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Remarks regarding eigenvalues and fixed points in some algebras obtained by the Cayley-Dickson process

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Abstract. Some results regarding eigenvalues, companion matrices and fixed points in some algebras obtained by the Cayley-Dickson process are presented in this paper. The paper contains several examples which emphasize the obtained results.

1. Introduction

In this paper we present some results regarding eigenvalues and fixed points in some algebras obtained by the Cayley-Dickson process.

The paper is organized as follows:

-In Introduction we present some general definitions and properties regarding these algebras.

-In Section 2 we present some aspects regarding eigenvalues of some division algebras obtained by the Cayley-Dickson process.

-In Section 3 we present some properties of the companion matrices and fixed points for a polynomial with coefficients in a real division quaternion algebra.

In the following, we suppose that K is a commutative field with $charK \neq 2$ and $\mathfrak{A}$ is an algebra over the field K. An algebra $\mathfrak{A}$ is called *alternative* if $x^2y = x(xy)$ and $xy^2 = (xy)y$, for all $x, y \in \mathfrak{A}$, *flexible* if $x(yx) = (xy)x = xyx$, for all $x, y \in \mathfrak{A}$ and *power associative* if the subalgebra $< x >$ of $\mathfrak{A}$ generated by any element $x \in \mathfrak{A}$ is associative. Each alternative algebra is a flexible algebra and a power associative algebra. A unitary algebra $\mathfrak{A} \neq K$ such that we have $x^2 + \gamma_x x + \delta_x = 0$ for each $x \in \mathcal{A}$, with $\gamma_x, \delta_x \in K$, is called a *quadratic algebra*. A finite-dimensional algebra $\mathfrak{A}$ is *a division* algebra if and only if $\mathfrak{A}$ does not contain zero divisors.

The Cayley-Dickson process is an iterative process with which are obtained generalizations of octonion and quaternion algebras over an arbitrary field K.

For a finite dimensional unitary algebra $\mathfrak{A}$ over the field K, we consider the following linear map, named *scalar involution*,

$$\bar{\ } : \mathfrak{A} \to \mathfrak{A}, a \to \bar{a},$$

with the properties

$$\overline{ab} = \bar{b}\bar{a}, \bar{\bar{a}} = a,$$

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and

$$a+\overline{a}, a\overline{a} \in K\cdot 1 \text{ for all } a,b\in\mathfrak{A}.$$

The element $\overline{a}$ is called the *conjugate* of the element a. With this element, the linear form

$$\mathbf{t}: A \to K,\ \mathbf{t}(a) = a+\overline{a}$$

and the quadratic form

$$\mathbf{n}: A \to K,\ \mathbf{n}(a) = a\overline{a}$$

can be defined. These forms are called the *trace* and the *norm* of the element a, respectively. From here, it results that an algebra $\mathfrak{A}$ with a scalar involution is quadratic, that means for each $x\in\mathfrak{A}$, we have $x^2+t(x)x+n(x)=0$.

For $\beta\in K$, a fixed non-zero element, we define the following algebra multiplication on the vector space $\mathfrak{A}\oplus\mathfrak{A}$:

$$(a_1,a_2)(b_1,b_2) = \left(a_1b_1+\beta\overline{b}_2a_2, a_2\overline{b_1}+b_2a_1\right). \tag{1.3.}$$

We obtain an algebra structure over $\mathfrak{A}\oplus\mathfrak{A}$, denoted by $(\mathfrak{A},\beta)$. This algebra is called the *algebra obtained from the algebra $\mathfrak{A}$ by the Cayley-Dickson process.* It is clear that $\dim(\mathfrak{A},\beta) = 2\dim\mathfrak{A}$.

Let $x\in(\mathfrak{A},\beta)$, $x=(a_1,a_2)$. The map

$$^{-}: (\mathfrak{A},\beta)\to(\mathfrak{A},\beta)\ ,\ x\to\bar{x} = (\overline{a}_1,-a_2)\,,$$

is a scalar involution of the algebra $(\mathfrak{A},\beta)$, extending the involution $^{-}$ of the algebra $\mathfrak{A}$. Let

$$\mathbf{t}(x) = \mathbf{t}(a_1)$$

and

$$\mathbf{n}(x) = \mathbf{n}(a_1) - \beta\mathbf{n}(a_2)$$

be the *trace* and the *norm* of the element $x\in(\mathfrak{A},\beta)$, respectively.

If we start from $\mathfrak{A}=K$ and we apply this process t times, $t\geq 1$, we obtain an algebra over K,

$$\mathfrak{A}_t = \left(\frac{\beta_1,...,\beta_t}{K}\right). \tag{1.4.}$$

On this algebra, the set $\{1,f_2,...,f_n\}$, $n=2^t$, generates a basis with the properties:

$$f_i^2 = \beta_i 1,\ \beta_i\in K, \beta_i\neq 0,\ i=2,...,n \tag{1.5.}$$

and

$$f_if_j = -f_jf_i = \gamma_{ij}f_k,\ \gamma_{ij}\in K,\ \gamma_{ij}\neq 0, i\neq j, i,j= 2,...n, \tag{1.6.}$$

γ_{ij} and f_k being uniquely determined by f_i and f_j.

