ given by

$$(\gamma) : \Big(\nu(v)\cos v, \nu(v)\sin v, h(v)\Big), \tag{28}$$

for

$$\nu(v) = \frac{\ell(u_0)\Big(d_1\cos v_0 + d_2\sin v_0\Big) - d_3\Big(h(v) - h(v_0)\Big)}{d_1\cos v + d_2\sin v}. \tag{29}$$

The distance between points $\tilde{A} = \gamma(v_1)$ and $\tilde{B} = \gamma(v_2)$ is

$$dis(\tilde{A}, \tilde{B}) = \int_{v_1}^{v_2} \sqrt{1 + \nu^2(v) + \nu_v^2(v) + h_v^2(v)}dv. \tag{30}$$

3. Methodological approach

The unit sphere is the surface with the largest number of symmetries, and as such returns the largest number of third-coordinate capabilities. In order to further limit the range of third-coordinate values due to additional conditions, the surface within the sphere must be determined. Based on this selection, it is possible to retain the first two coordinates of the spherical fuzzy numbers and limit the values of the third coordinate. In this way, the hesitancy function can be adjusted to describe the observed situation in a much better way.

3.1. Basic definitions and properties for Surface Fuzzy Numbers

The decision-making is a complex task involving multiple data sets, often both accurate (objective) and uncertain (subjective) enabling the decision-maker to deal with imprecise, partially known, and uncertain information. When comparing the criteria, sub-criteria, or alternatives, the decision-makers give their judgements using fuzzy numbers and their numerous extensions to describe the influence of one criteria over another in a precise and good manner.

In the sequel, we give an extended table of levels of importance of linguistic terms. After that, we will generalize the concept of Spherical Fuzzy Sets [6, 8, 11, 13]

Levels of importance of linguistic terms	(μ, ν, π)	SI
Level 1	$\big(\mu_1, \nu_1, \pi_1(t)\big)$	$2k+1$
$\vdots$	$\vdots$	$\vdots$
Level k	$\big(\mu_k, \nu_k, \pi_k(t)\big)$	3
Level $k+1$	$\big(0.5, 0.4, 0.4\big)$	1
Level $k+2$	$\big(\nu_k, \mu_k, \pi_k(t)\big)$	3^{-1}
$\vdots$	$\vdots$	
Level $2k+1$	$\big(\nu_1, \mu_1, \pi_1(t)\big)$	$(2k+1)^{-1}$

Table 2: Score indices, Linguistic measures of criteria, and Surface Fuzzy Sets

The π_S, $S = 1, \ldots, k+1$ are functions $\pi_S : [0,1] \to [0,1]$ whose values are the third coordinates of points $\big(\mu_S, \nu_S, \pi_S(t)\big)$ inside the rotational surface such that $\big(\mu_S, \nu_S, \pi_S(0)\big)$, $\big(\mu_S, \nu_S, \pi_S(1)\big)$ are points on the surface.

The ordered triples $\big(\mu, \nu, \pi(t)\big)$ are *Surface Fuzzy Sets*.

The basic operations with Surface Fuzzy Sets $\tilde{A}_S = \big(\mu_{\tilde{A}_S}, \nu_{\tilde{A}_S}, \pi_{S^*_{\tilde{A}_S}}(t)\big)$ and $\tilde{B}_S = \big(\mu_{\tilde{B}_S}, \nu_{\tilde{B}_S}, \pi_{S^*_{\tilde{B}_S}}(t)\big)$ at the level S, $S = 1, \ldots, 2k+1$, where $(k+1+r)^* = r$ and $(k+1-r)^* = k+1-r$, $r = 1, \ldots, k$, are

- **Union operation**

$$\tilde{A}_S \cup \tilde{B}_S = \Big\{ \max\big\{\mu_{\tilde{A}_S}, \mu_{\tilde{B}_S}\big\}, \min\big\{\nu_{\tilde{A}_S}, \nu_{\tilde{B}_S}\big\} \\ \min\Big\{\Big(1 - \sqrt{\min{}^2\big\{\mu_{\tilde{A}_S}, \mu_{\tilde{B}_S}\big\} + \max{}^2\big\{\nu_{\tilde{A}_S, \tilde{B}_S}\big\}}, \max\big\{\pi^*_{\tilde{A}_S}(t), \pi^*_{\tilde{B}_S}(t)\big\}\Big\}. \tag{31}$$

