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The A-Davis-Wielandt Berezin number of semi Hilbert operators with some related inequalities

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Abstract. In this article, the concept of the A-Davis-Wielandt Berezin number is introduced for positive operator A. Some upper and lower bounds for the A-Davis-Wielandt Berezin number are proved. Moreover, some inequalities related to the concept of the Davis-Wielandt Berezin number are obtained, which are generalizations of known results. Among them, it is shown that

$$ber^2_{dw}(S) \leq \inf_{\gamma\in\mathbb{C}}\{(2\|Re(\gamma)Re(S)+Im(\gamma)Im(S)\|+\|S^*S-2Re(\bar{\gamma}S)\|)^2+2\|Re(\bar{\gamma}S)\|-|\gamma|^2+ber^2(S-\gamma I)\},$$

where $S\in B(\mathcal{H}(\Omega))$. Also, we determined the exact value of the A-Davis–Wielandt Berezin number of some special type of operator matrices.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H};\langle\cdot,\cdot\rangle)$ with the identity operator $1_{\mathcal{H}}$ in $\mathcal{B}(\mathcal{H})$. For $S\in\mathcal{B}(\mathcal{H})$, we denote by $\mathcal{R}(S)$ and $\mathcal{N}(S)$ the range and the null space of S, respectively. Every where this paper, we suppose that $A\in\mathcal{B}(\mathcal{H})$ is positive operator. Recall that A is called positive, denoted by $A\geq 0$, if $\langle Ax,x\rangle\geq 0$ for all $x\in\mathcal{H}$. Such an operator A induces positive semidefinite sesquilinear form as follows:

$$\begin{aligned}\langle .,.\rangle_A :&\mathcal{H}\times\mathcal{H}\longrightarrow\mathbb{C}\\ &(x,y)\mapsto\langle x,y\rangle_A=\langle Ax,y\rangle,\end{aligned}$$

and $\|x\|_A=\sqrt{\langle x,x\rangle_A},\ \ x\in\mathcal{H}$, is the seminorm induced by the above sesquilinear form. This make $\mathcal{H}$ into a semi-Hilbertian space. Since $\|x\|_A=0$ if and only if $x\in\mathcal{N}(A)$, then $\|.\|_A$ is a norm on $\mathcal{H}$ if and only if A is injective. Also, $(\mathcal{H},\|.\|_A)$ is complete if and only if $\mathcal{R}(A)$ is closed in $\mathcal{H}$.

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For $S \in \mathcal{B}(\mathcal{H})$, an operator $R \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of S if for every $x, y \in \mathcal{H}$, we have $\langle Sx, y\rangle_A = \langle x, Ry\rangle_A$, *i.e.*, $AR = S^*A$.
The set of all operator which admit A-adjoints is denoted by $\mathcal{B}_A(\mathcal{H})$. If $S \in \mathcal{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = S^*A$ is a distinguished A-adjoints operator of S, which is denoted by $S^{\sharp_A}$. Note that $S^{\sharp_A} = A^{\dagger}S^*A$ in which $A^{\dagger}$ is the Moore-Penrose inverse of A [2]. It is useful that if $S \in \mathcal{B}_A(\mathcal{H})$, then $AS^{\sharp A} = S^*A$.
An operator $S \in \mathcal{B}(\mathcal{H})$ is called A-selfadjoint if AS is selfadjoint, i.e., $AS = S^*A$ and it is called A-positive if $AS \geq 0$. An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be A-bounded if there exists $c > 0$ such that $\|Sx\|_A \leq c\|x\|_A$, for all $x \in \mathcal{H}$. We denote by $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, the collection of all A-bounded operators, i.e.,

$$\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) = \{S \in \mathcal{B}(\mathcal{H}) : \exists c > 0 \ \ s.t. \ \|Sx\|_A \leq c\|x\|_A, \ \forall \ x \ \in \mathcal{H}\}.$$

Note that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ are two subalgebras of $\mathcal{B}(\mathcal{H})$ which are neither closed nor dense in $\mathcal{B}(\mathcal{H})$. Moreover, the inclusions

$$\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$$

hold with equality if A is one-to-one and has closed range [2],[3],[26].
An operator $S \in \mathcal{B}_A(\mathcal{H})$ is called A-normal if $SS^{\sharp_A} = S^{\sharp_A}S$.
An operator $U \in \mathcal{B}_A(\mathcal{H})$ is called A-unitary if $\|Ux\|_A = \|U^{\sharp_A}x\|_A = \|x\|_A$ for all $x \in \mathcal{H}$. In [3], showed that an operator $U \in \mathcal{B}_A(\mathcal{H})$ is A-unitary if and only if $U^{\sharp_A}U = (U^{\sharp_A})^{\sharp_A}U^{\sharp_A} = P_A$, where P_A denotes the projection onto $\overline{\mathcal{R}(A)}$.
For $S \in \mathcal{B}_A(\mathcal{H})$, we write $|S|_A^2 = S^{\sharp_A}S$, $Re_A(S) = \frac{1}{2}(S + S^{\sharp_A})$ and $Im_A(S) = \frac{1}{2i}(S - S^{\sharp_A})$. For $T, S \in \mathcal{B}_A(\mathcal{H})$, $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$, $\|TS\|_A \leq \|T\|_A\|S\|_A$ and $\|Sx\|_A \leq \|S\|_A\|x\|_A$, for all $x \in \mathcal{H}$ [3].

The function k on $\Omega \times \Omega$ defined by $k(z, \mu) = k_\mu(z)$ is called the reproducing kernel of $\mathcal{H}$, see [18]. It was shown that $k_\mu(z)$ can be represented by

$$k_\mu(z) = \sum_{n=1}^{\infty} \overline{e_n(\mu)} e_n(z)$$

for any orthonormal basis $\{e_n\}_{n\geq 1}$ of $\mathcal{H}$.
Let $\widehat{k_\mu} = \frac{k_\mu}{\|k_\mu\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For a given a bounded linear operator S on $\mathcal{H}$, the Berezin symbol(or Berezin transform) of S is the bounded function $\widetilde{S}$ on Ω defined by

$$\widetilde{S}(\mu) = \langle S\widehat{k_\mu}(z), \widehat{k_\mu}(z)\rangle, \ \mu \in \Omega.$$

Berezin number of an operator S are defined, respectively, by

$$\text{Ber}(S) = \left\{\widetilde{S}(\mu) : \mu \in \Omega\right\} = \text{Range}(\widetilde{S}),$$

and

$$\text{ber}(S) = \sup\left\{|\gamma| : \gamma \in \text{Ber}(S)\right\} = \sup_{\mu\in\Omega}\left|\widetilde{S}(\mu)\right|.$$

The Berezin norm of an operator $S \in \mathcal{B}(\mathcal{H})$ is defined by

$$\|S\|_{\text{Ber}} := \sup_{\mu\in\Omega}\left\|S\widehat{k_\mu}\right\|.$$

For more details, see [6, 15, 17, 21] and the references therein.
Recall that the numerical range, the numerical radius, and the Crawford number of $S \in \mathcal{B}(\mathcal{H})$ are defined respectively, by

$$W(S) := \{\langle Sx, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\},$$

$$w(S) := \sup\{|\langle Sx, x\rangle| : \langle Sx, x\rangle \in W(S)\},$$

and

$$C(S) := \inf\{|\langle Sx, x\rangle| : x \in \mathcal{H}, \| x \|= 1\}.$$

Clearly, $\mathrm{Ber}(S) \subset W(S)$ and $\mathrm{ber}(S) \leq w(S)$. Suppose that $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$. The A-operator seminorm, the A-numerical rang, the A-numerical radius and the A-Crawford number of S are defined, respectively as follows:

$$\|S\|_A = \sup\left\{\frac{\|Sx\|_A}{\|x\|_A} : x \neq 0,\ x \in \overline{\mathcal{R}(A)}\right\} = \sup\{\|Sx\|_A : x \in \mathcal{H},\ \|x\|_A = 1\},$$

$$W_A(S) := \{\langle Sx, x\rangle_A : x \in \mathcal{H}, \text{ and } \|x\|_A = 1\},$$

$$w_A(S) := \sup\{|c| : c \in W_A(S)\},$$

and

$$C_A(S) := \inf\{|c| : c \in W_A(S)\}.$$

It is well known that $\|\,.\,\|_A$ and $w_A(\cdot)$ are equivalent seminorm on $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, in which

$$\frac{1}{2}\|S\|_A \leq w_A(S) \leq \|S\|_A.$$

The first inequality becomes equality if $AS^2 = 0$, and the second inequality becomes equality if S is A-normal [13]. One of the most less common celebrated generalization of the numerical range, and the numerical radius is the Davis-Wielandt shell, and its radius of $S \in \mathcal{B}(\mathcal{H})$, which are defined in [10, 11, 25], as follows:

$$DW(S) := \{(\langle Sx, x\rangle, \|Sx\|) : x \in \mathcal{H}, \|x\| = 1\},$$

and

$$dw(S) = \sup\{\sqrt{|\langle Sx, x\rangle|^2 + \|Sx\|^4} : x \in \mathcal{H}, \|x\| = 1\}. \tag{1}$$