From the above, it results that $x\in\mathfrak{A}_t$ can be written under the form

$$x = x' + x'' f_{2^{t-1}}, \tag{1.8.}$$

where x' and $x''\in\mathfrak{A}_{t-1} = \left(\frac{\beta_1,...,\beta_{t-1}}{K}\right)$.

For other details, the reader is referred to [SCH; 66] and [SCH; 54].

We remark that, from the above described process, for $t = 2$ we obtain the generalized quaternion algebras and for $t = 3$ we obtain the generalized octonion algebras.

The generalized quaternion algebras

We consider two elements $\alpha, \beta \in K$ and we define a generalized quaternion algebra, denoted by $\mathbb{H}(\alpha, \beta) = \left(\frac{\alpha,\beta}{K}\right)$, with basis $\{1, f_1, f_2, f_3\}$ and multiplication given in the following table:

$\cdot$	1	f_1	f_2	f_3
1	1	f_1	f_2	f_3
f_1	f_1	α	f_3	αf_2
f_2	f_2	$-f_3$	β	$-\beta f_1$
f_3	f_3	$-\alpha f_2$	βf_1	$-\alpha\beta$

If $a \in \mathbb{H}(\alpha, \beta)$, $a = a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3$, then $\bar{a} = a_0 - a_1 f_1 - a_2 f_2 - a_3 f_3$ is called the *conjugate* of the element a. For $a \in \mathbb{H}(\alpha, \beta)$, the *trace*, respectively, the *norm* of the element $a \in \mathbb{H}(\alpha, \beta)$ are:

$$\mathbf{t}(a) = a + \overline{a} \in K$$

and

$$\mathbf{n}(a) = a\overline{a} = a_0^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_3^2 \in K.$$

It results that $a^2 - \mathbf{t}(a)\, a + \mathbf{n}(a) = 0, \forall a \in \mathbb{H}(\alpha, \beta)$, therefore the generalized quaternion algebra is a *quadratic algebra*.

The algebra $\mathbb{H}(\alpha, \beta)$ is a *division* algebra if, for $x \in \mathbb{H}(\alpha, \beta)$, the relation $\mathbf{n}(x) = 0$ implies $x = 0$. A quaternion non-division algebra is called a *split* algebra.

With the above notations, we remark that $\mathbb{H}(-1, -1) = \left(\frac{-1,-1}{\mathbb{R}}\right)$ is a division algebra. In whole the paper, we denote this algebra with $\mathbb{H}$.

The generalized octonion algebras

We consider three elements $\alpha, \beta, \gamma \in K$ and we define a generalized octonion algebra denoted $\mathbb{O}(\alpha, \beta, \gamma)$, with basis $\{1, f_1, ..., f_7\}$ and multiplication given in the following table:

$\cdot$	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
1	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	f_1	α	f_3	αf_2	f_5	αf_4	$-f_7$	$-\alpha f_6$
f_2	f_2	$-f_3$	β	$-\beta f_1$	f_6	f_7	βf_4	βf_5
f_3	f_3	$-\alpha f_2$	βf_1	$-\alpha\beta$	f_7	αf_6	$-\beta f_5$	$-\alpha\beta f_4$
f_4	f_4	$-f_5$	$-f_6$	$-f_7$	γ	$-\gamma f_1$	$-\gamma f_2$	$-\gamma f_3$
f_5	f_5	$-\alpha f_4$	$-f_7$	$-\alpha f_6$	γf_1	$-\alpha\gamma$	γf_3	$\alpha\gamma f_2$
f_6	f_6	f_7	$-\beta f_4$	βf_5	γf_2	$-\gamma f_3$	$-\beta\gamma$	$-\beta\gamma f_1$
f_7	f_7	αf_6	$-\beta f_5$	$\alpha\beta f_4$	γf_3	$-\alpha\gamma f_2$	$\beta\gamma f_1$	$\alpha\beta\gamma$

The algebra $\mathbb{O}(\alpha, \beta, \gamma)$ is a non-commutative and a non-associative algebra, but it is *alternative*, *flexible* and *power-associative*.

If $a \in \mathbb{O}(\alpha, \beta, \gamma)$, $a = a_0 + a_1 f_1 + a_2 f_2 + a_3 f_3 + a_4 f_4 + a_5 f_5 + a_6 f_6 + a_7 f_7$, then $\bar{a} = a_0 - a_1 f_1 - a_2 f_2 - a_3 f_3 - a_4 f_4 - a_5 f_5 - a_6 f_6 - a_7 f_7$ is called the *conjugate* of the element a. For $a \in \mathbb{O}(\alpha, \beta, \gamma)$, we define the *trace*, respectively, the *norm* of the element $a \in \mathbb{O}(\alpha, \beta, \gamma)$:

$$\mathbf{t}(a) = a + \overline{a} \in K$$

and

$$\mathbf{n}(a) = a\overline{a} = a_0^2 - \alpha a_1^2 - \beta a_2^2 + \alpha\beta a_3^2 - \gamma a_4^2 + \alpha\gamma a_5^2 + \beta\gamma a_6^2 - \alpha\beta\gamma a_7^2 \in K.$$

It follows that $a^2 - \mathbf{t}(a)a + \mathbf{n}(a) = 0, \forall a \in a \in \mathbb{O}(\alpha,\beta,\gamma)$, therefore the generalized octonion algebra is a *quadratic* algebra.

