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Bounded factorization property for ℓ-Köthe spaces

Murat Hayrettin Yurdakul[a,*], Emre Taştüner[a]

[a] *Middle East Technical University*

Abstract. Let ℓ denote a Banach sequence space with a monotone norm in which the canonical system $(e_n)_n$ is an unconditional basis. We show that the existence of an unbounded continuous linear operator T between ℓ-Köthe spaces $\lambda^\ell(A)$ and $\lambda^\ell(C)$ which factors through a third ℓ-Köthe space $\lambda^\ell(B)$ causes the existence of an unbounded continuous quasidiagonal operator from $\lambda^\ell(A)$ into $\lambda^\ell(C)$ factoring through $\lambda^\ell(B)$ as a product of two continuous quasidiagonal operators. Using this result, we study when the triple $(\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C))$ satisfies the bounded factorization property $\mathcal{BF}$ (which means that all continuous linear operators from $\lambda^\ell(A)$ into $\lambda^\ell(C)$ factoring through $\lambda^\ell(B)$ are bounded). As another application, we observe that the existence of an unbounded factorized operator for a triple of ℓ-Köthe spaces, under some additional assumptions, causes the existence of a common basic subspace at least for two of the spaces.

1. Introduction

Dragilev [3] and Nurlu [6] proved that if X and Y are nuclear ℓ_1-Köthe spaces and there exists a continuous linear unbounded operator $T : X \to Y$, then there exists a continuous unbounded quasidiagonal operator $D : X \to Y$. Djakov and Ramanujan [1] sharpened this result by omitting the nuclearity condition. The ℓ-Köthe version of that result in [1] has recently been obtained in [11] by Uyanık and Yurdakul.
On the other hand, Nurlu and Terzioğlu [7] proved (under some conditions) that the existence of an unbounded continuous linear operator between nuclear ℓ_1-Köthe spaces X and Y implies the existence of a common basic subspace of X and Y; this result was generalized by Djakov and Ramanujan [1] to the non-nuclear case (see [11] also). In these works, Dragilev's theorem plays a crucial role.
Zahariuta in [13] observed that if the matrices of ℓ_1-Köthe spaces X and Y satisfy the conditions d_2, d_1, respectively, then every continuous linear operator from X into Y is bounded. This phenomenon was studied extensively by many authors; the most comprehensive result is due to Vogt [12], where all pairs of Fréchet spaces with this property are characterized.
Terzioğlu and Zahariuta [10] characterized those triples (X, Y, Z) of Fréchet spaces such that each continuous linear operator $T : X \to Z$ which factors through Y is automatically bounded.
The aim of the present work is to prove a factorization analogue of Dragilev's theorem [3] and its generalizations [1, 11]. Namely, we prove that if there is an unbounded continuous linear operator $T : \lambda^\ell(A) \to \lambda^\ell(C)$ which factors through $\lambda^\ell(B)$, then, in fact, there exists an unbounded continuous quasidiagonal operator $D : \lambda^\ell(A) \to \lambda^\ell(C)$ that factors through $\lambda^\ell(B)$ as a product of two continuous quasidiagonal operators.

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* Corresponding author: Murat Hayrettin Yurdakul
Email addresses: myur@metu.edu.tr (Murat Hayrettin Yurdakul), tastuner@metu.edu.tr (Emre Taştüner)

Terzioğlu, Yurdakul and Zahariuta [9] obtained the ℓ_1-Köthe version of our result by using the characterization of the bounded factorization property [10]. Our proof is the factorized analogue of the proof of Proposition 1 in [1].
Using this result, we study when the triple $(\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C))$ satisfies the bounded factorization property. Also, exactly as in [9], we show that the existence of an unbounded factorized operator for a triple of ℓ-Köthe spaces causes that, under some additional conditions, these spaces (or at least two of them) have a common basic subspace.

2. Bounded factorization property and ℓ-Köthe spaces

We denote by $L(X, Y)$ and $LB(X, Y)$ the spaces of all continuous linear operators and of all bounded linear operators from the locally convex space X into the locally convex space Y. If for each $S \in L(X, Y)$ and $R \in L(Y, Z)$ we have $T = RS \in LB(X, Z)$, we say (X, Y, Z) has the *bounded factorization property* and write $(X, Y, Z) \in \mathcal{BF}$ [10]. We simply write $(X, Y) \in \mathcal{B}$ when $L(X, Y) = LB(X, Y)$.
Notice that if $(X, Y) \in \mathcal{B}$ or $(Y, Z) \in \mathcal{B}$, then $(X, Y, Z) \in \mathcal{BF}$; and if $(X, Z) \in \mathcal{B}$, then $(X, Y, Z) \in \mathcal{BF}$ for any Y.
The bounded factorization property is essential in the isomorphic classification of Cartesian products of locally convex spaces. See for example [2].
Dealing with several Fréchet spaces we always use the same notation $\{|\cdot|_p,\ p \in \mathbf{N}\}$ for a system of seminorms defining their topologies and $\{|\cdot|_p^*,\ p \in \mathbf{N}\}$ for the corresponding system of polar norms in the dual spaces.
For any operator $T \in L(E, F)$ we consider the following operator seminorms