Unlike the numerical radius, the Davis-Wielandt radius is not a norm. It has many properties that you can refer to reference [23, 24, 27].
The A-Davis-Wielandt shell and the A-Davis-Wielandt radius of an operator $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ are defined, respectively in [14], as follows:

$$DW_A(S) := \{(\langle Sx, x\rangle_A, \langle Sx, Sx\rangle_A) : x \in \mathcal{H}, \|x\|_A = 1\},$$

and

$$dw_A(S) := \sup\{\sqrt{|\langle Sx, x\rangle_A|^2 + \|Sx\|_A^4} : x \in \mathcal{H}, \|x\|_A = 1\}. \tag{2}$$

It is easy to see that the A-Davis-Wielandt radius of an operator $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ satisfying the following inequality:

$$\max(w_A(S)\,,\,\|S\|_A^2) \leq dw_A(S) \leq \sqrt{w_A^2(S) + \|S\|_A^4}. \tag{3}$$

The Crawford Berezin number, and the minimum Berezin modulus of the operator S are defined by

$$C_{\mathrm{Ber}}(S) := \inf\{|\widetilde{S}(\mu)| : \mu \in \Omega\}, \text{ and } \quad m_{\mathrm{Ber}}(S) := \inf\{\|S\widehat{k_\mu}\| : \mu \in \Omega\},$$

respectively.
Also, the concepts Davis-Wielandt Berezin set and Davis-Wielandt Berezin number have been introduced in [1], as follows:

$$Ber_{dw}(S) := \{(\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle, \langle S\widehat{k_\mu}, S\widehat{k_\mu}\rangle) \;:\; \mu \in \Omega\},$$

and

$$ber_{dw}(S) := \sup_{\mu\in\Omega}\{\sqrt{|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2 + \|S\widehat{k_\mu}\|^4}\}.$$

We can clearly see that $ber_{dw}(S)$ is a generalization of $ber(S)$, and also $dw(S) \leq ber_{dw}(S)$. The following properties of $ber_{dw}(S)$, $S \in \mathcal{B}(\mathcal{H}(\Omega))$ are also known:
(i) $ber_{dw}(\cdot)$ does not define a norm on $\mathcal{B}(\mathcal{H}(\Omega))$ but it is unitarily invariant, i.e., $ber_{dw}(U^*SU) = ber_{dw}(S)$ for any unitarily operator $U \in \mathcal{B}(\mathcal{H}(\Omega))$.
(ii)

$$\max\left(ber(S)\,,\,\|S\|_{Ber}^2\right) \leq ber_{dw}(S) \leq \sqrt{ber^2(S) + \|S\|_{Ber}^4}. \tag{4}$$

The paper is organized as follows: In the next section, we obtain several new inequalities for the Davis-Wielandt Berezin number of bounded linear operators on $\mathcal{H}(\Omega)$. In Section 3 for positive operator A, we introduce an extension of the Davis–Wielandt Berezin number and finally we give some inequalities for the A- Davis–Wielandt Berezin number of operator matrices.

2. Some inequalities of the Davis–Wielandt Berezin number

In this section, we give some inequalities of the Davis-Wielandt Berezin number. At first, we give a generalization of (4).
From the norm properties of vectors $s, t \in \mathcal{H}$, it can be shown that [12]:

$$\|s\|^2\|t\|^2 - |\langle s, t\rangle|^2 = \|s - \gamma t\|^2\|t\|^2 - |\langle s - \gamma t, t\rangle|^2, \quad (\gamma \in \mathbb{C}). \tag{5}$$

Theorem 2.1. *Assume that $S \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$\begin{aligned} ber_{dw}^2(S) \leq \inf_{\gamma\in\mathbb{C}}\{&(2\|Re(\gamma)Re(S) + Im(\gamma)Im(S)\| + \|S^*S - 2Re(\bar{\gamma}S)\|)^2 + 2\|Re(\bar{\gamma}S)\| \\ &- |\gamma|^2 + ber^2(S - \gamma I)\}. \end{aligned} \tag{6}$$

Proof. Let $\widehat{k_\mu} \in \mathcal{H}(\Omega)$ be a normalized reproducing kernel, and $\gamma \in \mathbb{C}$. By the Cartesian decomposition of S, we have

$$\begin{aligned}
\|S\widehat{k_\mu}\|^2 &= (\langle Re(S)\widehat{k_\mu},\widehat{k_\mu}\rangle)^2 - (\langle Re(S-\gamma I)\widehat{k_\mu},\widehat{k_\mu}\rangle)^2 + (\langle Im(S)\widehat{k_\mu},\widehat{k_\mu}\rangle)^2 - (\langle Im(S-\gamma I)\widehat{k_\mu},\widehat{k_\mu}\rangle)^2 \\
&\quad + \|S\widehat{k_\mu} - \gamma\widehat{k_\mu}\|^2 \\
&= \langle (2Re(S) - Re(\gamma)I)\widehat{k_\mu},\widehat{k_\mu}\rangle\langle Re(\gamma)\widehat{k_\mu},\widehat{k_\mu}\rangle + \langle (2Im(S) - Im(\gamma)I)\widehat{k_\mu},\widehat{k_\mu}\rangle\langle Im(\gamma)\widehat{k_\mu},\widehat{k_\mu}\rangle \\
&\quad + \|S\widehat{k_\mu} - \gamma\widehat{k_\mu}\|^2 \\
&= 2Re(\gamma)\langle Re(S)\widehat{k_\mu},\widehat{k_\mu}\rangle + 2Im(\gamma)\langle Im(S)\widehat{k_\mu},\widehat{k_\mu}\rangle - (Re(\gamma))^2 - (Im(\gamma))^2 + \|S\widehat{k_\mu} - \gamma\widehat{k_\mu}\|^2 \\
&= 2(Re(\gamma)\langle Re(S)\widehat{k_\mu},\widehat{k_\mu}\rangle + Im(\gamma)\langle Im(S)\widehat{k_\mu},\widehat{k_\mu}\rangle) - |\gamma|^2 + \langle S\widehat{k_\mu} - \gamma\widehat{k_\mu}, S\widehat{k_\mu} - \gamma\widehat{k_\mu}\rangle \\
&= 2(Re(\gamma)\langle Re(S)\widehat{k_\mu},\widehat{k_\mu}\rangle + Im(\gamma)\langle Im(S)\widehat{k_\mu},\widehat{k_\mu}\rangle) + \langle (S^*S - 2Re(\bar{\gamma}S))\widehat{k_\mu},\widehat{k_\mu}\rangle \\
&\leq 2\|Re(\gamma)Re(S) + Im(\gamma)Im(S)\| + \|S^*S - 2Re(\bar{\gamma}S)\|.
\end{aligned}$$

On the other hand, by applying (5), we have

$$\begin{aligned}
|\langle S\widehat{k_\mu},\widehat{k_\mu}\rangle|^2 &= \|S\widehat{k_\mu}\|^2 - \|S\widehat{k_\mu} - \gamma\widehat{k_\mu}\|^2 + |\langle S\widehat{k_\mu} - \gamma\widehat{k_\mu},\widehat{k_\mu}\rangle|^2 \\
&= 2\langle Re(\bar{\gamma}S)\widehat{k_\mu},\widehat{k_\mu}\rangle - |\gamma|^2 + |\langle S\widehat{k_\mu} - \gamma\widehat{k_\mu},\widehat{k_\mu}\rangle|^2 \\
&\leq 2\|Re(\bar{\gamma}S)\| - |\gamma|^2 + ber^2(S-\gamma I).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\langle S\widehat{k_\mu},\widehat{k_\mu}\rangle|^2 + \|S\widehat{k_\mu}\|^4 &\leq 2\|Re(\bar{\gamma}S)\| - |\gamma|^2 + ber^2(S-\gamma I) + (2\|Re(\gamma)Re(S) + Im(\gamma)Im(S)\| \\
&\quad + \|S^*S - 2Re(\bar{\gamma}S)\|)^2.
\end{aligned}$$

By taking the supremum over $\mu \in \Omega$, and infimum over $\gamma \in \mathbb{C}$, we deduce that

$$\begin{aligned}
ber_{dw}^2(S) \leq \inf_{\gamma\in\mathbb{C}}\{&(2\|Re(\gamma)Re(S) + Im(\gamma)Im(S)\| + \|S^*S - 2Re(\bar{\gamma}S)\|)^2 + 2\|Re(\bar{\gamma}S)\| \\
&- |\gamma|^2 + ber^2(S-\gamma I)\}.
\end{aligned}$$

□

Remark 2.2. *By considering $\gamma = 0$ in* (6)*, we have*

$$\begin{aligned}
ber_{dw}(S) &\leq \sqrt{ber^2(S) + \|S^*S\|^2} \\
&\leq \sqrt{ber^2(S) + \|S\|^4}.
\end{aligned}$$

So (6) *is a generalization of* (4).