If, for $x \in \mathbb{O}(\alpha,\beta,\gamma)$, the relation $\mathbf{n}(x) = 0$ implies $x = 0$, then the algebra $\mathbb{O}(\alpha,\beta,\gamma)$ is a *division* algebra. With the above notations, we remark that $\mathbb{O}(-1,-1,-1) = \left(\frac{-1,-1,-1}{\mathbb{R}}\right)$, denoted $\mathbb{O}$, is a division algebra (see [SCH; 66] and [SCH; 54]).

Remark 1.1. ([La; 04]) As we remarked above, quaternion and octonion algebras can be with division or split algebras.

i) A quaternion algebra over $\mathbb{R}$ is isomorphic to $\mathbb{H}$ (it is a division algebra) or with $\mathcal{M}_2(\mathbb{R}) \simeq \mathbb{H}_{\mathbb{R}}(1,-1)$, the ring of real quadratic matrices. An isomorphism between two quaternion algebras is a ring homomorphism which fixes the scalars. It is clear that $\mathbb{H}_{\mathbb{R}}(\alpha,\beta)$ is a division algebra if $\alpha < 0, \beta < 0$. If $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is the canonical basis in $\mathbb{H}$, we can find the following basis $\{1, \sqrt{-a}\mathbf{i}, \sqrt{-b}\mathbf{j}, \sqrt{ab}\mathbf{ij}\}$ in $\mathbb{H}_{\mathbb{R}}(\alpha,\beta)$, therefore we find the isomorphism between $\mathbb{H}$ and $\mathbb{H}_{\mathbb{R}}(\alpha,\beta)$.

ii) Similar with the above, on $\mathbb{R}$, the octonion algebras can be with division, isomorphic to $\mathbb{O}(-1,-1,-1)$ or split, isomorphic to $\mathbb{O}(1,1,1)$.

2. Remarks regarding the left and the right spectrum

First of all, we remark that algebras $\mathfrak{A}_t$ obtained by the Cayley-Dickson process, described above, in general, are not division algebras for all $t \geq 1$. But there are fields on which, if we apply the Cayley-Dickson process, the obtained algebras $\mathfrak{A}_t$ are division algebras for all $t \geq 1$. (see [BR; 67] and [FL: 13])

With the above remark, let $\mathfrak{A}_t$ be a division algebra obtained by the Cayley-Dickson process, and $A \in \mathcal{M}_n(\mathfrak{A}_t)$, be a matrix of order n. Since these algebras are not commutative, the notion of left and right eigenvalues are distinct.

Definition 2.1. i) An element $\lambda \in \mathfrak{A}_t$ is called a *left eigenvalue* for the matrix A if there is a nonzero matrix $X \in \mathcal{M}_{n\times 1}(\mathfrak{A}_t)$ such that

$$AX = \lambda X.$$

The set of distinct left eigenvalues is called *the left spectrum of the matrix A* and is denoted $\sigma_L(A)$.

ii) An element $\lambda \in \mathfrak{A}_t$ is called a *right eigenvalue* for the matrix A if there is a nonzero matrix $X \in \mathcal{M}_{n\times 1}(\mathfrak{A}_t)$ such that

$$AX = X\lambda.$$

The set of distinct right eigenvalues is called *the right spectrum of the matrix A* and is denoted $\sigma_R(A)$.

We remark that if a right eigenvalue λ is in K, therefore λ is a left eigenvalue and viceversa.

The study of the solutions of quternionic equations are very important in the study of the spectrum(left and right) for the quaternionic matrices (see [HS;02], [MS; 08], [Mi; 11], [PS; 04], [Sz; 09], etc.). The results of the below proposition will be often used in this paper.

Proposition 2.2. ([HS;02], Theorem 2.3) *Solutions to the quadratic equation*

$$x^2 + bx + c = 0, b, c \in \mathbb{H}$$

are:

Case 1. If $b, c \in \mathbb{R}$ *and* $b^2 < 4c$, *then*

$$x = \frac{1}{2}(-b + rf_1 + sf_2 + tf_3)$$

with $r^2 + s^2 + t^2 = 4c - b^2$ *where* $r, s, t \in \mathbb{R}$.