$$|T|_{p,q} = \sup\{|Tx|_p : |x|_q \leq 1\}, \quad p, q \in \mathbf{N},$$

which may take the value $+\infty$. In particular, for any one-dimensional operator $T = x' \otimes y$, $x' \in E'$, $y \in F$, we have $|T|_{p,q} = |x'|_q^* \cdot |y|_p$. Notice that $T \in L(E, F)$ means that for some function $\sigma : \mathbb{N} \to \mathbb{N}$, we have $|T|_{p,\sigma(p)} < \infty$ for every $p \in \mathbb{N}$. Also T is bounded (i.e. $T \in LB(E, F)$) if there exists $r \in \mathbb{N}$ such that $|T|_{q,r} < \infty$ for every $q \in \mathbb{N}$.
Following [4], we denote by ℓ a Banach sequence space in which the canonical system $(e_n)_n$ is an unconditional basis. The norm $\|\cdot\|$ is called monotone if $\|x\| \leq \|y\|$ whenever $|x_n| \leq |y_n|$, $x = (x_n)_n, y = (y_n)_n \in \ell$, $n \in \mathbb{N}$. Let Λ be the class of such spaces with monotone norm. In particular, $\ell_p, c_0 \in \Lambda$.
It is known that every Banach space with an unconditional basis $(x_n)_n$ has a monotone norm which is equivalent to its original norm.
Indeed, it is enough to put
$\|.\| = \sup\limits_{|\alpha_n| \leq 1} \left| \sum\limits_n x_n'(.)\alpha_n x_n \right|$, where $|.|$ is the original norm of the Banach space and x_n' is the coefficient functional corresponding to x_n for each $n \in \mathbb{N}$.
Let $\ell \in \Lambda$ and $\|.\|$ be a monotone norm in ℓ.
If $A = (a_n^k)$ is a Köthe matrix, the ℓ-Köthe space $\lambda^\ell(A)$ is the Fréchet space of all sequences of scalars (x_n) such that $(x_n a_n^k) \in \ell$ for all $k \in \mathbb{N}$ with the topology generated by the seminorms $\|(x_n)\|_k = \|(x_n a_n^k)\|$, $k \in \mathbb{N}$.
Notice that $\|e_n\|_k = a_n^k$, $n, k \in \mathbb{N}$. We always assume that the matrix $A = (a_n^k)$ satisfies $a_n^k \leq a_n^{k+1}$, $n, k \in \mathbb{N}$.
An operator $T \in L(\lambda^\ell(A), \lambda^\ell(B))$ is quasidiagonal if $T(e_n) = t_n e_{\sigma(n)}$, $n \in \mathbb{N}$, for some bijective map $\sigma : \mathbb{N} \to \mathbb{N}$ and scalar sequence (t_n).

3. Main Results

Our main result characterizes the bounded factorization property for triples of ℓ-Köthe spaces in terms of quasidiagonal operators, which is a natural generalization of Dragilev's theorem [1, 3, 11].

Proposition 3.1. *If $(\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C)) \notin \mathcal{BF}$, then there are continuous quasidiagonal operators $D_1 : \lambda^\ell(A) \to \lambda^\ell(B)$ and $D_2 : \lambda^\ell(B) \to \lambda^\ell(C)$ such that $D = D_2 D_1$ is unbounded.*