- **Intersection operation**

$$\tilde{A}_S \cap \tilde{B}_S = \Big\{ \min\big\{\mu_{\tilde{A}_S}, \mu_{B_S}\big\}, \max\big\{\nu_{\tilde{A}_S}, \nu_{\tilde{B}_S}\big\} \\ \min\Big\{\Big(1 - \sqrt{\max{}^2\big\{\mu_{\tilde{A}_S}, \mu_{\tilde{B}_S}\big\} + \min{}^2\big\{\nu_{\tilde{A}_S, \tilde{B}_S}\big\}}, \min\big\{\pi^*_{\tilde{A}_S}(t), \pi^*_{\tilde{B}_S}(t)\big\}\Big\}. \tag{32}$$

- **Addition operation**

$$\tilde{A}_S \oplus \tilde{B}_S = \Big\{ \sqrt{\mu^2_{\tilde{A}_S} + \mu^2_{\tilde{B}_S} - \mu^2_{\tilde{A}_S}\mu^2_{\tilde{B}_S}}, \nu_{\tilde{A}_S}\nu_{\tilde{B}_S}, \\ \sqrt{\Big(1 - \mu^2_{\tilde{B}_S}\Big)\pi^2_{\tilde{A}_S}(t) + \Big(1 - \mu^2_{\tilde{A}_S}\Big)\pi^2_{\tilde{B}_S}(t) - \pi^2_{\tilde{A}_S}(t)\pi^2_{\tilde{B}_S}(t)} \Big\}. \tag{33}$$

- **Multiplication operation**

$$\tilde{A}_S \otimes \tilde{B}_S = \Big\{ \mu_{\tilde{A}_S}\mu_{\tilde{B}_S}, \sqrt{\nu^2_{\tilde{A}_S} + \nu^2_{\tilde{B}_S} - \nu^2_{\tilde{A}_S}\nu^2_{\tilde{B}_S}}, \\ \sqrt{\Big(1 - \nu^2_{\tilde{B}_S}\Big)\pi^2_{\tilde{A}_S}(t) + \Big(1 - \nu^2_{\tilde{A}_S}\Big)\pi^2_{\tilde{B}_S}(t) - \pi^2_{\tilde{A}_S}(t)\pi^2_{\tilde{B}_S}(t)} \Big\}. \tag{34}$$

- **Multiplication by a scalar operation, for $\lambda > 0$:**

$$\lambda \cdot A_S = \left\{ \sqrt{1-\left(1-\mu_{\tilde{A}_S}^2\right)^{\lambda}}, \nu_{\tilde{A}_S}^{\lambda}, \sqrt{\left(1-\mu_{\tilde{A}_S}^2\right)^{\lambda}-\left(1-\mu_{\tilde{A}_S}^2-\pi_{\tilde{A}_S}^2(t)\right)} \right\}. \tag{35}$$

- **Power of $\tilde{A}_S$ operation, for $\lambda > 0$:**

$$A_S^{\lambda} = \left\{ \mu_{\tilde{A}_S}^{\lambda}, \sqrt{1-\left(1-\nu_{\tilde{A}_S}^2\right)^{\lambda}}, \sqrt{\left(1-\nu_{\tilde{A}_S}^2\right)^{\lambda}-\left(1-\nu_{\tilde{A}_S}^2-\pi_{\tilde{A}_S}^2(t)\right)^{\lambda}} \right\}. \tag{36}$$

As in [6], the next equations hold

$$\tilde{A}_S \oplus \tilde{B}_S = \tilde{B}_S \oplus \tilde{A}_S, \tag{37}$$
$$\tilde{A}_S \otimes \tilde{B}_S = \tilde{B}_S \otimes \tilde{A}_S, \tag{38}$$
$$\lambda\left(\tilde{A}_S \oplus \tilde{B}_S\right) = \lambda \tilde{A}_S \oplus \lambda \tilde{B}_S, \tag{39}$$
$$\lambda_1 \tilde{A}_S \oplus \lambda_2 \tilde{A}_S = (\lambda_1 + \lambda_2)\tilde{A}_S, \tag{40}$$
$$\left(\tilde{A}_S \otimes \tilde{B}_S\right)^{\lambda} = \tilde{A}_S^{\lambda} \otimes \tilde{B}_S^{\lambda}, \tag{41}$$
$$\tilde{A}_S^{\lambda_1} \otimes \tilde{A}_S^{\lambda_2} = \tilde{A}_S^{\lambda_1+\lambda_2}, \tag{42}$$

for $\lambda, \lambda_1, \lambda_2 > 0$.