Case 2. If $b, c \in \mathbb{R}$ *and* $b^2 \geq 4c$, *then*

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

Case 3. If $b \in \mathbb{R}$, $c \notin \mathbb{R}$, *then*

$$x = \frac{-b}{2} \pm \frac{\theta}{2} \mp \frac{c_1}{\theta} f_1 \mp \frac{c_2}{\theta} f_2 \mp \frac{c_3}{\theta} f_3,$$

where $c = c_0 + c_1 f_1 + c_2 f_2 + c_3 f_3$ and

$$\theta = \sqrt{\frac{b^2 - 4c_0 + \sqrt{(b^2 - 4c_0)^2 + 16\left(c_1^2 + c_2^2 + c_3^2\right)}}{2}}.$$

Case 4. If $b \notin \mathbb{R}$, *then*

$$x = \frac{(-Re(b))}{2} - (b' + T)^{-1}(c' - N)$$

with $b' = b - Re(b) = Im(b)$, $c' = c - \frac{(Re(b))}{2}(b - \frac{(Re(b))}{2})$, *where* (T, N) *are chosen in the following way:*

i) $T = 0, N = (\Gamma \pm \sqrt{\Gamma^2 - 4\Psi})/2$, *if* $\Delta = 0, \Gamma^2 \geq 4\Psi$;

ii) $T = \pm\sqrt{2\sqrt{\Psi} - \Gamma}, N = \sqrt{\Psi}$ *if* $\Delta = 0, \Gamma^2 < 4\Psi$;

iii) $T = \pm\sqrt{z}, N = (T^3 + \Gamma T + \Delta)/2T$ *if* $\Delta \neq 0$ *and* z *is the unique positive solution to the equation*

$$z^3 + 2\Gamma z^2 + (\Gamma^2 - 4\Psi)z - \Delta^2 = 0,$$

where $\Gamma = |b'|^2 + 2Re(c')$, $\Psi = |c'|^2$ *and* $\Delta = 2Re(\overline{b'}c')$.

For quaternionic matrices over real division quaternion algebra, it is known that the right spectrum are always nonempty. In [BR; 51], Theorem 1, the author proved that every matrix with coefficients in $\mathbb{H}$ has at least a right eigenvalue. After more than 30 years, in [WO; 85], Wood has proved a similar result for the left eigenvalues. These results explain the huge number of papers devoted to the study of the left or right spectrum for quaternionic and octonionic matrices (See [HU; 01], [TI; 00], etc).

If we consider $\mathbb{H}_{\mathbb{Q}}(-1, -1)$, the quaternion division algebra over rational field $\mathbb{Q}$, the above mentioned results regarding the existence of left and right eigenvalues are not always true, as we can see in the following example.

Example 2.3. We consider the matrix

$$A = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{k} & 0 \end{pmatrix}, A \in \mathcal{M}_2(\mathbb{H}).$$

We compute the left and the right spectrum. For the left spectrum, we have

$$\left(\begin{array}{cc} 0 & \mathbf{i} \\ \mathbf{k} & 0 \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right)=\lambda\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right).$$

It results the following equations: $\mathbf{i}x_2 = \lambda x_1$ and $\mathbf{k}x_1 = \lambda x_2$. We obtain that $x_2 = -\mathbf{i}\lambda x_1$ and $\mathbf{k}x_1 = -\lambda\mathbf{i}\lambda x_1$. Therefore, $\mathbf{k} = -\lambda\mathbf{i}\lambda$ and $\mathbf{ik} = -(\mathbf{i}\lambda)^2$ and $\mathbf{j} = (\mathbf{i}\lambda)^2$. Denoting $y^2 = j$, since $y^2 - 2y_0 y + \mathbf{n}(y) = 0$, we get $y^2 = 2y_0 y - \left(y_0^2 + y_1^2 + y_2^2 + y_3^2\right) = \mathbf{j}$, where $y = y_0 + y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$. It results $y_0^2 - y_1^2 - y_2^2 - y_3^2 = 0, y_1 = 0, 2y_0y_2 = 1$ and $y_3 = 0$. We get the solutions $y \in \{\frac{\sqrt{2}}{2}(1+\mathbf{j}), -\frac{\sqrt{2}}{2}(1+\mathbf{j})\}$. Therefore, $\lambda \in \{-\frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{k}), \frac{\sqrt{2}}{2}(\mathbf{i}+\mathbf{k})\}$.

For the right spectrum, we have

$$\left(\begin{array}{cc} 0 & \mathbf{i} \\ \mathbf{k} & 0 \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right)=\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right)\lambda.$$

It results the following equations: $\mathbf{i}x_2 = x_1\lambda$ and $\mathbf{k}x_1 = x_2\lambda$. We obtain that $x_2 = -\mathbf{i}x_1\lambda$ and $\mathbf{k}x_1 = -\mathbf{i}x_1\lambda^2$. Therefore, $-\mathbf{j}x_1 = x_1\lambda^2$. We have $-x_1^{-1}\mathbf{j}x_1 = \lambda^2$ and we obtain $\lambda^4 = \left(-x_1^{-1}\mathbf{j}x_1\right)\left(-x_1^{-1}\mathbf{j}x_1\right)$, thus $\lambda^4 + 1 = 0$. Denoting $y = \lambda^2$, the solutions in $\mathbb{H}$ of the equation

$$y^2 + 1 = 0$$

are of the form $y = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, with $a, \beta, \gamma \in \mathbb{R}$, such that $\alpha^2 + \beta^2 + \gamma^2 = 1$.