Proof. Let $T = RS : \lambda^\ell(A) \to \lambda^\ell(C)$ be a linear continuous unbounded operator which factors through $\lambda^\ell(B)$. Then $R : \lambda^\ell(B) \to \lambda^\ell(C)$ is also unbounded, because otherwise T would be bounded. Now, we want to argue according to the following observation in the spirit of the lemma in [8]:
Let U_k, V_k, W_k denote the closed unit balls defined by the k^{th} seminorms on $\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C)$, respectively. We start with an arbitrary ball W_1 in $\lambda^\ell(C)$. Using the continuity of R we find a ball V_1 in $\lambda^\ell(B)$ such that $R(V_1) \subset W_1$, and by the continuity of S we find a ball U_1 in $\lambda^\ell(A)$ such that $S(U_1) \subset V_1$. Since R is unbounded, $R(V_1)$ is not absorbed by, say, the ball W_2 contained in W_1 and so $T(U_1)$ is not absorbed by W_2. For this W_2, we use the continuity of R to find a ball $V_2 \subset V_1$ in $\lambda^\ell(B)$ and the continuity of S to find a ball $U_2 \subset U_1$ in $\lambda^\ell(A)$ such that $R(V_2) \subset W_2$ and $S(U_2) \subset V_2$. Since $R(V_2)$ is not a bounded set, we can find a ball, say, W_3 in $\lambda^\ell(C)$ such that $R(V_2) \not\subset \lambda W_3$ for all $\lambda > 0$. Hence, using the continuities of R and S and unboundedness of R alternately, we find decreasing sequences of balls $(U_k)_k, (V_k)_k, (W_k)_k$ in $\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C)$, respectively, such that $R(V_k) \subset W_k$, $S(U_k) \subset V_k$ and $T(U_k) \not\subset \lambda W_{k+1}$ for all $k \in \mathbb{N}$.
Keeping the observation above in our minds, without loss of generality, we may assume that
(i) $\|Tx\|_k \leq \frac{1}{2^k}\|x\|_k$ for all $x \in \lambda^\ell(A)$, $k = 1, 2, 3, \ldots$
(ii) $\sup_n \dfrac{\|Te_n\|_{k+1}}{\|e_n\|_k} = \infty$, $k = 1, 2, 3, \ldots$
(iii) $\sup_\ell \dfrac{\|R\widetilde{e_\ell}\|_{k+1}}{\|\widetilde{e_\ell}\|_k} = \infty$, $k = 1, 2, 3, \ldots$
where $(e_n)_n$, $(\widetilde{e_\ell})_\ell$ and $(\widetilde{\widetilde{e_v}})_v$ denote the canonical bases in $\lambda^\ell(A)$, $\lambda^\ell(B)$ and $\lambda^\ell(C)$, respectively.
Indeed, one may obtain these by using appropriate multipliers and passing to a subsequence of seminorms, if necessary.
Let $(k_j)_j$ be a sequence of integers such that each k appears in it infinitely many times and in view of (ii) choose inductively an increasing subsequence $(n_j)_j$ such that
(iv) $\dfrac{\|Te_{n_j}\|_{k_j+1}}{\|e_{n_j}\|_{k_j}} \geq 2^j$ for all j.
Let $S(e_n) = \sum_\ell \widetilde{\theta}_{n\ell}\widetilde{e_\ell}$ and $R(\widetilde{e_\ell}) = \sum_v \theta_{\ell v}\widetilde{\widetilde{e_v}}$.
Then $T(e_n) = \sum_\ell \widetilde{\theta}_{n\ell} R(\widetilde{e_\ell}) = \sum_\ell \widetilde{\theta}_{n\ell}(\theta_{\ell 1}, \theta_{\ell 2}, \theta_{\ell 3}, \ldots)$

So, $T(e_n) = (\widetilde{\theta}_{n1}\theta_{11}, \widetilde{\theta}_{n1}\theta_{12}, \widetilde{\theta}_{n1}\theta_{13}, \ldots) + (\widetilde{\theta}_{n2}\theta_{21}, \widetilde{\theta}_{n2}\theta_{22}, \widetilde{\theta}_{n2}\theta_{23}, \ldots) + \ldots = \left(\sum_\ell \widetilde{\theta}_{n\ell}\theta_{\ell 1}, \sum_\ell \widetilde{\theta}_{n\ell}\theta_{\ell 2}, \sum_\ell \widetilde{\theta}_{n\ell}\theta_{\ell 3}, \ldots\right)$

i.e. $T(e_n) = \sum_v \left(\sum_\ell \widetilde{\theta}_{n\ell}\theta_{\ell v}\right)\widetilde{\widetilde{e_v}}$.
Consider

$$\sup_{|\alpha_v|\leq 1}\left|\sum_v \left(\sum_\ell \widetilde{\theta}_{n\ell}\theta_{\ell v}\right)\alpha_v \left(\sup_k \frac{c_v^k}{b_\ell^k}\right)\left(\sup_k \frac{b_\ell^k}{a_n^k}\right)\widetilde{\widetilde{e_v}}\right| \leq \sup_{|\alpha_v|\leq 1}\left|\sum_v \left(\sum_\ell \widetilde{\theta}_{n\ell}\theta_{\ell v}\right)\alpha_v \sum_k \frac{c_v^k}{a_n^k}\widetilde{\widetilde{e_v}}\right|$$

$$\leq \sum_k \frac{1}{a_n^k}\sup_{|\alpha_v|\leq 1}\left|\sum_v \left(\sum_\ell \widetilde{\theta}_{n\ell}\theta_{\ell v}\right)\alpha_v c_v^k \widetilde{\widetilde{e_v}}\right| \leq \sum_k \frac{\|Te_n\|_k}{\|e_n\|_k} \leq \sum_k \frac{1}{2^k} \leq 1.$$