Theorem 2.3. *Let $S \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$ber_{dw}^2(S) \leq \min_{0\leq\theta\leq2\pi} ber^2(|S|^2 + e^{i\theta}S) + 2\|S\|^2 ber(S). \tag{7}$$

Proof. Let $\widehat{k_\mu} \in \mathcal{H}(\Omega)$ be a normalized reproducing kernel. Then

$$\begin{aligned}
|\langle S\widehat{k_\mu},\widehat{k_\mu}\rangle|^2 + \|S\widehat{k_\mu}\|^4 &= |\langle S\widehat{k_\mu},\widehat{k_\mu}\rangle + \langle S\widehat{k_\mu}, S\widehat{k_\mu}\rangle|^2 - 2Re(\langle S\widehat{k_\mu}, S\widehat{k_\mu}\rangle\langle S\widehat{k_\mu},\widehat{k_\mu}\rangle) \\
&\leq |\langle (S^2 + S)\widehat{k_\mu},\widehat{k_\mu}\rangle|^2 + 2\|S\widehat{k_\mu}\|^2 \\
&\leq ber^2(|S|^2 + S) + 2\|S\|^2 ber(S).
\end{aligned}$$

By taking the supremum over $\mu \in \Omega$, we have

$$ber_{dw}^2(S) \le ber^2(|S|^2 + S) + 2\|S\|^2 ber(S).$$

If we replace S with $e^{i\theta}S$ for $\theta \in [0, 2\pi]$, and taking minimum over θ, the desired result obtained. □

Theorem 2.4. *Suppose that $S \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$ber_{dw}^2(S) \ge \max\left\{\left(1 + m_{\text{Ber}}^2(S)\right) ber^2(S), \left(1 + \|S^2\|_{Ber}\right) C_{Ber}^2(S)\right\},$$

where $C_{Ber}(S) := \inf\{|\widetilde{S}(\mu)| : \mu \in \Omega\}$, and $m_{Ber}(S) := \inf\{\|S\widehat{k}_\mu\| : \mu \in \Omega\}$.

Proof. Let $\widehat{k}_\mu \in \mathcal{H}(\Omega)$ be a normalized reproducing kernel. Hence, we have

$$\begin{aligned}
|\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 + \|S\widehat{k}_\mu\|^4 &\ge |\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 + |\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 \|S\widehat{k}_\mu\|^2 \\
&\ge \left(1 + \|S\widehat{k}_\mu\|^2\right) C_{\text{Ber}}^2(S).
\end{aligned}$$

By taking the supremum over normalized reproducing kernels $\widehat{k}_\mu \in \mathcal{H}(\Omega)$, we have

$$ber_{dw}^2(S) \ge \left(1 + \|S\|_{\text{Ber}}^2\right) C_{\text{Ber}}^2(S). \tag{8}$$

Also, we have

$$\begin{aligned}
|\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 + \|S\widehat{k}_\mu\|^4 &= |\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 + \|S\widehat{k}_\mu\|^2 \|S\widehat{k}_\mu\|^2 \\
&\ge |\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 + |\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2 m_{\text{Ber}}^2(S) \\
&= \left(1 + m_{\text{Ber}}^2(S)\right) |\langle S\widehat{k}_\mu, \widehat{k}_\mu\rangle|^2.
\end{aligned}$$

By taking the supremum over normalized reproducing kernels $\widehat{k}_\mu \in \mathcal{H}(\Omega)$, we deduce that

$$ber_{dw}^2(S) \ge \left(1 + m_{\text{Ber}}^2(S)\right) ber^2(S). \tag{9}$$

Combining (8) and (9), we have the required result. □

We need a sequence of lemmas to prove our results.

Lemma 2.5. *[19] Let $s, t \ge 0$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.*
Then, $st \le \frac{s^p}{p} + \frac{t^q}{q} \le (\frac{s^{pr}}{p} + \frac{t^{qr}}{q})^{\frac{1}{r}}$ for $r \ge 1$. For $r = 1$, we recapture the Power-Mean inequality, which reads

$$s^\alpha t^{1-\alpha} \le \alpha s + (1-\alpha)\, t \le (\alpha s^p + (1-\alpha)\, t^p)^{\frac{1}{p}} \tag{10}$$

for all $\alpha \in [0, 1]$ and $p \ge 1$.

The next lemma follows from the spectral theorem for positive operators and Jensen's inequality, see [22].

Lemma 2.6. *(McCarty inequality). Let $S \in \mathcal{B}(\mathcal{H})$, $S \ge 0$ and $x \in \mathcal{H}$ be a unit vector. Then*
(a) $\langle S^r x, x\rangle \le \langle Sx, x\rangle^r$ for $0 < r \le 1$;
(b) $\langle Sx, x\rangle^r \le \langle S^r x, x\rangle$ for $r \ge 1$.

Lemma 2.7. *[22, Theorem 1] Let $S \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$ be any vectors.*
(a) If f, g are non-negative continuous functions on $[0, \infty)$ satisfying the relation $f(t)g(t) = t(t \in [0, \infty))$, then

$$\left|\langle Sx, y\rangle\right| \leq \left\|f(|S|)\, x\right\| \left\|g(|S^*|)\, y\right\|;$$

(b) If $0 \leq \alpha \leq 1$, then

$$\left|\langle Sx, y\rangle\right|^2 \leq \langle |S|^{2\alpha}\, x, x\rangle\langle |S^*|^{2(1-\alpha)}\, y, y\rangle.$$

Theorem 2.8. *Assume that $S \in \mathcal{B}(\mathcal{H}(\Omega))$. Then for $0 \leq \alpha \leq 1$,*

$$ber^2_{dw}(S) \leq \frac{1}{2} ber\left(|S|^{4\alpha} + |S^*|^{4(1-\alpha)} + 2|S|^4\right). \tag{11}$$

Proof. Let $\widehat{k_\mu} \in \mathcal{H}(\Omega)$ be a normalized reproducing kernel. Then

$$\begin{aligned}
|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2 + \|S\widehat{k_\mu}\|^4 &\leq \langle |S|^{2\alpha}\widehat{k_\mu}, \widehat{k_\mu}\rangle\langle |S^*|^{2(1-\alpha)}\widehat{k_\mu}, \widehat{k_\mu}\rangle + \|S\widehat{k_\mu}\|^4 \\
&\qquad \text{(by Lemma 2.7(b))} \\
&\leq \frac{1}{2}\left(\langle |S|^{2\alpha}\widehat{k_\mu}, \widehat{k_\mu}\rangle^2 + \langle |S^*|^{2(1-\alpha)}\widehat{k_\mu}, \widehat{k_\mu}\rangle^2\right) + \|S\widehat{k_\mu}\|^4 \\
&\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&\leq \frac{1}{2}\left(\langle |S|^{4\alpha}\widehat{k_\mu}, \widehat{k_\mu}\rangle + \langle |S^*|^{4(1-\alpha)}\widehat{k_\mu}, \widehat{k_\mu}\rangle\right) + \langle |S|^4\widehat{k_\mu}, \widehat{k_\mu}\rangle \\
&\qquad \text{(by Lemma 2.6(b))} \\
&\leq \frac{1}{2} ber\left(|S|^{4\alpha} + |S^*|^{4(1-\alpha)} + 2|S|^4\right).
\end{aligned}$$

By taking the supremum over $\lambda \in \Omega$, we have

$$ber^2_{dw}(S) \leq \frac{1}{2} ber\left(|S|^{4\alpha} + |S^*|^{4(1-\alpha)} + 2|S|^4\right).$$

□

In the next theorem, we give another inequality which gives the upper bound for the Davis-Wielant Berezin number of bounded linear operators.