Theorem 3.1. *For Surface Fuzzy Sets $A_1 = \left(\mu_1, \nu_1, \pi_1(t)\right)$, $A_2 = \left(\mu_2, \nu_2, \pi_2(t)\right)$, $A_3 = \left(\mu_3, \nu_3, \pi_3(t)\right)$, the operations $\oplus$ and $\otimes$ are associative.*

For a Surface Fuzzy Set $\left(\mu, \nu, \pi(t)\right)$, the next equations hold

$$\left(\mu, \nu, \pi(t)\right) \oplus (0,1,0) = (0,1,0) \oplus \left(\mu, \nu, \pi(t)\right) = \left(\mu, \nu, \pi(t)\right), \tag{43}$$
$$\left(\mu, \nu, \pi(t)\right) \otimes (1,0,0) = (1,0,0) \otimes \left(\mu, \nu, \pi(t)\right) = \left(\mu, \nu, \pi(t)\right), \tag{44}$$

i.e. Surface Fuzzy Sets $e_0 = (0,1,0)$ and $e_1 = (1,0,0)$ are neutrals for operations $\oplus$ and $\otimes$, respectively.

The next equalities hold

$$e_0 \oplus e_0 = e_0 \quad \textit{and} \quad e_1 \otimes e_1 = e_1. \tag{45}$$

For Surface Fuzzy Sets $\left(\mu, \nu, \pi(t)\right) \notin \{e_0, e_1\}$, it does not exist a Surface Fuzzy Set s_f such that $\left(\mu, \nu, \pi(t)\right) \oplus s_f = e_0$ or $\left(\mu, \nu, \pi(t)\right) \otimes s_f = e_1$.

For random Surface Fuzzy Sets A_1 and A_2 the only Surface Fuzzy Sets A_3 which satisfy identity $(A_1 \oplus A_2\big) \otimes A_3 = \left(A_1 \otimes A_3\right) \oplus \left(A_2 \otimes A_3\right)$ are $A_{3_1} = (1,0,0)$ and $A_{3_2} = (0,1,0)$.

Proof. With respect to definition of operation $\oplus$ given by (33), one obtains

$$\left(\mu_1, \nu_1, \pi_1(t)\right) \oplus \left(\mu_2, \nu_2, \pi_2(t)\right) = \left\{ \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \nu_1\nu_2, \sqrt{(1-\mu_2^2)\pi_1^2(t) + (1-\mu_1^2)\pi_2^2(t) - \pi_1^2(t)\pi_2^2(t)} \right\}. \tag{33'}$$

The second coordinates of Surface Fuzzy Sets $A_l = \left(A_1 \oplus A_2\right) \oplus A_3$ and $A_r = A_1 \oplus \left(A_2 \oplus A_3\right)$ are trivially equal. The first coordinate of surface fuzzy number A_l is $\sqrt{\left(\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}\right)^2 + \mu_3^2 - \left(\sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}\right)^2 \mu_3^2}$, which is equal to

$$A_{l_1} = \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2 + \mu_3^2 - \mu_1^2\mu_3^2 - \mu_2^2\mu_3^2 + \mu_1^2\mu_2^2\mu_3^2}.$$

After changing μ_2^2 in (33′) by $\mu_2^2+\mu_3^2-\mu_2^2\mu_3^2$, we obtain that the first coordinate of $A_1\oplus\left(A_2\oplus A_3\right)$ is $\sqrt{\mu_1^2+\left(\sqrt{\mu_2^2+\mu_3^2-\mu_2^2\mu_3^2}\right)^2-\mu_1^2\left(\sqrt{\mu_2^2+\mu_3^2-\mu_2^2\mu_3^2}\right)^2}$, i.e.