Thus, for each $j = 1, 2, \ldots$, we obtain in view of (iv)

(v) $$\sup_{|\alpha_v|\leq 1}\left|\sum_v \left(\sum_\ell \widetilde{\theta}_{n_j\ell}\theta_{\ell v}\right)\alpha_v \left(\sup_k \frac{c_v^k}{b_\ell^k}\right)\left(\sup_k \frac{b_\ell^k}{a_{n_j}^k}\right)\widetilde{\widetilde{e_v}}\right| \leq 1 \leq 2^{-j}\sup_{|\alpha_v|\leq 1}\left|\sum_v \left(\sum_\ell \widetilde{\theta}_{n_j\ell}\theta_{\ell v}\right)\alpha_v \frac{c_v^{k_j+1}}{a_{n_j}^{k_j}}\widetilde{\widetilde{e_v}}\right|.$$

Hence, there is v_j such that

(vi) $$\left(\sup_k \frac{c_{v_j}^k}{b_\ell^k}\right)\left(\sup_k \frac{b_\ell^k}{a_{n_j}^k}\right) \leq \frac{1}{2^j}\frac{c_{v_j}^{k_j+1}}{a_{n_j}^{k_j}}$$

Otherwise, we obtain a contradiction to (v) by monotonicity of $\|.\|$.
Notice that (vi) holds for any ℓ.

Because of (iii) we would choose inductively an increasing subsequence (ℓ_j) such that $\dfrac{\|R\widetilde{e}_{\ell_j}\|_{k_j+1}}{\|\widetilde{e}_{\ell_j}\|_{k_j}} \geq 2^j$ for all $j = 1, 2, \ldots$.

Let $\lambda_j = \sup_k \dfrac{c^k_{v_j}}{b^k_{\ell_j}}$, $\mu_j = \sup_k \dfrac{b^k_{\ell_j}}{a^k_{n_j}}$ so that

(vii) $\lambda_j \mu_j \leq \dfrac{1}{2^j} \dfrac{c^{k_j+1}_{v_j}}{a^{k_j}_{n_j}}$.

Consider the quasidiagonal operator $D_1 : \lambda^\ell(A) \to \lambda^\ell(B)$ defined by $D_1 e_{n_j} = \mu_j^{-1} \widetilde{e}_{\ell_j}$, $j = 1, 2, \ldots$; $D_1 e_n = 0$ if $n \neq n_j$, and the quasidiagonal operator $D_2 : \lambda^\ell(B) \to \lambda^\ell(C)$ defined by $D_2 \widetilde{e}_{\ell_j} = \lambda_j^{-1} \widetilde{\widetilde{e}}_{v_j}$, $j = 1, 2, \ldots$; $D_2 \widetilde{e}_\ell = 0$ if $\ell \neq \ell_j$.

Hence, the quasidiagonal operator $D : \lambda^\ell(A) \to \lambda^\ell(C)$ is defined by

$De_{n_j} = D_2 D_1 e_{n_j} = (\lambda_j \mu_j)^{-1} \widetilde{\widetilde{e}}_{v_j} = \left(\sup_k \dfrac{c^k_{v_j}}{a^k_{n_j}} \right)^{-1} \widetilde{\widetilde{e}}_{v_j} =: t_j^{-1} \widetilde{\widetilde{e}}_{v_j}$, $j = 1, 2, \ldots$; $De_n = 0$ if $n \neq n_j$.

If $x = \sum_j x_{n_j} e_{n_j} \in \lambda^\ell(A)$, then $D_1 x = \sum_j x_{n_j} (\mu_j)^{-1} \widetilde{e}_{\ell_j}$.

Since $|x_{n_j} (\mu_j)^{-1} b^k_{\ell_j}| \leq |x_{n_j} a^k_{n_j}|$ for all j, by monotonicity of $\|.\|$, we obtain that $\|(x_{n_j} (\mu_j)^{-1} b^k_{\ell_j})\| \leq \|(x_{n_j} a^k_{n_j})\|$, i.e. $\|D_1 x\|_k \leq \|x\|_k$ for all k. Hence, D_1 is continuous.

If $x = \sum_j x_{\ell_j} \widetilde{e}_{\ell_j} \in \lambda^\ell(B)$, then $D_2 x = \sum_j x_{\ell_j} (\lambda_j)^{-1} \widetilde{\widetilde{e}}_{v_j}$.

Since $|x_{\ell_j} (\lambda_j)^{-1} c^k_{v_j}| \leq |x_{\ell_j} b^k_{\ell_j}|$ for all j, by monotonicity of $\|.\|$, we obtain that $\|(x_{\ell_j} (\lambda_j)^{-1} c^k_{v_j})\| \leq \|(x_{\ell_j} b^k_{\ell_j})\|$, i.e. $\|D_2 x\|_k \leq \|x\|_k$ for all k. Hence, D_2 is continuous. So, D is continuous (or it can be shown similarly).