Theorem 2.9. *Suppose that $S \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$ber^2_{dw}(S) \leq \frac{1}{2} ber\left((|S| + |S^*|)^2 + 2|S|^4\right) - C_{\mathrm{Ber}}(|S|)C_{\mathrm{Ber}}(|S^*|). \tag{12}$$

Proof. Let $\widehat{k_\mu} \in \mathcal{H}(\Omega)$ be a normalized reproducing kernel. Then

$$\begin{aligned}
|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2 + \|S\widehat{k_\mu}\|^4 &\leq \langle |S|\widehat{k_\mu}, \widehat{k_\mu}\rangle\langle |S^*|\widehat{k_\mu}, \widehat{k_\mu}\rangle + \|S\widehat{k_\mu}\|^4 \\
&\qquad \text{(by Lemma 2.7(b))} \\
&\leq \frac{1}{2}\left(\langle |S|\widehat{k_\mu}, \widehat{k_\mu}\rangle^2 + \langle |S^*|\widehat{k_\mu}, \widehat{k_\mu}\rangle^2\right) + \|S\widehat{k_\mu}\|^4 \\
&\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{1}{2}\left(\langle |S|\widehat{k_\mu}, \widehat{k_\mu}\rangle + \langle |S^*|\widehat{k_\mu}, \widehat{k_\mu}\rangle\right)^2 + \|S\widehat{k_\mu}\|^4 - \langle |S|\widehat{k_\mu}, \widehat{k_\mu}\rangle\langle |S^*|\widehat{k_\mu}, \widehat{k_\mu}\rangle \\
&= \frac{1}{2}\langle (|S| + |S^*|)\widehat{k_\mu}, \widehat{k_\mu}\rangle^2 + \langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle^2 - \langle |S|\widehat{k_\mu}, \widehat{k_\mu}\rangle\langle |S^*|\widehat{k_\mu}, \widehat{k_\mu}\rangle \\
&\leq \frac{1}{2} ber\left(|S| + |S^*| + 2|S|^4\right) - \inf\langle |S|\widehat{k_\mu}, \widehat{k_\mu}\rangle \inf\langle |S^*|\widehat{k_\mu}, \widehat{k_\mu}\rangle.
\end{aligned}$$

By taking the supremum over $\mu \in \Omega$, we have

$$ber^2_{dw}(S) \leq \frac{1}{2} ber\left(|S| + |S^*| + 2|S|^4\right) - C_{\mathrm{Ber}}(|S|)C_{\mathrm{Ber}}(|S^*|).$$

□

Theorem 2.10. *Let $S \in \mathcal{B}(\mathcal{H}(\Omega))$. Let f, g be non-negative continuous functions on $[0,\infty)$, which are satisfying the relation $f(t)g(t) = t(t \in [0,\infty))$. Then for $r \geq 1$ and $0 \leq \alpha \leq 1$, we have*

$$ber^2_{dw}(S) \leq \frac{2^r}{4} ber\left(\alpha M + (1-\alpha)N + f^{2r}(|S|^4) + g^{2r}(|S|^4)\right),$$

where

$$M = f^{2r}(|S|^2) + g^{2r}(|S|^2), \qquad and \qquad N = f^{2r}(|S^*|^2) + g^{2r}(|S^*|^2).$$

Proof. *Suppose that $\widehat{k_\mu} \in \mathcal{H}(\Omega)$ is a normalized reproducing kernel. Then, we deduce that*

$$\begin{aligned}
\left(|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2 + \|S\widehat{k_\mu}\|^4\right)^r &= \left(|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2 + |\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2\right)^r \\
&\leq \frac{2^r}{2}\left(|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r} + |\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r}\right) \\
&\qquad \text{(by convexity of } f(t) = t^r) \\
&= \frac{2^r}{2}\left(\alpha|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r} + (1-\alpha)|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r} + |\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r}\right) \\
&\leq \frac{2^r}{2}\left(\alpha\|S\widehat{k_\mu}\|^{2r} + (1-\alpha)\|S^*\widehat{k_\mu}\|^{2r} + |\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r}\right) \\
&\leq \frac{2^r}{2}\left(\alpha\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle^r + (1-\alpha)\langle |S^*|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle^r + |\langle |S|^4\widehat{k_\mu}, \widehat{k_\mu}\rangle|^r\right) \\
&\qquad \text{(by Lemma 2.6(b))} \\
&\leq \frac{2^r}{2}\Big[\alpha\langle f^2(|S|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle^{\frac{r}{2}}\langle g^2(|S|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle^{\frac{r}{2}} + (1-\alpha)\langle f^2(|S^*|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle^{\frac{r}{2}} \\
&\qquad \langle g^2(|S^*|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle^{\frac{r}{2}} + \langle f^2(|S|^4)\widehat{k_\mu}, \widehat{k_\mu}\rangle^{\frac{r}{2}}\langle g^2(|S|^4)\widehat{k_\mu}, \widehat{k_\mu}\rangle^{\frac{r}{2}}\Big] \\
&\qquad \text{(by Lemma 2.7(a))} \\
&\leq \frac{2^r}{2}\Big[\frac{\alpha}{2}\left(\langle f^2(|S|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle^r + \langle g^2(|S|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle^r\right) \\
&\qquad + \frac{(1-\alpha)}{2}\left(\langle f^2(|S^*|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle^r + \langle g^2(|S^*|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle^r\right) \\
&\qquad + \frac{1}{2}\left(\langle f^2(|S|^4)\widehat{k_\mu}, \widehat{k_\mu}\rangle^r + \langle g^2(|S|^4)\widehat{k_\mu}, \widehat{k_\mu}\rangle^r\right)\Big] \\
&\qquad \text{(by the arithmetic-geometric mean inequality)} \\
&\leq \frac{2^r}{2}\Big[\frac{\alpha}{2}\left(\langle f^{2r}(|S|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle + \langle g^{2r}(|S|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle\right) \\
&\qquad + \frac{(1-\alpha)}{2}\left(\langle f^{2r}(|S^*|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle + \langle g^{2r}(|S^*|^2)\widehat{k_\mu}, \widehat{k_\mu}\rangle\right) \\
&\qquad + \frac{1}{2}\left(\langle f^{2r}(|S|^4)\widehat{k_\mu}, \widehat{k_\mu}\rangle + \langle g^{2r}(|S|^4)\widehat{k_\mu}, \widehat{k_\mu}\rangle\right)\Big] \\
&\qquad \text{(by Lemma 2.6(b))} \\
&= \frac{2^r}{4}\left\langle(\alpha M + (1-\alpha)N + f^{2r}(|S|^4) + g^{2r}(|S|^4))\widehat{k_\mu}, \widehat{k_\mu}\right\rangle \\
&\leq ber(\alpha M + (1-\alpha)N + f^{2r}(|S|^4) + g^{2r}(|S|^4)).
\end{aligned}$$

Taking the supremum over $\mu \in \Omega$, we get the desired result. □

Considering $f(t) = t^{\gamma}$, and $g(t) = t^{1-\gamma}$, $0 \leq \gamma \leq 1$ in theorem 2.10, we get the following corollary.

Corollary 2.11. *Assume that $S \in \mathcal{B}(\mathcal{H}(\Omega))$.Then for $r \geq 1$ and $0 \leq \gamma \leq 1$, the following inequality holds:*

$$ber^2_{dw}(S) \leq \frac{2^r}{4} ber\left(\alpha X + (1-\alpha)Y + |S|^{8r\gamma} + |S|^{8r(1-\gamma)}\right),$$

where

$$X = \left(|S|^{4r\gamma} + |S|^{4r(1-\gamma)}\right), Y = \left(|S^*|^{4r\gamma} + |S^*|^{4r(1-\gamma)}\right).$$

Now, we need the following lemma to prove the next theorem.