$$A_{r_1}=\sqrt{\mu_1^2+\mu_2^2+\mu_3^2-\mu_2^2\mu_3^2-\mu_1^2\mu_2^2-\mu_1^2\mu_3^2+\mu_1^2\mu_2^2\mu_3^2}=A_{l_1}.$$

Based on (33′), the squares of third coordinates of A_l and A_r are

$$\begin{aligned}A_{l_3}^2&=\pi_1^2(t)-\mu_2^2\pi_1^2(t)-\mu_3^2\pi_1^2(t)+\mu_2^2\mu_3^2\pi_1^2(t)+\pi_2^2(t)-\mu_1^2\pi_2^2(t)-\mu_3^2\pi_2^2(t)+\mu_1^2\mu_3^2\pi_2^2(t)-\pi_1^2(t)\pi_2^2(t)\\&+\pi_3^2(t)+\mu_3^2\pi_1^2(t)\pi_2^2(t)-\mu_1^2\pi_3^2(t)-\mu_2^2\pi_3^2(t)+\mu_1^2\mu_2^2\pi_3^2(t)-\pi_1^2(t)\pi_3^2(t)+\mu_2^2\pi_1^2(t)\pi_3^2(t)-\pi_2^2(t)\pi_3^2(t)\\&+\mu_1^2\pi_2^2(t)\pi_3^2(t)+\pi_1^2(t)\pi_2^2(t)\pi_3^2(t)=A_{r_3},\end{aligned}$$

which completes the proof of associativity of the operation $\oplus$.

After changing $\mu_k\leftrightarrow\nu_k$, $k=1,2,3$, in the proof of the associativity of $\oplus$, we confirm the associativity of the operation $\otimes$.

The next equalities hold,

$$\begin{aligned}&\left(\mu,\nu,\pi(t)\right)\oplus(0,1,0)=\left\{\sqrt{\mu^2+0^2-\mu^2\cdot0},\nu\cdot1,\sqrt{(1-0^2)\pi^2(t)+(1-\mu^2)\cdot0-\pi^2(t)\cdot0}\right\}=\left(\mu,\nu,\pi(t)\right),\\&\left(\mu,\nu,\pi(t)\right)\otimes(1,0,0)=\left\{\mu\cdot1,\sqrt{\nu^2+0^2-\nu^2\cdot0^2},\sqrt{(1-0^2)\pi^2+(1-\nu^2)\cdot0^2-\pi^2\cdot0^2}\right\}=\left(\mu,\nu,\pi(t)\right),\end{aligned}\tag{46}$$

which confirms the validity of equations (43, 44).

Because the solution of equation $x^2+y^2-x^2\cdot y^2=0$ by x is $x=x\cdot(-1+x^2)^{-1/2}$, which is a complex number for $x\in(0,1)$, the inverse surface fuzzy numbers for $\left(\mu,\nu,\pi(t)\right)\notin\{e_0,e_1\}$, with respect to operations $\oplus$, $\otimes$ does not exist. For Surface Fuzzy Sets $A_{S_1}=\left(\mu_1,\nu_1,\pi_1(t)\right)$, $A_{S_2}=\left(\mu_2,\nu_2,\pi_2(t)\right)$, $A_{S_3}=\left(\mu_3,\nu_3,\pi_3(t)\right)$, the first two components of $(A_{S_1}\oplus A_{S_2})\otimes A_{S_3}$ and $\left(A_{S_1}\otimes A_{S_3}\right)\oplus\left(A_{S_2}\otimes A_{S_3}\right)$ are

$$\begin{cases}&\text{first component}&\text{second component}\\\left(A_{S_1}\oplus A_{S_2}\right)\otimes A_{S_3}: &\mu_3\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2}&\sqrt{\nu_1^2\nu_2^2+\nu_3^2-\nu_1^2\nu_2^2\nu_3^2}\\\left(A_{S_1}\otimes A_{S_3}\right)\oplus\left(A_{S_2}\otimes A_{S_3}\right): &\mu_3\sqrt{\mu_1^2+\mu_2^2-\mu_1^2\mu_2^2\mu_3^2}&\sqrt{\nu_1^2+\nu_3^2-\nu_1^2\nu_3^2}\cdot\sqrt{\nu_2^2+\nu_3^2-\nu_2^2\nu_3^2}\end{cases}\tag{47}$$

For random μ_1, μ_2, the previous first components are equal if and only if $\mu_3=0$ or $\mu_3=1$.