In addition, D is unbounded, because if k is fixed, then for some subsequence (j_s) we have $k_{j_s} = k$, $s = 1, 2, 3, \ldots$ and by (vii), $\dfrac{\|De_{n_{j_s}}\|_{k+1}}{\|e_{n_{j_s}}\|_k} \geq 2^{j_s} \to \infty$ as $s \to \infty$.

□

The next theorem gives a necessary and sufficient condition for $(\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C)) \in \mathcal{BF}$. Formally in ℓ_1-Köthe case this condition coincides with the one given by Terzioğlu, Zahariuta (see [10], Theorem 3.5), but its sufficiency in our case cannot be obtained directly for a general map, since continuity at any e_n does not imply continuity at $x \in \lambda^\ell(A)$. Proposition 3.1 gets rid of this difficulty.

Theorem 3.2. *We have $(\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C)) \in \mathcal{BF}$ if and only if for each non-decreasing map $\pi : \mathbb{N} \to \mathbb{N}$ there exists $r \in \mathbb{N}$ such that for all $q \in \mathbb{N}$ there exist $s \in \mathbb{N}$ and $C > 0$ so that the inequality*

(viii) $\dfrac{c^q_i}{a^r_j} \leq C \max_{k=1,\ldots,s} \left(\dfrac{c^k_i}{b^{\pi(k)}_v} \right) \max_{k=1,\ldots,s} \left(\dfrac{b^k_v}{a^{\pi(k)}_j} \right)$ *holds for all $i, j, v \in \mathbb{N}$.*

Notice that Theorem 3.2 above is the factorized analogue of Theorem 2.2 in [11]. In its proof we will use the following result from [10].

Proposition 3.3. *For Fréchet spaces E, F, G we have $(E, G, F) \in \mathcal{BF}$ if and only if for each non-decreasing map $\pi : \mathbb{N} \to \mathbb{N}$ there exists $r \in \mathbb{N}$ such that for all $q \in \mathbb{N}$ there exist $s = s(q) \in \mathbb{N}$ and $C = C(q) > 0$ so that the following inequality*

$\|T\|_{q,r} \leq C \max_{k=1,\ldots,s} (\|R\|_{k,\pi(k)}) \max_{k=1,\ldots,s} (\|S\|_{k,\pi(k)})$

is satisfied for every $R \in L(G,F), S \in L(E,G)$ where $T = RS$.

Now we are ready to prove Theorem 3.2.

Proof. Suppose $(\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C)) \in \mathcal{BF}$.
Let $R = e_v^{'} \otimes e_i$, $S = e_j^{'} \otimes e_v$ so that $T = RS = e_j^{'} \otimes e_i$ is an operator of rank one. Note that
$\|R\|_{k,\pi(k)} = \frac{c_i^k}{b_v^{\pi(k)}}$, $\|S\|_{k,\pi(k)} = \frac{b_v^k}{a_j^{\pi(k)}}$, and $\|T\|_{q,r} = \frac{c_i^q}{a_j^r}$.
Then results follows from Proposition 3.3 above.
In view of Proposition 3.1 it is enough to prove the converse for quasidiagonal operators.
Let $S(e_j) = s_j\widetilde{e}_{v(j)}$, $R(\widetilde{e_v}) = t_v\widetilde{\widetilde{e}}_{i(v)}$, and $T(e_j) = RS(e_j) = s_j t_{v(j)}\widetilde{\widetilde{e}}_{i(v(j))}$, $j \in \mathbb{N}$ define a continuous quasidiagonal operator on $\lambda^\ell(A)$ to $\lambda^\ell(C)$ which factors through $\lambda^\ell(B)$.
We determine $\pi : \mathbb{N} \to \mathbb{N}$ such that $\|S\|_{k,\pi(k)} < \infty$ and $\|R\|_{k,\pi(k)} < \infty$ for each $k \in \mathbb{N}$ (Remember our observation at the beginning of the proof of our Proposition 3.1) and find $r \in \mathbb{N}$ such that for every q there exists $C > 0$ and $s \in \mathbb{N}$ so that the relation (viii) holds. We observe that

$$\|T\|_{q,r} = \sup_j \frac{|s_j||t_{v(j)}|c^q_{i(v(j))}}{a_j^r},$$

$$\|S\|_{k,\pi(k)} = \sup_j \frac{|s_j|b^k_{v(j)}}{a_j^{\pi(k)}} \text{ and } \|R\|_{k,\pi(k)} = \sup_j \frac{|t_{v(j)}|c^k_{i(v(j))}}{b^{\pi(k)}_{v(j)}}.$$