Lemma 2.12. *[8] Let $S \in \mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$ with $\|x\| = 1$. Assume that f, g are non-negative continuous functions on $[0,\infty)$ which are satisfying the relation $f(t)g(t) = t(t \in [0,\infty))$. Then for $r \geq 1$, we have*

$$|\langle Sx, x\rangle|^{2r} \leq \frac{1}{2}|\langle Sx, x\rangle|^r + \frac{1}{8}\left|\langle (f^{2r}(|S|^2) + g^{2r}(|S|^2) + f^{2r}(|S^*|^2) + g^{2r}(|S^*|^2))x, x\rangle\right|.$$

Theorem 2.13. *Suppose that $S \in \mathcal{B}(\mathcal{H}(\Omega))$. Let f, g be non-negative continuous functions on $[0,\infty)$ which are satisfying the relation $f(t)g(t) = t(t \in [0,\infty))$. Then for $r \geq 1$, and $0 \leq \alpha \leq 1$, the following inequalitiy holds:*

$$ber^{2r}_{dw}(S) \leq \frac{2^r}{2}\left[\frac{\alpha}{2} ber^r(S^2) + \|\frac{\alpha}{8}Q + (1-\alpha)|S^*|^{2r} + (1-\frac{\alpha}{2})|S|^{4r}\|_{\mathrm{Ber}}\right], \tag{13}$$

where

$$Q = f^{2r}(|S|^2) + g^{2r}(|S|^2) + f^{2r}(|S^*|^2) + g^{2r}(|S^*|^2) + 2\left(f^{2r}(|S|^4) + g^{2r}(|S|^4)\right).$$

Proof. *Assume that $\widehat{k_\mu} \in \mathcal{H}(\Omega)$ is a normalized reproducing kernel. Then, we have*

$$\begin{aligned}
&\left(|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2 + \|S\widehat{k_\mu}\|^4_{Ber}\right)^r \\
&= \left(|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2 + |\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2\right)^r \\
&\leq \frac{2^r}{2}\left(|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r} + |\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r}\right) \\
&\qquad\qquad (\textit{by convexity of } f(t) = t^r) \\
&= \frac{2^r}{2}\left[\alpha|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r} + (1-\alpha)|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r} + \alpha|\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r} + (1-\alpha)|\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r}\right] \\
&\leq \frac{2^r}{2}\Big[\frac{\alpha}{2}|\langle S^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^r + \frac{\alpha}{8}\langle\left(f^{2r}(|S|^2) + g^{2r}(|S|^2) + f^{2r}(|S^*|^2) + g^{2r}(|S^*|^2)\right)\widehat{k_\mu}, \widehat{k_\mu}\rangle \\
&\qquad + (1-\alpha)\|S^*\|^{2r}_{ber} + (1-\alpha)\langle |S|^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^{2r} + \frac{\alpha}{2}|\langle |S|^4\widehat{k_\mu}, \widehat{k_\mu}\rangle|^r + \frac{\alpha}{4}\langle\left(f^{2r}(|S|^4) + g^{2r}(|S|^4)\right)\widehat{k_\mu}, \widehat{k_\mu}\rangle\Big], \\
&\qquad\qquad (\textit{by Lemma 2.12}) \\
&\leq \frac{2^r}{2}\Big[\frac{\alpha}{2}\left(|\langle S^2\widehat{k_\mu}, \widehat{k_\mu}\rangle|^r + \langle |S|^{4r}\widehat{k_\mu}, \widehat{k_\mu}\rangle\right) + (1-\alpha)\left(\langle |S^*|^{2r}\widehat{k_\mu}, \widehat{k_\mu}\rangle + \langle |S|^{4r}\widehat{k_\mu}, \widehat{k_\mu}\rangle\right) + \frac{\alpha}{8}\langle Q\widehat{k_\mu}, \widehat{k_\mu}\rangle\Big], \\
&\qquad\qquad (\textit{by Lemma 2.6(b)}) \\
&\leq \frac{2^r}{2}\Big[\frac{\alpha}{2} ber^r(S^2) + \|\frac{\alpha}{8}Q + (1-\alpha)|S^*|^{2r} + (1-\frac{\alpha}{2})|S|^{4r}\|_{\mathrm{Ber}}\Big].
\end{aligned}$$

By taking the supremum over $\mu \in \Omega$, we get the result. □

Next, we give bounds for the Davis-Wielandt Berezin involving the generalized Aluthge transform. The generalized Aluthge transform of S, denoted by $\tilde{S}_t$, is defined as follows:

$$\tilde{S}_t = |S|^t U |S|^{1-t} \quad , \quad 0 \leq t \leq 1.$$

Here U is the partial isometry associated with the polar decomposition of S, and so $S = U|S|$, $kerS = kerU$. We note that $\tilde{S}_0 = U^*U^2|S|$, $\tilde{S}_1 = |S|UU^*U = |S|U$ and $\tilde{S_{\frac{1}{2}}} = |S|^{\frac{1}{2}}U|S|^{\frac{1}{2}} = \tilde{S}$(the Aluthge transform of S). It is Know [5] that $S^2 = 0$ if and only if $\tilde{S}_t = 0$ for all $t \in [0,1]$. Now, we prove the following lemma.

Lemma 2.14. *Let $S \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$ber\left(S^2\right) = \|S\|_{\text{Ber}} \left(\min_{0 \leq t \leq 1} \|\tilde{S}_t\|_{\text{Ber}}\right).$$

Proof. *Suppose that $\widehat{k_\mu} \in \mathcal{H}(\Omega)$ is a normalized reproducing kernel. Then, applying polar decomposition of S, i.e., $S = U|S|$, where U is the partial isometry associated with the polar decomposition of T, we get*

$$|\langle S^2\widehat{k_\mu}, \widehat{k_\mu}\rangle| = |\langle U|S|^{1-t}|S|^t U|S|^{1-t}|S|^t\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2 = |\langle U|S|^{1-t}\tilde{S}_t|S|^t\widehat{k_\mu}, \widehat{k_\mu}\rangle|^2 \leq \|S\|_{\text{Ber}}\|\tilde{S}_t\|_{\text{Ber}}.$$

Taking supremum over the normalized reproducing kernel $\widehat{k_\mu}$, we have

$$ber\left(S^2\right) = \|S\|_{\text{Ber}}\|\tilde{S}_t\|_{\text{Ber}}.$$

This holds for all $t \in [0,1]$, and so we get the required inequality by taking minimum over $t \in [0,1]$. □

In the next result, we give an upper bound for the Davis-Wielandt Berezin number involving the operator norm of the generalized Aluthge transform.

Corollary 2.15. *Assume that $S \in \mathcal{B}(\mathcal{H}(\Omega))$.Then*

$$ber^2_{dw}(S) \leq \frac{1}{2}\|S\|_{\text{Ber}}\left(\min_{0 \leq t \leq 1} \|\tilde{S}_t\|_{\text{Ber}}\right) + \frac{1}{4}\left\||S|^2 + |S^*|^2 + 4|S|^4\right\|_{\text{Ber}}.$$

Proof. *By considering $r = 1$, $\alpha = 1$, $f(t) = t^\gamma$, $g(t) = t^{1-\gamma}$ and $\gamma = \frac{1}{2}$ in Theorem 2.13 we have*

$$ber^2_{dw}(S) \leq \frac{1}{2}ber\left(S^2\right) + \frac{1}{4}\left\||S|^2 + |S^*|^2 + 4|S|^4\right\|_{\text{Ber}},$$

now we deduce the required inequality by applying Lemma 2.14. □

Remark 2.16. *We note that if $S^2 = 0$, then $\tilde{S}_t = 0$, and so*

$$ber^2_{dw}(S) \leq \frac{1}{4}\left\||S|^2 + |S^*|^2 + 4|S|^4\right\|_{\text{Ber}} \leq \frac{1}{4}\left\||S|^2 + |S^*|^2\right\|_{\text{Ber}} + \|S\|^4_{\text{Ber}} = ber^2(S) + \|S\|^4_{\text{Ber}}.$$

The inequalities obtained in Theorems 2.13 and 2.15 are better than the right hand inequality in (4).