After squaring and equalizing the second components from (47), we obtain equality

$$\nu_3^2\left(1-\nu_3^2\right)\left(1-\nu_1^2-\nu_2^2+\nu_1^2\nu_2^2\right)=0.\tag{48}$$

For random ν_1, ν_2, the equality (48) holds if and only if $\nu_3=0$ or $\nu_3=1$.

Because $\mu_3^2+\nu_3^2+\left(\pi_3(t)\right)^2\le1$, the only Surface Fuzzy Sets for which both the first and the second components presented in (47) are equal are $(1,0,0)$ and $(0,1,0)$.

For $\left(\mu_3,\nu_3,\pi_3(t)\right)=(1,0,0)$ or $\left(\mu_3,\nu_3,\pi_3(t)\right)=(0,1,0)$, after some computing, one obtains that the third coordinates of $\left(A_{S_1}\oplus A_{S_2}\right)\otimes A_{S_3}$ and $\left(A_{S_1}\otimes A_{S_3}\right)\oplus\left(A_{S_2}\otimes A_{S_3}\right)$ are equal. □

Based on $\omega=(\omega_1,\ldots,\omega_n)$, $\omega_i\in[0,1]$, $\sum_{i=1}^n\omega_i=1$, Surface Weighted Geometric Mean (S_fWGM) is

$$\begin{aligned}S_fWGM_\omega\left(A_{S_1},\ldots,A_{S_n}\right)&=A_{S_1}^{\omega_1}+\ldots+A_{S_n}^{\omega_n}\\&=\Big\{\prod_{i=1}^n\mu_{A_{S_i}}^{\omega_i},\sqrt{1-\prod_{i=1}^n(1-\nu_{A_{S_i}}^2)^{\omega_i}},\\&\sqrt{\prod_{i=1}^n\left(1-\nu_{A_{S_i}}^2\right)^{\omega_i}-\prod_{i=1}^n\left(1-\nu_{A_{S_i}}^2-\pi_{A_{S_i}}^2(t)\right)^{\omega_i}}\Big\}.\end{aligned}\tag{49}$$

Corresponding Score and Accuracy functions are

$$Ss\left(\tilde{A}_S\right) = \left(\mu_{\tilde{A}_S} - \pi_{\tilde{A}_S}(t)\right)^2 - \left(\nu_{\tilde{A}_S} - \pi_{\tilde{A}_S}(t)\right), \tag{50}$$

$$Sa\left(\tilde{A}_S\right) = \mu^2_{\tilde{A}_S} + \nu^2_{\tilde{A}_S} + \pi^2_{\tilde{A}_S}(t). \tag{51}$$

As in [6], the next statements are equivalent

$$\tilde{A}_S < \tilde{B}_S, \tag{52}$$

$$Ss\left(\tilde{A}_S\right) < Ss\left(\tilde{B}_S\right), \tag{53}$$

$$Ss\left(\tilde{A}_S\right) = Ss\left(\tilde{B}_S\right) \wedge Sa\left(\tilde{A}_S\right) < Sa\left(\tilde{B}_S\right). \tag{54}$$

4. Surface Fuzzy Sets and AHP

Decision-makers, n of them, give their opinions about m criteria. These results are organized as matrices $M_1, \ldots, M_n$, of the types $m \times m$.

The corresponding crisp matrix is

$$M_c = \left[M_{ij}\right], \tag{55}$$

$i, j = 1, \ldots, m$.