Then, using (viii) we get

$$\|T\|_{q,r} = \sup_j \frac{|s_j||t_{v(j)}|c^q_{i(v(j))}}{a_j^r} \leq C \sup_j \left(|s_j||t_{v(j)}| \max_{k=1,\dots,s} \left(\frac{c^k_{i(v(j))}}{b^{\pi(k)}_{v(j)}} \right) \max_{k=1,\dots,s} \left(\frac{b^k_{v(j)}}{a_j^{\pi(k)}} \right) \right)$$

$$\leq C \max_{k=1,\dots,s} \left(\sup_j \frac{|t_{v(j)}|c^k_{i(v(j))}}{b^{\pi(k)}_{v(j)}} \right) \max_{k=1,\dots,s} \left(\sup_j \frac{|s_j|b^k_{v(j)}}{a_j^{\pi(k)}} \right)$$

$$= C \max_{k=1,\dots,s} \|R\|_{k,\pi(k)} \max_{k=1,\dots,s} \|S\|_{k,\pi(k)} < \infty.$$

Hence, T is bounded.

□

Vogt characterized the pairs $(\lambda(A), \lambda^\infty(B)) \in \mathcal{B}$ ([12]: Satz 1.5). The relation $(\lambda(A), \lambda(B)) \in \mathcal{B}$ was investigated by a different approach in [1] and the relation $(\lambda^\ell(A), \lambda^\ell(B)) \in \mathcal{B}$ was obtained in [11] similarly. A complete characterization of this case is an immediate by-product of our previous theorem.

Corollary 3.4. *We have $(\lambda^\ell(A), \lambda^\ell(B)) \in \mathcal{B}$ if and only if for each non-decreasing $\pi : \mathbb{N} \to \mathbb{N}$ there exists $r \in \mathbb{N}$ such that for each $q \in \mathbb{N}$ we can find $C > 0$ and $s \in \mathbb{N}$ so that the inequality*
$\frac{b_i^q}{a_j^r} \leq C \max_{k=1,\dots,s} \left(\frac{b_i^k}{a_j^{\pi(k)}} \right)$ *holds for all $i, j \in \mathbb{N}$.*

4. Common Subspaces

Following [9], we say that a pair (F, E) of Fréchet spaces satisfies the condition $\mathcal{S}$ if there is a mapping $\tau : \mathbb{N} \to \mathbb{N}$ such that for each pair $p, r \in \mathbb{N}$ there exists a constant $C = C(p, r)$ such that the estimate
(ix) $\|T\|_{r,\tau(p)} \leq C \max(\|T\|_{\tau(p),p}, \|T\|_{\tau(r),r})$
holds for every one-dimensional operator $T = e' \otimes f$, where $e' \in E'$, $f \in F$. A pair of ℓ-Köthe spaces $E = \lambda^\ell(A)$

and $F = \lambda^\ell(B)$ satisfies the condition $\mathcal{S}$ if the condition (ix) holds for the operators $T = e_i' \otimes e_j, i, j \in \mathbb{N}$ ([5]). If the estimate (ix) is true for arbitrary bounded operators $T \in L(E, F)$ then we write $(F, E) \in \overline{\mathcal{S}}$.
Again following [9], a triple of Fréchet spaces (F, G, E) satisfies the condition $\mathcal{SF}$ (we then write $(F, G, E) \in \mathcal{SF}$) if for any one-dimensional operator $T = RS$, with both $S \in L(E, G)$ and $R \in L(G, F)$ also one-dimensional, the inequality
(x) $\|T\|_{r,\tau(p)} \le C \max(\|R\|_{\tau(p),p}, \|R\|_{\tau(r),r}) \max(\|S\|_{\tau(p),p}, \|S\|_{\tau(r),r})$
holds with the same requisites as in (ix).
If the condition (x) holds for an arbitrary bounded operator $T = RS$, with $S \in L(E, G)$ and $R \in L(G, F)$ we will write $(F, G, E) \in \overline{\mathcal{SF}}$.
We note that if $E = G$ or $G = F$ the condition $(F, G, E) \in \mathcal{SF}$ reduces simply to $(F, E) \in \mathcal{S}$ as well as $(F, G, E) \in \overline{\mathcal{SF}}$ does so to $(F, E) \in \overline{\mathcal{S}}$.
The following example shows that $\mathcal{SF}$ is strictly weaker than $\mathcal{S}$. Here we use the notation $\Lambda_\alpha(a) = \lambda(exp(\alpha_p a_i))$ with $\alpha_p \nearrow \alpha \le \infty$, $a = (a_i)$.
Notice that the finite type power series space $\Lambda_1(a)$ has d_2-matrix and infinite type power series space $\Lambda_\infty(a)$ has d_1-matrix.