From Proposition 2.2, the solution in $\mathbb{H}$ of the equation

$$z^2 = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k},$$

where $a, \beta, \gamma \in \mathbb{R}$, such that $\alpha^2 + \beta^2 + \gamma^2 = 1$ are of the form $\delta_1\frac{\sqrt{2}}{2} + \delta_2\frac{\sqrt{2}}{2}\alpha\mathbf{i} + \delta_2\frac{\sqrt{2}}{2}\beta\mathbf{j} + \delta_2\frac{\sqrt{2}}{2}\gamma\mathbf{k}$, or of the form $\delta_2\frac{\sqrt{2}}{2} + \delta_2\frac{\sqrt{2}}{2}\alpha\mathbf{i} + \delta_2\frac{\sqrt{2}}{2}\beta\mathbf{j} + \delta_2\frac{\sqrt{2}}{2}\gamma\mathbf{k}$, with $\delta_1, \delta_2 \in \{-1, 1\}$ and $\delta_1\delta_2 = -1$, therefore an infinite number of solutions.

But, if we consider $A \in \mathcal{M}_2(\mathbb{H}_{\mathbb{Q}}(-1,-1))$ we have that $\sigma_L(A) = \sigma_R(A) = \emptyset$.

As we remarked above, the right spectrum of a quaternionic matrix was studied more than the left spectrum. Since the left spectrum is not always easy to found, the left eigenvalues were not so studied. It is clear that these two notions, left and right spectrum, are different and a left eigenvalue is not always a right eigenvalue and vice-versa, as we can see in the below examples.

Example 2.4. From Remark 1.1, the algebra $\mathfrak{A}_2 = \left(\frac{-4,-9}{\mathbb{R}}\right)$ is a division quaternion algebra. We consider the matrix

$$A = \left(\begin{array}{cc} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{array}\right)$$

and we compute the left and the right spectrum.

For the left spectrum, we obtain

$$\left(\begin{array}{cc} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{array}\right)\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right)=\lambda\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right)$$

and the following equations $\mathbf{j}x_2 = \lambda x_1$, $-\mathbf{j}x_1 = \lambda x_2$, $\lambda, x_1, x_2 \in \mathfrak{A}_2$ are obtained. From here, we have that $x_2 = -\frac{1}{9}\mathbf{j}\lambda x_1$ and $-\mathbf{j}x_1 = -\frac{1}{9}\lambda\mathbf{j}\lambda x_1$, which implies $9\mathbf{j} = \lambda\mathbf{j}\lambda$, therefore $(\mathbf{j}\lambda)^2 = -81$. We denote $y = \mathbf{j}\lambda$ and we obtain equation $y^2 + 81 = 0$. The solutions in $\mathfrak{A}_2$ of the equation

$$y^2 + 81 = 0$$

are of the form $y = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, with $a, \beta, \gamma \in \mathbb{R}$, such that $4\alpha^2 + 9\beta^2 + 36\gamma^2 = 81$. Since $\lambda = -\frac{1}{9}\mathbf{j}y$, it results that $\sigma_L(A)$ is infinite.

For the right spectrum, relation

$$\begin{pmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\lambda,$$

implies that $\mathbf{j}x_2 = x_1\lambda$ and $-\mathbf{j}x_1 = x_2\lambda$, $\lambda, x_1, x_2 \in \mathfrak{A}_2$. We have $x_2 = -\frac{1}{9}\mathbf{j}x_1\lambda$ and $-\mathbf{j}x_1 = -\frac{1}{9}\mathbf{j}x_1\lambda\lambda$, therefore $\lambda^2 = 9$, with $\lambda \in \mathfrak{A}_2$. It results that $\lambda \in \{-3, 3\}$. We obtain $\sigma_R(A) = \{-3, 3\}$. It is clear that $\sigma_R(A) \subset \sigma_L(A)$.

For the quaternion real division algebra the following result was proved.

Proposition 2.5. (Theorem 4.5, [HU; 01]). *If $A \in \mathcal{M}_2(\mathbb{H})$ and $\sigma_L(A)$ and $\sigma_R(A)$ are both finite, then $\sigma_L(A) = \sigma_R(A)$.*

In the following, we will generalize this result. Let $\mathfrak{A}_t$ be a division algebra obtained by the Cayley-Dickson process, and $A \in \mathcal{M}_n(\mathfrak{A}_t)$.

Proposition 2.6. *Let $A \in \mathcal{M}_n(K)$ such that $\sigma_R(A) \neq \emptyset$ and $\sigma_L(A) \neq \emptyset$, therefore the sets $\sigma_L(A)$ and $\sigma_R(A)$ have the same cardinal.*

Proof. Let $A \in \mathcal{M}_n(K)$ and $\lambda \in \sigma_L(A)$. It results that there is a nonzero matrix $X \in \mathcal{M}_{n\times 1}(\mathfrak{A}_t)$ such that $AX = \lambda X$. Taking the conjugate, we obtain

$$A\overline{X} = \overline{X}\,\overline{\lambda},$$

therefore $\overline{\lambda}$ is a right eigenvalue for the matrix A and conversely.