The normalized crisp matrix is

$$\overline{M}_c = \left[\overline{M}_{ij}\right], \tag{56}$$

for

$$\overline{M}_{ij} = \Big(\sum_{k=1}^{m} M_{ik}\Big)^{-1} M_{ij}. \tag{57}$$

If $f^1_{ij} = \left(\mu^1_{ij}, \nu^1_{ij}, \pi^1_{ij}(t)\right), \ldots, f^n_{ij}\left(\mu^m_{ij} \nu^m_{ij}, \pi^m_{ij}(t)\right)$ are fuzzy numbers which correspond to the decision-makers' opinions about correlation of the i-th and j-th criterium, the corresponding Integrated surface comparison fuzzy matrix is

$$F = \left[F_{ij}\right], \tag{58}$$

where

$$F_{ij} = \frac{1}{n}\left(f^1_{ij} + \ldots + f^n_{ij}\right). \tag{59}$$

The weights for r-th criteria, $r = 1, \ldots, m$, are $w_r = \left(\mu_r, \nu_r, \pi_r(t)\right)$, for

$$\mu_r = \left[1 - \prod_{i=1}^{n} \left(1 - \mu_i^2\right)^{\omega_i}\right]^{0.5}, \tag{60}$$

$$\nu_r = \prod_{i=1}^{n} \nu_i^{\omega_i}, \tag{61}$$

$$\pi_r(t) = \left[\prod_{i=1}^{n} \left(1 - \mu_i^2\right)^{\omega_i} - \prod_{i=1}^{n} \left(1 - \mu_i - \pi_r^2(t)\right)^{\omega_i}\right]^{0.5} \tag{62}$$

for $\omega_1, \dots, \omega_n > 0$, $\omega_1 + \dots + \omega_n = 1$.

The score function is

$$Ss_k(t) = \sqrt{\left|100 \cdot \left[\left(3\mu_k - \frac{\pi_k(t)}{2}\right)^2 - \left(\frac{\nu_k}{2} - \pi_k(t)\right)^2\right]\right|}. \tag{63}$$

The normalized weight of criteria is

$$\overline{S}s_k(t) = \frac{Ss_k(t)}{\sum_{p=1}^{m} Ss_p(t)}. \tag{64}$$

4.1. Example

In [22], authors deal with factors that affect the B2C e-commerce websites. Among five groups and nineteen sub-factors, we choose A1: Safe payment, A2: Account security, A3: Secure data sharing sub-criteria, belonging to Security, Privacy, and Authority group to apply introduced Surface Fuzzy Sets to within AHP.

The surface which will be used is ellipsoid

$$(e): \begin{cases} x = 0.9\cos\phi\cos\psi \\ y = 0.9\cos\phi\sin\psi \\ z = 0.7\sin\phi, \end{cases} \tag{65}$$

$\phi \in [0, 2\pi)$, $\psi \in [0, \pi)$.

At the point $(x_0, y_0) = (0.9\cos\phi_0\cos\psi_0, 0.9\cos\phi_0\sin\psi_0, 0.7\phi_0)$, the π-functions for any decision-maker are equal and they are

$$\pi_{ij}^k(t) = \pi_{ij}(t) = 0.7\sin\left(\arctan\frac{y_0}{x_0}\right)\cos\left(-\frac{\pi}{4} + \frac{\pi}{2}t\right). \tag{66}$$

We will have criteria and five decision makers.

The decision-makers' opinions are

$$\begin{aligned} &M_1 = \begin{bmatrix} 1 & 3 & 5 \\ 3^{-1} & 1 & 3 \\ 5^{-1} & 3^{-1} & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 5 & 3 \\ 5^{-1} & 1 & 5 \\ 3^{-1} & 5^{-1} & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 1 & 3 & 5 \\ 3^{-1} & 1 & 5 \\ 5^{-1} & 5^{-1} & 1 \end{bmatrix}, \\ &M_4 = \begin{bmatrix} 1 & 3 & 5 \\ 3^{-1} & 1 & 3 \\ 5^{-1} & 3^{-1} & 1 \end{bmatrix}, \quad M_5 = \begin{bmatrix} 1 & 5 & 3 \\ 5^{-1} & 1 & 3 \\ 3^{-1} & 3^{-1} & 1 \end{bmatrix}. \end{aligned} \tag{67}$$

The crisp matrix is

$$M_c = \begin{bmatrix} 1 & 3.8 & 4.2 \\ 0.26 & 1 & 3.8 \\ 0.24 & 0.26 & 1 \end{bmatrix}. \tag{68}$$

The normalized crisp matrix is

$$\overline{M}_c = \begin{bmatrix} 0.67 & 0.75 & 0.47 \\ 0.17 & 0.20 & 0.42 \\ 0.16 & 0.51 & 0.11 \end{bmatrix}. \tag{69}$$