3. The A-Davis–Wielandt Berezin inequalities for 2 × 2 operator matrices

Let $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}(\Omega))$ and $\widehat{k_\mu}$ be the normalized reproducing kernel of the space $\mathcal{H}(\Omega)$. For positive operator A, we define the A-Berezin symbol(or A-Berezin transform) of S, which is the bounded function $\widetilde{S}_A$ on $\mathcal{H}(\Omega)$, as follows:

$$\widetilde{S}_A(\mu) = \langle S\widehat{k_\mu}(z), \widehat{k_\mu}(z)\rangle_A, \ \mu \in \Omega.$$

Moreover, the A- Berezin set, the A-Berezin number, and the A-Berezin norm of the operator $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}(\Omega))$ introduce, respectively, as follows:

$$\mathrm{Ber}_A(S) := \left\{ \widetilde{S}_A(\mu) : \mu \in \Omega \right\} = \mathrm{Range}(\widetilde{S}_A),$$

$$\mathrm{ber}_A(S) := \sup \left\{ \left|\gamma\right| : \gamma \in \mathrm{Ber}_A(S) \right\} = \sup_{\mu \in \Omega} \left| \widetilde{S}_A(\mu) \right|,$$

and

$$\|S\|_{A-\mathrm{Ber}} := \sup_{\mu \in \Omega} \left\| S\widehat{k_\mu} \right\|_A .$$

Through the following, the A-Crawford Berezin number, and the A-minimum Berezin modulus of the operator $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}(\Omega))$, define as;

$$C_{A-\mathrm{Ber}}(S) := \inf\{|\widetilde{S}_A(\mu)| : \mu \in \Omega\}, \text{ and } \quad m_{A-\mathrm{Ber}}(S) := \inf\{\|S\widehat{k_\mu}\|_A : \mu \in \Omega\},$$

respectively.
Now, we introduce concepts A-Davis-Wielandt Berezin set and A-Davis-Wielandt Berezin number for $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H}(\Omega))$ and positive operator A, as follows:

$$Ber_{A-dw}(S) = \left\{ (\langle S\widehat{k_\mu}, \widehat{k_\mu} \rangle_A, \langle S\widehat{k_\mu}, S\widehat{k_\mu} \rangle_A), \mu \in \Omega \right\},$$

and

$$ber_{A-dw}(S) = \sup_{\mu \in \Omega} \left\{ \sqrt{|\langle S\widehat{k_\mu}, \widehat{k_\mu} \rangle_A|^2 + \|S\widehat{k_\mu}\|_A^4} \right\}.$$

Clearly, $ber_{A-dw}(S)$ is a generalization of $ber_A(S)$. Also, $ber_{A-dw}(\cdot)$ is weakly unitarily invariant. Indeed,

$$\begin{aligned} ber_{A-dw}(U^{\sharp_A} SU) &= \sup_{\mu \in \Omega} (|\langle U^{\sharp_A} SU\widehat{k_\mu}, \widehat{k_\mu} \rangle_A|^2 + \|U^{\sharp_A} SU\widehat{k_\mu}\|_A^4)^{\frac{1}{2}} \\ &= \sup_{\mu \in \Omega} (|\langle SU\widehat{k_\mu}, U\widehat{k_\mu} \rangle_A|^2 + \|S\widehat{k_\mu}\|_A^4)^{\frac{1}{2}} \\ &= \sup_{\eta \in \Omega} (|\langle S\widehat{k_\eta}, \widehat{k_\eta} \rangle_A|^2 + \|S\widehat{k_\mu}\|_A^4)^{\frac{1}{2}} \\ &= ber_{A-dw}(S) \end{aligned}$$

for any A-unitarily operator $U \in \mathcal{B}_A(\mathcal{H}(\Omega))$.
Also for the A-Davis-Wielandt Berezin number of operator $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, we have

$$\max\left(ber_A(S), \|S\|_{A-\mathrm{Ber}}^2\right) \le ber_{A-dw}(S) \le \sqrt{ber_A^2(S) + \|S\|_{A-\mathrm{Ber}}^4}. \tag{14}$$

In the following, we give an upper bound for the A-Davise-Wielandt Berezin number of operators in $\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, which is similar to upper bound for the A-Davise-Wielandt radius of operators in $\mathcal{B}(\mathcal{H})$, introduced in [4, 7].
First, we recall the following lemma.

Lemma 3.1. *[4]Let $s, t, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then*

$$|\langle s, e \rangle_A \langle e, t \rangle_A| \le \frac{1}{2} \left(|\langle s, t \rangle_A| + \|s\|_A \|t\|_A \right).$$

Theorem 3.2. *Suppose that $S \in \mathcal{B}_A(\mathcal{H}(\Omega))$. Then the following inequalities hold:*
(1) $ber^2_{A-dw}(S) \le \left\| |S|^2_A + \left(|S|^2_A\right)^{\sharp_A} |S|^2_A \right\|_{A-\mathrm{Ber}}$;
(2) $ber^2_{A-dw}(S) \le \frac{1}{2}\left(ber_A(S^2) + \|S\|^2_{A-\mathrm{Ber}}\right) + \|S\|^4_{A-\mathrm{Ber}}$.

Proof. *Let $\widehat{k_\mu}$ be the A-normalized reproducing kernel of the space $\mathcal{H}$. Then, applying Lemma 3.1, we have*

$$\begin{aligned}
|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle_A|^2 + \|S\widehat{k_\mu}\|^4_A &= |\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle_A \langle \widehat{k_\mu}, S\widehat{k_\mu}\rangle_A| + \langle |S|^2_A\widehat{k_\mu}, \widehat{k_\mu}\rangle_A \langle \widehat{k_\mu}, |S|^2_A\widehat{k_\mu}\rangle_A \\
&\le \frac{1}{2}\left(\|S\widehat{k_\mu}\|^2_A + \langle S\widehat{k_\mu}, S\widehat{k_\mu}\rangle_A\right) + \frac{1}{2}\left(\left\||S|^2_A\widehat{k_\mu}\right\|^2_A + \langle |S|^2_A\widehat{k_\mu}, |S|^2_A\widehat{k_\mu}\rangle_A\right) \\
&= \langle |S|^2_A\widehat{k_\mu}, \widehat{k_\mu}\rangle_A + \langle \left(|S|^2_A\right)^{\sharp_A} |S|^2_A\widehat{k_\mu}, \widehat{k_\mu}\rangle_A \\
&= \langle |S|^2_A + \left(|S|^2_A\right)^{\sharp_A} |S|^2_A\widehat{k_\mu}, \widehat{k_\mu}\rangle_A.
\end{aligned}$$

By taking supremum over all A-normalized reproducing kernels of the space $\mathcal{H}$, we gets (1). *Also, by considering $|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle_A|^2 = |\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle_A \langle \widehat{k_\mu}, S^{\sharp_A}\widehat{k_\mu}\rangle_A|$ and then using Lemma 3.1, we get* (2). □

Lemma 3.3. *[4]Assume that $s, t, e \in \mathcal{H}$ with $\|e\|_A = 1$. Then*

$$\|s\|^2_A\|t\|^2_A - |\langle s, t\rangle_A|^2 \ge 2|\langle s, e\rangle_A \langle e, t\rangle_A| (\|s\|_A\|t\|_A - |\langle s, t\rangle_A|).$$

Theorem 3.4. *Let $S \in \mathcal{B}_A(\mathcal{H})$. Then*

$$\begin{aligned}
ber^2_{A-dw}(S) \le 3\left\|(|S|^2_A)^{\sharp_A}|S|^2_A + |S|^2_A\right\|_{A-\mathrm{Ber}} &- C_{A-\mathrm{Ber}}(|S|^2_A + S)m_{A-\mathrm{Ber}}(|S|^2_A + S) \\
&- C_{A-\mathrm{Ber}}(|S|^2_A - S)m_{A-\mathrm{Ber}}(|S|^2_A - S).
\end{aligned}$$

Proof. *Assume that $\widehat{k_\mu}$ is the A-normalized reproducing kernel of the space $\mathcal{H}(\Omega)$. Then, applying Lemmas 3.1 and 3.3, we have*

$$\begin{aligned}
|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle_A|^2 &\le \| S\widehat{k_\mu} \|^2_A \| \widehat{k_\mu} \|^2_A - 2|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle_A \langle \widehat{k_\mu}, \widehat{k_\mu}\rangle_A| \left(\|S\widehat{k_\mu}\|_A\|\widehat{k_\mu}\|_A - |\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle_A|\right) \\
&= \|S\widehat{k_\mu}\|^2_A + 2|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle_A||\langle \widehat{k_\mu}, S\widehat{k_\mu}\rangle_A| - 2|\langle S\widehat{k_\mu}, \widehat{k_\mu}\rangle_A| \left\|S\widehat{k_\mu}\right\|_A \\
&\le \|S\widehat{k_\mu}\|^2_A + \|S\widehat{k_\mu}\|^2_A + \langle S\widehat{k_\mu}, S\widehat{k_\mu}\rangle_A - 2C_{A-\mathrm{Ber}}(S)\|S\widehat{k_\mu}\|_A \\
&\le 3\langle |S|^2_A\widehat{k_\mu}, \widehat{k_\mu}\rangle_A - 2C_{A-\mathrm{Ber}}(S)m_{A-\mathrm{Ber}}(S).
\end{aligned}$$