Definition 2.7. Let $\mathfrak{A}_t$ be a division algebra obtained by the Cayley-Dickson process and $\mathcal{A} \subseteq \mathfrak{A}_t$ be an associative subalgebra of $\mathfrak{A}_t$. An element $\lambda \in \mathcal{A}$ is called a *local $\mathcal{A}$-left eigenvalue* for the matrix A if there is a nonzero matrix $X \in \mathcal{M}_{n\times 1}(\mathcal{A})$ such that

$$AX = \lambda X.$$

The set of distinct local $\mathcal{A}$-left eigenvalues is called *the local $\mathcal{A}$-left spectrum of the matrix A in $\mathcal{A}$*, denoted $\sigma_L^{\mathcal{A}}(A)$.

An element $\lambda \in \mathcal{A}$ is called a *local $\mathcal{A}$-right eigenvalue* for the matrix A if there is a nonzero matrix $X \in \mathcal{M}_{n\times 1}(\mathcal{A})$ such that

$$AX = X\lambda.$$

The set of distinct right eigenvalues is called *the local $\mathcal{A}$- right spectrum of the matrix A*, denoted $\sigma_R^{\mathcal{A}}(A)$.

Example 2.8. We consider the real division octonion algebra $\mathfrak{A}_3 = \mathbb{O} = \left(\frac{-1,-1,-1}{\mathbb{R}}\right)$. In this algebra, the elements f_3, f_5, f_6 associate and not commute, therefore generate an associative and noncommutative subalgebra of $\mathfrak{A}_3$, denoted by $\mathcal{A}$. Let $A = \begin{pmatrix} 0 & f_3 \\ f_5 & 0 \end{pmatrix}\mathcal{M}_n(\mathcal{A})$. We compute the local $\mathcal{A}$-left and right spectrum of the matrix A.

For the local $\mathcal{A}$-left spectrum, we have $\begin{pmatrix} 0 & f_3 \\ f_5 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. We obtain the system $\begin{cases} f_3x_2 = \lambda x_1 \\ f_5x_1 = \lambda x_2 \end{cases}$. We have $x_2 = -f_3\lambda x_1$ and $f_5x_1 = \lambda x_2$. It results $f_5x_1 = -\lambda f_3\lambda x_1$, then $f_5 = -\lambda f_3\lambda$, therefore $f_3f_5 = -f_3\lambda f_3\lambda$. We denote $y = f_3\lambda$ and we get $y^2 = f_6$. It results $2y_0y - n(y) = f_6$, therefore $2y_0^2 - n(y) = 0$ and $2y_0y_6 = 1, y_3 = 0, y_5 = 0, y_0^2 - y_6^2 = 0$. We get $y_0 = y_6 = \frac{\sqrt{2}}{2}$ or $y_0 = y_6 = -\frac{\sqrt{2}}{2}$. We obtain $f_3\lambda = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}f_6$ or $f_3\lambda = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}f_6$, therefore $\lambda \in \{\frac{\sqrt{2}}{2}f_3 + \frac{\sqrt{2}}{2}f_5, -\frac{\sqrt{2}}{2}f_3 - \frac{\sqrt{2}}{2}f_5\}$.

For the local $\mathcal{A}$-right spectrum, we obtain the system$\begin{cases} f_3x_2 = x_1\lambda \\ f_5x_1 = x_2\lambda \end{cases}$. It results, $x_2 = -f_3x_1\lambda$ and $f_5x_1 = -f_3x_1\lambda^2$, therefore $f_6x_1 = -x_1\lambda^2$. From [Ti; 99], we have $\lambda^2 = 2\lambda_0\lambda - n(\lambda)$, therefore $2\lambda_0^2 - n(\lambda) = 0$ and $4(\lambda_0^2\lambda_3^2 + \lambda_0^2\lambda_5^2 + \lambda_0^2\lambda_6^2) = 1$.It results $\lambda_0^2 = \lambda_3^2 + \lambda_5^2 + \lambda_6^2$ and $4\lambda_0^4 = 1$. We obtain $\lambda_0 \in \{-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\}$ and $\lambda_3^2 + \lambda_5^2 + \lambda_6^2 = \frac{1}{4}$, with infinite many solutions.
With the above notations we can generalize Proposition 2.5.