The integrated surface fuzzy comparison matrix is

$$M_F = \begin{bmatrix} (0.5, 0.4, 0.4), \big(0.64, 0.36, \pi(t)\big), \big(0.66, 0.34, \pi(t)\big) \\ \big(0.36, 0.64, \pi(t)\big), (0.5, 0.4, 0.4), \big(0.64, 0.36, \pi(t)\big) \\ \big(0.34, 0.66, \pi(t)\big), \big(0.36, 0.64, \pi(t)\big), (0.5, 0.4, 0.4) \end{bmatrix}. \tag{70}$$

Surface AHP-weights are

$$SW_{AHP} = \begin{bmatrix} A_1: & \left(0.61, 0.37, \sqrt{0.63 - 0.84 \cdot \left(0.56 - \pi^2(t)\right)^{1/3} \cdot \left(0.59 - \pi^2(t)\right)^{1/3}}\right) \\ A_2: & \left(0.52, 0.45, \sqrt{0.73 - 0.84 \cdot \left(0.59 - \pi^2(t)\right)^{1/3} \cdot \left(0.87 - \pi^2(t)\right)^{1/3}}\right) \\ A_3: & \left(0.41, 0.55, \sqrt{0.83 - 0.84 \cdot \left(0.87 - \pi^2(t)\right)^{1/3} \cdot \left(0.88 - \pi^2(t)\right)^{1/3}}\right) \end{bmatrix}. \tag{71}$$

In the case of $M_F = \begin{bmatrix} A_1: & (a_1, b_1, c_1) \\ A_2: & (a_2, b_2, c_2) \\ A_3: & (a_3, b_3, c_3) \end{bmatrix}$, the corresponding crisp numbers are

$$S(\tilde{w}) = \begin{bmatrix} \sqrt{\left|100\left(\left(3a_1 - \frac{c_1}{2}\right)^2 - \left(\frac{b_1}{2} - c_1\right)^2\right)\right|} \\ \sqrt{\left|100\left(\left(3a_2 - \frac{c_2}{2}\right)^2 - \left(\frac{b_2}{2} - c_2\right)^2\right)\right|} \\ \sqrt{\left|100\left(\left(3a_3 - \frac{c_3}{2}\right)^2 - \left(\frac{b_3}{2} - c_3\right)^2\right)\right|} \end{bmatrix}. \tag{72}$$

The graphics for crisp weights of three analyzed criteria are presented in figure below.

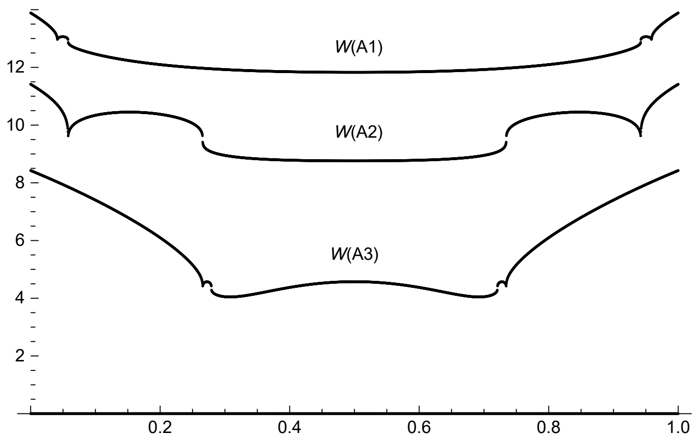

Figure 1: Crisp weights of analyzed criteria

The crisp weight corresponding first criteria decreases from $t = 0$ to $t = 0.5$. For $t > 0.5$, the crisp weight increases. This minimum is equal 11.8282.

The crisp weight corresponding second criteria has minimum at $t = 0.5$ and this minimum is 8.76022.

The crisp weight corresponding third criteria has minimum at $t = 0.692172$ which is 4.04456.

The crisp values of criteria $A1$, $A2$, $A3$, for different values $t \in [0, 1]$, are listed in Table 4.1.