Using the above inequality, we deduce that

$$\begin{aligned}
&|\langle S\widehat{k_\mu},\widehat{k_\mu}\rangle_A|^2+\|S\widehat{k_\mu}\|_A^2\\
&\quad=\frac{1}{2}\left(\left|\|S\widehat{k_\mu}\|_A^2+\langle S\widehat{k_\mu},\widehat{k_\mu}\rangle_A\right|^2+\left|\|S\widehat{k_\mu}\|_A^2-\langle S\widehat{k_\mu},\widehat{k_\mu}\rangle_A\right|^2\right)\\
&\quad=\frac{1}{2}\left(\left|\langle(|S|_A^2+S)\widehat{k_\mu},\widehat{k_\mu}\rangle_A\right|^2+\left|\langle(|S|_A^2-S)\widehat{k_\mu},\widehat{k_\mu}\rangle_A\right|^2\right)\\
&\quad\le\frac{1}{2}\{3\left\langle\left||S|_A^2+S\right|_A^2\widehat{k_\mu},\widehat{k_\mu}\right\rangle_A-2C_{A-\mathrm{Ber}}(|S|_A^2+S)m_{A-\mathrm{Ber}}(|S|_A^2+S)\\
&\quad+3\left\langle\left||S|_A^2-S\right|_A^2\widehat{k_\mu},\widehat{k_\mu}\right\rangle_A-2C_{A-\mathrm{Ber}}(|S|_A^2-S)m_{A-\mathrm{Ber}}(|S|_A^2-S)\}\\
&\quad=\frac{3}{2}\left\langle\left(\left||S|_A^2+S\right|_A^2+\left||S|_A^2-S\right|_A^2\right)\widehat{k_\mu},\widehat{k_\mu}\right\rangle_A-C_{A-\mathrm{Ber}}(|S|_A^2+S)m_{A-\mathrm{Ber}}(|S|_A^2+S)\\
&\quad-C_{A-\mathrm{Ber}}(|S|_A^2-S)m_{A-\mathrm{Ber}}(|S|_A^2-S)\\
&\quad=3\left\langle\left((|S|_A^2)^{\sharp_A}|S|_A^2+|S|_A^2\right)\widehat{k_\mu},\widehat{k_\mu}\right\rangle_A\\
&\quad-C_{A-\mathrm{Ber}}(|S|_A^2+S)m_{A-\mathrm{Ber}}(|S|_A^2+S)-C_{A-\mathrm{Ber}}(|S|_A^2-S)m_{A-\mathrm{Ber}}(|S|_A^2-S).
\end{aligned}$$

Now, we take the supremum over all A-unit vectors in $\mathcal{H}(\Omega)$ and get the required inequality. □

Now, we try to determine the exact value of the A-Davis-Wieladt-Berezin number of some 2×2 operator matrices in $\mathcal{B}_{A^{\frac{1}{2}}}\left(\mathcal{H}\bigoplus\mathcal{H}\right)$.

Theorem 3.5. *Assume that $X\in\mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, and $S=\begin{pmatrix} I & X\\ 0 & 0\end{pmatrix}$. Then*

$$ber_{A-dw}(S)=\begin{cases}\sqrt{2} & \|X\|_{\mathrm{A-Ber}}=0\\ (\cos\theta_0+\|X\|_{A-\mathrm{Ber}}\sin\theta_0)\left(\cos\theta_0^2+(\cos\theta_0+\|X\|_{A-\mathrm{Ber}}\sin\theta_0)^2\right)^{\frac{1}{2}} & \|X\|_{\mathrm{A-Ber}}\neq 0,\end{cases}$$

where
$\theta_0=\arctan(\beta+\gamma-\frac{p}{3})$, $\alpha=\frac{1}{27}(2p^3-9pq+27r)$, $\beta=(-\frac{\alpha}{2}+\sqrt{s})^{\frac{1}{3}}$, $\gamma=(-\frac{\alpha}{2}-\sqrt{s})^{\frac{1}{3}}$,
$s=\frac{1}{2^4 3^3 b^6}(8b^8+20b^6+45b^4+61b^2+28)$, $r=\frac{3}{2b}$, $q=-\frac{2b^2-2}{b^2}$, $p=-\frac{2b^2-5}{2b}$, *and* $b=\|X\|_{A-\mathrm{Ber}}$.

Proof. *Let $\mathcal{H}=\bigoplus_{i=1}^{2}\mathcal{H}(\Omega_i)$. For every $\mu=(\mu_1,\mu_2)\in\Omega_1\times\Omega_2$, let $\widehat{\mathbf{k}}_\mu=\begin{pmatrix}\widehat{k_{\mu_1}}\\ \widehat{k_{\mu_2}}\end{pmatrix}$ be an A-normalized reproducing kernel in the space $\mathcal{H}(\Omega)$. Then*

$$\langle S\widehat{\mathbf{k}}_\mu,\widehat{\mathbf{k}}_\mu\rangle_A=\langle\widehat{k_{\mu_1}}+X\widehat{k_{\mu_2}},\widehat{k_{\mu_1}}\rangle_A,$$

and

$$\langle S\widehat{\mathbf{k}}_\mu,S\widehat{\mathbf{k}}_\mu\rangle_A=\langle\widehat{k_{\mu_1}}+X\widehat{k_{\mu_2}},\widehat{k_{\mu_1}}+X\widehat{k_{\mu_2}}\rangle_A.$$

Now, we have

$$\begin{aligned}|\langle S\widehat{\mathbf{k}}_\mu, \widehat{\mathbf{k}}_\mu\rangle_A|^2 + |\langle S\widehat{\mathbf{k}}_\mu, S\widehat{\mathbf{k}}_\mu\rangle_A|^2 &\leq \|\widehat{k}_{\mu_1} + X\widehat{k}_{\mu_2}\|_A^2 \|\widehat{k}_{\mu_1}\|_A^2 + \|\widehat{k}_{\mu_1} + X\widehat{k}_{\mu_2}\|_A^4 \\ &= \|\widehat{k}_{\mu_1} + X\widehat{k}_{\mu_2}\|_A^2 \left(\|\widehat{k}_{\mu_1}\|_A^2 + \|\widehat{k}_{\mu_1} + X\widehat{k}_{\mu_2}\|_A^2\right) \\ &\leq \sup_{\|\widehat{k}_{\mu_1}\|_A^2 + \|\widehat{k}_{\mu_2}\|_A^2 = 1} \left(\|\widehat{k}_{\mu_1}\|_A + \|X\|_{A-\mathrm{Ber}}\|\widehat{k}_{\mu_2}\|_A\right)^2 \\ &\quad \times \left(\|\widehat{k}_{\mu_1}\|_A^2 + (\|\widehat{k}_{\mu_1}\|_A + \|X\|_{A-\mathrm{Ber}}\|\widehat{k}_{\mu_2}\|_A)^2\right) \\ &= \sup_{0\in[0,\frac{\pi}{2}]} (\cos\theta + \|X\|_{A-\mathrm{Ber}} \sin\theta)^2 \\ &\quad \times \left(\cos^2\theta + (\cos\theta + \|X\|_{A-\mathrm{Ber}}\sin\theta)^2\right).\end{aligned}$$

At first, we suppose that $\|X\|_{A-\mathrm{Ber}} = 0$. *Hence*

$$\sup_{0\in[0,\frac{\pi}{2}]} (\cos\theta + \|X\|_{A-\mathrm{Ber}} \sin\theta)^2 \left(\cos^2\theta + (\cos\theta + \|X\|_{A-\mathrm{Ber}}\sin\theta)^2\right) = 2.$$