Proposition 2.9. *Let $\mathfrak{A}_t$ be a division algebra obtained by the Cayley-Dickson process and $\mathcal{A} \subseteq \mathfrak{A}_t$ be an associative and noncommutative subalgebra of $\mathfrak{A}_t$. If $A \in \mathcal{M}_n(\mathcal{A})$ and $\sigma_R^{\mathcal{A}}(A)$ is finite, then $\sigma_R^{\mathcal{A}}(A) \subseteq \sigma_L^{\mathcal{A}}(A)$.*

Proof. Supposing that $\lambda \in \sigma_R^{\mathcal{A}}(A)$, therefore there is a nonzero matrix $X \in \mathcal{M}_{n\times 1}(\mathcal{A})$ such that $AX = X\lambda$. For $y \in \mathcal{A}, y \neq 0$, we obtain that $AXy = Xyy^{-1}\lambda y$. Since $\mathcal{A}$ is associative, it results that $A(Xy) = (Xy)(y^{-1}\lambda y)$. From here, we get that $y^{-1}\lambda y \in \sigma_R^{\mathcal{A}}(A)$. Therefore, $<\lambda>\subseteq \sigma_R^{\mathcal{A}}(A)$, where $<\lambda>= \{q \in \mathcal{A} \ / \ q = w^{-1}\lambda w, w \in \mathcal{A}, w \neq 0\}$. If $\lambda \notin K$, then $<\lambda>\nsubseteq K$ and $<\lambda>$ contains infinitely many distinct elements. Therefore, $<\lambda>$ is an infinite set. This is false, since $\sigma_R^{\mathcal{A}}(A)$ is supposed finite. Therefore $\lambda \in K, <\lambda>= \{\lambda\}, \sigma_R^{\mathcal{A}}(A) \subset K$ and, in this situation, a right eigenvalue is also a right eigenvalue. From here, we get that $\sigma_R^{\mathcal{A}}(A) \subseteq \sigma_L^{\mathcal{A}}(A)$.

3. Remarks regarding the companion matrix and fixed points for a polynomial with coefficients in division quaternion algebra $\mathbb{H}_{\mathbb{R}}(\alpha, \beta)$

Definition 3.1. [SP; 01]. If $\mathbb{H}_{\mathbb{R}}(\alpha, \beta)$ is a division quaternion algebra and the polynomial $p \in \mathbb{H}_{\mathbb{R}}(\alpha, \beta)[x]$, $p(x) = x^n + a_{n-1}x^{n-1} + ... + a_0$, the following matrix

$$C(p) = \begin{pmatrix} 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \\ -a_0 & -a_1 & ... & -a_{n-1} \end{pmatrix}$$

is called *the companion matrix* associated to the polynomial $p(x)$.

Remark 3.2. ([SP; 01], Proposition 1 and Corollary 1) If λ is a left eigenvalue associated to the companion matrix $C(p)$, then:
i) λ is a root of the polynomial $p(x)$;
ii) $w = \left(1, \lambda, \lambda^2, ..., \lambda^{m-1}\right)^t$ is a left eigenvector;
iii) λ is also a right eigenvalue.
It is clear that, due to the Remark 1.1, these results are true for all generalized real division quaternion algebras.

Example 3.3. We consider the polynomial $p(x) = x^2 + \mathbf{i}x + \mathbf{j}$,with $C(p) = \begin{pmatrix} 0 & 1 \\ -\mathbf{j} & -\mathbf{i} \end{pmatrix}$. We compute the left eigenvalues of the matrix $C(p)$. We have
$\begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{j} & -\mathbf{i} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. It results the following equations: $x_2 = \lambda x_1$ and $-\mathbf{j}x_1 - \mathbf{i}x_2 = \lambda x_2$. We obtain that $b = \mathbf{i}$ and $c = \mathbf{j}$. From Proposition 2.1, since $T = \pm 1, N = 1, \ \Delta = 0, \Gamma^2 < 4\Psi$, therefore, the solutions of this equation are $\lambda_1 = -(\mathbf{i}{+}1)^{-1}(\mathbf{j}{-}1)$ and $\lambda_2 = -(\mathbf{i}{-}1)^{-1}(\mathbf{j}{-}1)$. We have $\lambda_1 = \frac{1}{2}(\mathbf{i}{-}1)(\mathbf{j}{-}1) = \frac{1}{2}(1 - \mathbf{i} - \mathbf{i} + \mathbf{k})$ and $\lambda_2 = \frac{1}{2}(\mathbf{i}{+}1)(\mathbf{j}{-}1) = \frac{1}{2}(-1 - \mathbf{i} + \mathbf{j} + \mathbf{k})$. For $\lambda_1 = \frac{1}{2}(1 - \mathbf{i} - \mathbf{j} + \mathbf{k})$, the associated eigenvector is $w_1 = \left(1, \frac{1}{2} - \frac{\mathbf{i}}{2} - \frac{\mathbf{j}}{2} + \frac{\mathbf{k}}{2}\right)^t$ and for $\lambda_2 = \frac{1}{2}(-1 - \mathbf{i} + \mathbf{j} + \mathbf{k})$, the associated eigenvector is $w_2 = \left(1, -\frac{1}{2} - \frac{\mathbf{i}}{2} + \frac{\mathbf{j}}{2} + \frac{\mathbf{k}}{2}\right)^t$.