Example 4.1. *Let $a = (a_i)$ be a positive sequence increasing to ∞. Since $(\Lambda_1(a), \Lambda_\infty(a)) \in \mathcal{B}$ ([13]), we have $(\Lambda_1(a), \Lambda_\infty(a), \Lambda_1(a)) \in \mathcal{BF}$ trivially. So we have $(\Lambda_1(a), \Lambda_\infty(a), \Lambda_1(a)) \in \overline{\mathcal{SF}}$ (hence $(\Lambda_1(a), \Lambda_\infty(a), \Lambda_1(a)) \in \mathcal{SF}$) by Proposition 7 in [9].*
However, $(\Lambda_1(a), \Lambda_\infty(a)) \notin \mathcal{S}$.

In what follows we shall denote by $\lambda^\ell(A)_L$ the basic subspace of an ℓ-Köthe space $\lambda^\ell(A)$ which is the closed linear envelope of $\{e_n : n \in L\}$, $L \subset \mathbb{N}$.
Suppose now $(\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C)) \notin \mathcal{BF}$ and $(\lambda^\ell(C), \lambda^\ell(A)) \in \mathcal{S}$. By Proposition 3.1, we know that there are $S : \lambda^\ell(A) \to \lambda^\ell(B)$; $S(e_i) = t_i\widetilde{e}_{\sigma(i)}$, $i \in \mathbb{N}$, and $R : \lambda^\ell(B) \to \lambda^\ell(C)$; $R\widetilde{e}_v = s_v\widetilde{\widetilde{e}}_{\rho(v)}$, $v \in \mathbb{N}$, with some bijective maps σ and ρ on $\mathbb{N}$ such that $T = RS$ is an unbounded quasidiagonal operator. By Corollary 2.3 in [11] (see also Proposition 3 in [1]) there exists infinite subsets J and I of $\mathbb{N}$ such that T maps $\lambda^\ell(A)_J$ isomorphically onto $\lambda^\ell(C)_I$. Then one can easily check that for $N = \sigma(J) = \rho^{-1}(I)$ both $S : \lambda^\ell(A)_J \to \lambda^\ell(B)_N$ and $R : \lambda^\ell(B)_N \to \lambda^\ell(C)_I$ are also isomorphisms. We have therefore proved that:

Proposition 4.2. *Suppose that $(\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C)) \notin \mathcal{BF}$ and $(\lambda^\ell(C), \lambda^\ell(A)) \in \mathcal{S}$. Then there is a common basic subspace for all three spaces.*

Now proceeding exactly as in [9], we consider a generalization of Djakov-Ramanujan's result ([1], Proposition 3) in the context of factorization.

Theorem 4.3. *Suppose that $(\lambda^\ell(A), \lambda^\ell(B), \lambda^\ell(C)) \notin \mathcal{BF}$ and $(\lambda^\ell(C), \lambda^\ell(B), \lambda^\ell(A)) \in \mathcal{SF}$. Then one of the pairs $(\lambda^\ell(A), \lambda^\ell(B))$ or $(\lambda^\ell(B), \lambda^\ell(C))$ has a common basic subspace.*

Proof. By Proposition 3.1, there exists quasidiagonal operators $S : \lambda^\ell(A) \to \lambda^\ell(B)$ and $R : \lambda^\ell(B) \to \lambda^\ell(C)$ with bijective σ and ρ (as above) such that $T = RS$ is unbounded. Without loss of generality we assume in what follows that all three operators are identity embeddings, since otherwise we can get this property by considering a new triple of ℓ-Köthe spaces obtained from the original one by some permutations and normalizations of their canonical bases (note that the property $\mathcal{SF}$ is preserved under such reconstruction).
When applied to the above embeddings, the condition $\mathcal{SF}$ gives the following:
there is a map $\tau : \mathbb{N} \to \mathbb{N}$ such that
(xi) $\dfrac{c_i^r}{a_i^{\tau(p)}} \le C \max\left(\dfrac{b_i^{\tau(p)}}{a_i^p}, \dfrac{b_i^{\tau(r)}}{a_i^r}\right) \max\left(\dfrac{c_i^{\tau(p)}}{b_i^p}, \dfrac{c_i^{\tau(r)}}{b_i^r}\right)$ for all $p, r, i \in \mathbb{N}$ with some constant $C = C(p, r)$.