Applying presented idea, we can see 2 the proportions of criteria A_i, $i = 1, 2, 3$, determining values for parameter t for which proportions obtain their extreme values. The maximum of ratios A_1/A_2 listed in Table 4.1 is 1.35021 for $t = 0.5$. The maximum of listed ratios A_1/A_3 in Table 4.1 is 2.94275 for $t = 0.3$ and $t = 0.7$. The maximum of listed ratios A_2/A_3 in Table 4.1 is 2.20869 for $t = 0.3$ and $t = 0.7$.

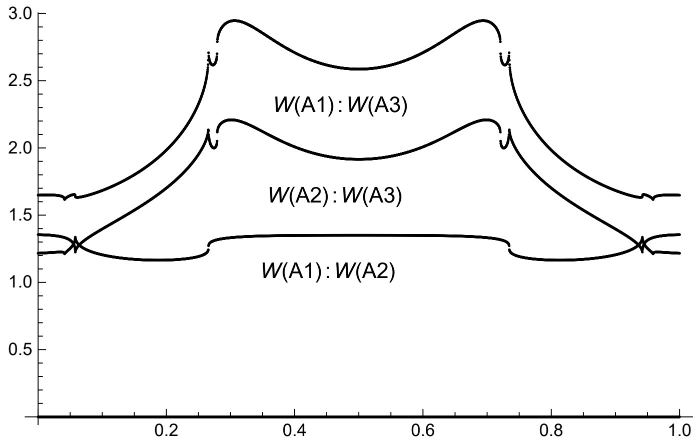

Figure 2: Proportions of crisp weights of analyzed criteria

	Crisp weights			Proportion of crisp values		
$t\backslash A1$	$A1$	$A2$	$A3$	$A1/A2$	$A1/A3$	$A2/A3$
0.0	13.8863	11.4153	8.42506	1.21646	1.64822	1.35493
0.1	12.4598	10.3395	7.40156	1.20507	1.6834	1.39693
0.2	12.0964	10.3755	6.09738	1.16586	1.98386	1.70163
0.3	11.9253	8.95058	4.05244	1.33235	2.94275	2.20869
0.4	11.8493	8.78391	4.37498	1.34898	2.70843	2.00776
0.5	11.8282	8.76022	4.57419	1.35021	2.58585	1.91514
0.6	11.8493	8.78391	4.37498	1.34898	2.70843	2.00776
0.7	11.9253	8.95058	4.05244	1.33235	2.94275	2.20869
0.8	12.0964	10.3755	6.09738	1.16585	1.98386	1.70163
0.9	12.4598	10.3395	7.40156	1.20507	1.6834	1.39693
1.0	13.8864	11.4153	8.41406	1.21646	1.64822	1.35493

5. Conclusion

In this paper, we improved the concept of Spherical Fuzzy Sets with Surface Fuzzy Sets. Unlike in the case of Spherical Fuzzy Sets, we used the third coordinate which is not constant. In Theorem 3.1, we proved the associativity and confirmed the commutativity of the operations $\oplus$, $\otimes$ specified for Surface Fuzzy Sets. In this theorem, we obtained the neutrals for operations $\oplus$ and $\otimes$ and proved that only these neutrals have the corresponding inverse elements. At the end of this paper, we obtained the results based on functional third coordinates instead of constant ones. This, and the generalization of Table of Score indices and Linguistic measures, empowers decision-makers with a more space in estimation of relative importance between criteria.

The advantage of this paper is reflected in considering the third coordinate, hesitancy, as a function instead of constant value. Calculating the extreme values of functions $\pi_i(t)$ enables us to choose extreme values points, improve condition (2) and empower decision-makers in describing the fuzzy part of data comprehensively and in detail. In this way, it is possible to determine essential relationship between uncertainty, imprecise and incomplete information in Multi-criteria decision-making and hesitation.

Even though our proposed method gives insight into some advantages in the field of Fuzzy Logic, there are limitations to this work. One of the main shortcomings of this paper, like in all AHP methods, concerns the existence of incomparable criteria. The top-down direction structure and comparisons of criteria from one level with all criteria from the upper level, might bring about unmatched sub-criteria. One of possible solutions to this problem could be use of Analytic Network Process, clustering elements in the hierarchy. Another limitation of this method is the impossibility of integration of arbitrary function $\pi(t)$, as well as possible shortening the hesitancy range caused by function selection.

In future, we will study Surface Fuzzy Sets with functionally expressed all three coordinates.

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