Proposition 3.4. *If* $\mathbb{H}_{\mathbb{R}}(\alpha,\beta)$ *is a division quaternion algebra, the matrices* A *of the form* $\begin{pmatrix} 0 & 1 \\ b & c \end{pmatrix}$, *with* $b, c \in \mathbb{H}_{\mathbb{R}}(\alpha,\beta) - \mathbb{R}$, *have* $\sigma_L(A) \subset \sigma_R(A)$. *The set* $\sigma_L(A)$ *has two elements and* $\sigma_R(A)$ *is always infinite.*

Proof. The matrix A is the companion matrix of the polynomial $p(x) = x^2 - cx - b$.From Proposition 2.2, Case 4, $p(x)$ has two roots. For a companion matrix a left eigenvalue is also a right eigenvalue, therefore $\sigma_L(A) \subset \sigma_R(A)$ and $\sigma_R(A)$ is not included in the ground field. From here, we have that $\sigma_R(A)$ is infinite.

Definition 3.5. ([CV; 21], Corollary 4.8**)** If $\mathbb{H}_{\mathbb{R}}(\alpha,\beta)$ is a division quaternion algebra and a polynomial $p \in \mathbb{H}_{\mathbb{R}}(\alpha,\beta)[x]$. *A fixed point* for the polynomial p is an element $x_0 \in \mathbb{H}_{\mathbb{R}}(\alpha,\beta)$ such that $p(x_0) = x_0$.

Proposition 3.6. *If* $\mathbb{H}_{\mathbb{R}}(\alpha,\beta)$ *is a division quaternion algebra, we consider the polynomial* $p \in \mathbb{H}_{\mathbb{R}}(\alpha,\beta)[x]$, $p(x) = x^2 + bx + c, b, c \in \mathbb{H}_{\mathbb{R}}(\alpha,\beta)$.

1) If $b, c \in \mathbb{R}$ *and* $(b-1)^2 < 4c$, *then* p *has an infinity of fixed points;*
2) If $b, c \in \mathbb{R}$ *and* $(b-1)^2 \geq 4c$, *or* $b \notin \mathbb{R}$, *then* p *has maximum two distinct fixed points;*
3) *If* $b \in \mathbb{R}$, $c \notin \mathbb{R}$ *then* p *has maximum* 16 *distinct fixed points.*

Proof. It is clear from Proposition 2.2.

If $\mathbb{H}_{\mathbb{R}}(\alpha,\beta)$ is a division quaternion algebra, we consider the equation

$$x^2 + xb + c = 0, \tag{3.1.}$$

with $b, c \in \mathbb{H}_{\mathbb{R}}(\alpha,\beta)$, which are not in the cases described in Proposition 2.2, but can be reduced to those situations. Let $p(x) = x^2 + xb + c$. We remark that $\overline{p(x)} = \overline{x^2 + xb + c} = \overline{x}^2 + \overline{b}\overline{x} + \overline{c}$. Therefore, if x_0 is a solution to the equation (3.1), then its conjugate $\overline{x}_0$ is a solution to the equation

$$\overline{x}^2 + \overline{b}\overline{x} + \overline{c} = 0, \tag{3.2.}$$

called *the conjugate* of the equation (3.1). Conversely, if x_0 is a solution to the equation (3.2), then $\overline{x}_0$ is a solution to the equation (3.1). From here, we conclude that the polynomial $p(x)$ and $\overline{p(x)}$ has the same number of fixed points.

Under the above conditions, we consider the equation

$$x^2 + Ax + xB + C = 0, \tag{3.3.}$$

with $A, B, C \in \mathbb{H}_{\mathbb{R}}(\alpha,\beta)$, which also are not in the cases described in Proposition 2.2, but can be reduced to those situations. We consider the polynomial $q(x) = x^2 + Ax + xB + C$. We have that $x^2 + Ax + xB + C = (x+B)^2 + (A-B)(x+B) - AB + C$. Denoting $y = x + B, \mu = A - B$ and $\eta = -AB + C$, it results the following equation

$$y^2 + \mu y + \eta = 0. \tag{3.4}$$

Therefore, if z is a solution to the equation (3.3), then $z + B$ is a solution to the equation (3.4) and if w is a solution to the equation (3.4), therefore $w - B$ is a solution to the equation (3.3). From here, we conclude that the polynomials $q(x)$ and $p(y) = y^2 + \mu y + \eta$ have the same number of fixed points.

Example 3.7. In $\mathbb{H}$ the polynomial $q(x) = x^2 + \mathbf{i}x + x\mathbf{i} + 1$ has the same number of fixed points as the polynomial $p(y) = y^2 + 2$. The fixed points for the polynomial p are the solution of the equation $y^2 + 2 = y$. Therefore, we obtain the equation $y^2 - y + 2 = 0$. From Proposition 2.2, the solutions are

$$y = \frac{1}{2}(-b + r\mathbf{i} + s\mathbf{j} + t\mathbf{k}),$$

with $r^2 + s^2 + t^2 = 7$ where $r, s, t \in \mathbb{R}$, that means an infinity number of fixed points. It results that the fixed points for the polynomial q are of the form $y - \mathbf{i}$,

$$x = \frac{1}{2}(-b + (r-2)\mathbf{i} + s\mathbf{j} + t\mathbf{k}),$$

with $r^2 + s^2 + t^2 = 7$ where $r, s, t \in \mathbb{R}$.

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