It now suffices to prove that there is an infinite set $I \subset \mathbb{N}$ such that $\lambda^\ell(A)_I = \lambda^\ell(B)_I$ or $\lambda^\ell(B)_I = \lambda^\ell(C)_I$. Suppose that this assertion is false. Then for each infinite set $I \subset \mathbb{N}$ and $m \in \mathbb{N}$ there is $r \geq m$ such that

(xii) $\liminf_{i \in \mathbb{I}} \frac{b_i^{\tau(r)}}{a_i^r} = \liminf_{i \in \mathbb{I}} \frac{c_i^{\tau(r)}}{b_i^r} = 0.$

We define inductively the sets $N_0 \supset N_1 \supset \ldots$ by

(xiii) $N_0 := \mathbb{N}, N_p := \left\{ i \in N_{p-1} : \max\left(\frac{b_i^{\tau(p)}}{a_i^p}, \frac{c_i^{\tau(p)}}{b_i^p} \right) \geq 1 \right\}, p \in \mathbb{N}$

with τ from (xi).

We claim that for each $p \in \mathbb{N}$ the embedding T is bounded on the basic subspace X_p of $\lambda^\ell(A)$ spanned by $\{e_i : i \in N_{p-1} \setminus N_p\}$. If that is not so, then for each $q \in \mathbb{N}$ there is an infinite subset $I_q \subset N_{p-1} \setminus N_p$ and $m(q) \in \mathbb{N}$ with

(xiv) $\lim_{i \in I_q} \frac{c_i^{m(q)}}{a_i^q} = \infty.$

For $I = I_q$ we find $r \geq m(q)$ such that (xii) holds.

Then there is an infinite set $J_q \subset I_q$ with

(xv) $\max\left(\frac{c_i^{\tau(r)}}{b_i^r}, \frac{b_i^{\tau(r)}}{a_i^r} \right) < 1, i \in J_q.$

On the other hand, by (xiii) we have

(xvi) $\max\left(\frac{c_i^{\tau(p)}}{b_i^p}, \frac{b_i^{\tau(p)}}{a_i^p} \right) < 1, i \in I_q.$

Applying now (xi) with $q = \tau(p)$ and r chosen above and taking into account the estimates (xv) and (xvi), we obtain $\frac{c_i^r}{a_i^q} \leq C$ for all $i \in J_q$, which contradicts (xiv).

This proves our claim that the embedding T is bounded on each X_p. Hence, for every $p \in \mathbb{N}$, the operator T must be unbounded on the basic subspace Y_p generated by $\{e_i : i \in N_p\}$, which, particularly, implies that N_p is an infinite set.

Now we construct a sequence $I = \{i_p\}$ so that $i_p \in N_p, i_{p+1} \neq i_p, p \in \mathbb{N}$.

Then due to (xiii), there is an infinite set $J \subset I$ such that at least one of the inequalities $a_i^p \leq b_i^{\tau(p)}$ or $b_i^p \leq c_i^{\tau(p)}$ holds for all $p \in \mathbb{N}$ and $i \in J$ such that $i \geq p$, which contradicts the assumption (xii). This completes the proof. □

References

[1] P. B. Djakov, M. S. Ramanujan, *Bounded and unbounded operators between Köthe spaces*, Studia Math., **152** (2002), 11–31.

[2] P. Djakov, T. Terzioğlu, M. Yurdakul, V. Zahariuta, *Bounded operators and isomorphisms of Cartesian products of Fréchet spaces,* Mich. Math. J., **45 (3)** (1998), 599–610.

[3] M.M. Dragilev, *Riesz classes and multiple regular-bases*, Func. Anal. and Func. Theory, Kharkov, **15** (1972), 65–77 (in Russian).

[4] M.M. Dragilev, *Bases in Köthe spaces*, Rostov, Russia: Rostov University Press, (1983).

[5] J. Krone, D. Vogt, *The splitting relation for Köthe spaces*, Math. Z. **190** (1985), 387–400.

[6] Z. Nurlu, *On pairs of Köthe spaces between which all operators are compact*, Math. Nachr. **122** (1985), 277–287.

[7] Z. Nurlu, T. Terzioğlu, *Consequences of the existence of a non-compact operator between nuclear Köthe spaces,* Manuscripta Math. **47** (1984), 1–12.

[8] T. Terzioğlu, M. Yurdakul, *Restrictions of unbounded continuous linear operators on Fréchet spaces*, Arch. Math. **46** (1986), 547–550.

[9] T. Terzioğlu, M. Yurdakul, V. Zahariuta, *Factorization of unbounded operators on Köthe spaces*, Studia Math. **161(I)** (2004), 61–70.

[10] T. Terzioğlu, V. Zahariuta, *Bounded factorization property for Fréchet spaces,* Math. Nachr., **253** (2003), 1–11.

[11] E. Uyanık, M. Yurdakul, *A remark on a paper of P.B. Djakov and M.S. Ramanujan*, Turk J Math. **43** (2019), 2494–2498.

[12] D. Vogt, *Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J.Reine. Angew. Math., **345** (1983), 182–200.

[13] V. Zahariuta, *On the isomorphism of Cartesian products of locally convex spaces,* Studia Math. **46** (1973), 201–221.