So, $ber_{A-dw}(S) \leq \sqrt{2}$.
Now, let $\widehat{\mathbf{k}}_\mu = \begin{pmatrix} \widehat{k}_{\mu_1} \\ 0 \end{pmatrix}$ *be an A-normalized reproducing kernel, i.e.,* $\|\widehat{k}_{\mu_1}\|_A = 1$. *Therefor*

$$\langle S\widehat{\mathbf{k}}_\mu, \widehat{\mathbf{k}}_\mu\rangle_A = \|\widehat{k}_{\mu_1}\|_A^2, \qquad \text{and} \qquad \langle S\widehat{\mathbf{k}}_\mu, S\widehat{\mathbf{k}}_\mu\rangle_A = \|\widehat{k}_{\mu_1}\|_A^2,$$

and so

$$\left(|\langle S\widehat{\mathbf{k}}_\mu, \widehat{\mathbf{k}}_\mu\rangle_A|^2 + |\langle S\widehat{\mathbf{k}}_\mu, S\widehat{\mathbf{k}}_\mu\rangle_A|^2\right)^{\frac{1}{2}} = \sqrt{2}.$$

Thus $ber_{A-dw}(S) = \sqrt{2}$.
Now, we assume that $\|X\|_{A-\mathrm{Ber}} \neq 0$. *So*

$$\begin{aligned}&\sup_{0\in[0,\frac{\pi}{2}]} (\cos\theta + \|X\|_{A-\mathrm{Ber}} \sin\theta)^2 \left(\cos^2\theta + (\cos\theta + \|X\|_{A-\mathrm{Ber}}\sin\theta)^2\right) \\ &\quad = \sup_{0\in[0,\frac{\pi}{2}]} (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}} \sin\theta_0)^2 \left(\cos^2\theta_0 + (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}}\sin\theta_0)^2\right),\end{aligned}$$

where $\theta_0 = \arctan(\beta + \gamma - \frac{p}{3})$, $\alpha = \frac{1}{27}(2p^3 - 9pq + 27r)$, $\beta = (-\frac{\alpha}{2} + \sqrt{s})^{\frac{1}{3}}$, $\gamma = (-\frac{\alpha}{2} - \sqrt{s})^{\frac{1}{3}}$,
$s = \frac{1}{2^4 3^3 b^6}(8b^8 + 20b^6 + 45b^4 + 61b^2 + 28)$, $r = \frac{3}{2b}$, $q = -\frac{2b^2-2}{b^2}$, $p = -\frac{2b^2-5}{2b}$, $b = \|X\|_{A-\mathrm{Ber}}$. *Therefore*

$$ber_{A-dw}(S) \leq \left(\sup_{0\in[0,\frac{\pi}{2}]} (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}} \sin\theta_0)^2 \left(\cos^2\theta_0 + (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}}\sin\theta_0)^2\right)\right)^{\frac{1}{2}}.$$

Now, we prove that there exists a sequence $\{\widehat{\mathbf{k}}_\mu^{(n)}\}$ *of A-normalized reproducing kernels in* $\bigoplus_{i=1}^2 \mathcal{H}(\Omega_i)$ *such that*

$$\begin{aligned}&\left(|\langle S\widehat{\mathbf{k}}_\mu^{(n)}, \widehat{\mathbf{k}}_\mu^{(n)}\rangle_A|^2 + |\langle S\widehat{\mathbf{k}}_\mu^{(n)}, S\widehat{\mathbf{k}}_\mu^{(n)}\rangle_A|^2\right)^{\frac{1}{2}} \\ &\quad = \left(\sup_{0\in[0,\frac{\pi}{2}]} (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}} \sin\theta_0)^2 \left(\cos^2\theta_0 + (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}}\sin\theta_0)^2\right)\right)^{\frac{1}{2}}.\end{aligned}$$

because $X \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, *there exists a sequence* $\{\widehat{\mathbf{y}}_\mu^{(n)}\}$ *of A-normalized reproducing kernels in* $\mathcal{H}(\Omega_i)$ *such that* $\lim_{n\to\infty} \|X\widehat{\mathbf{y}}_\lambda^{(n)}\|_A = \|X\|_{A-\mathrm{Ber}}$.

Let $\widehat{\mathbf{z}}_\mu^{(n)k} = \frac{1}{\sqrt{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2+k^2}}\begin{pmatrix} X\widehat{\mathbf{y}}_\mu^{(n)} \\ k\widehat{\mathbf{y}}_\mu^{(n)} \end{pmatrix}$, *where* $k \geq 0$. *Then*

$$
\begin{aligned}
|\langle S\widehat{\mathbf{z}}_\mu^{(n)k}, \widehat{\mathbf{z}}_\mu^{(n)k}\rangle_A|^2 + |\langle S\widehat{\mathbf{z}}_\mu^{(n)k}, S\widehat{\mathbf{z}}_\mu^{(n)k}\rangle_A|^2 &= \frac{(1+k^2)\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^4}{(\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2+k^2)^2}(1+(1+k)^2) \\
&= \left(\frac{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A}{\sqrt{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2+k^2}} + \frac{k\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A}{\sqrt{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2+k^2}} \right)^2 \\
&\times \left(\frac{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2}{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2+k^2} + \left(\frac{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A}{\sqrt{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2+k^2}} + \frac{k\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A}{\sqrt{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2+k^2}} \right)^2 \right).
\end{aligned}
$$

We can choose $k_0 \geq 0$ *such that* $\frac{\|X\|_{A-\mathrm{Ber}}}{\sqrt{\|X\|_{A-\mathrm{Ber}}^2+k_0^2}} = \cos\theta_0$, *and* $\frac{k_0}{\sqrt{\|X\|_{A-\mathrm{Ber}}^2+k_0^2}} = \sin\theta_0$. *Hence, by choosing* $\widehat{\mathbf{z}}_\mu^{(n)} = \frac{1}{\sqrt{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2+k_0^2}}\begin{pmatrix} X\widehat{\mathbf{y}}_\mu^{(n)} \\ k_0\widehat{\mathbf{y}}_\mu^{(n)} \end{pmatrix}$, *we have*

$$
\begin{aligned}
\lim_{n\to\infty}\left(|\langle S\widehat{\mathbf{z}}_\mu^{(n)}, \widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2 + |\langle S\widehat{\mathbf{z}}_\mu^{(n)}, S\widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2\right)^{\frac{1}{2}} &= (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}}\sin\theta_0)^2 \\
&\times \left(\cos^2\theta_0 + (\cos\theta_0 + \|X\|_{A-\mathrm{Ber}}\sin\theta_0)^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

This completes the proof. □

Theorem 3.6. *Suppose that* $X \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ *and* $Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$.
Then

$$
ber_{A-dw}(Y) = \begin{cases} 0 & \|X\|_{A-\mathrm{Ber}} = 0 \\ \frac{\|X\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|X\|_{A-\mathrm{Ber}}^2}} & \|X\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}} \\ \|X\|_{A-\mathrm{Ber}}^2 & \|X\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}}. \end{cases}
$$

Proof. Assume that $\mathcal{H} = \bigoplus_{i=1}^2 \mathcal{H}(\Omega_i)$. For every $\mu = (\mu_1, \mu_2) \in \Omega_1 \times \Omega_2$, let $\widehat{\mathbf{k}}_\mu = \begin{pmatrix} \widehat{k}_{\mu_1} \\ \widehat{k}_{\mu_2} \end{pmatrix}$ be an A-normalized reproducing kernel in the space $\mathcal{H}$. Then

$$
\langle Y\widehat{\mathbf{k}}_\mu, \widehat{\mathbf{k}}_\mu\rangle_A = \langle X\widehat{k}_{\mu_2}, \widehat{k}_{\mu_1}\rangle_A,
$$

and

$$
\langle Y\widehat{\mathbf{k}}_\mu, Y\widehat{\mathbf{k}}_\mu\rangle_A = \langle X\widehat{k}_{\mu_2}, X\widehat{k}_{\mu_2}\rangle_A,
$$

Now, we have

$$\begin{aligned}|\langle Y\widehat{\mathbf{k}}_\mu, \widehat{\mathbf{k}}_\mu\rangle_A|^2 + |\langle Y\widehat{\mathbf{k}}_\mu, Y\widehat{\mathbf{k}}_\mu\rangle_A|^2 &\leq \|X\widehat{k}_{\mu_2}\|_A^2\|\widehat{k}_{\mu_1}\|_A^2 + \|X\widehat{k}_{\mu_2}\|_A^4 \\ &\leq \sup_{\|\widehat{k}_{\mu_1}\|_A^2+\|\widehat{k}_{\mu_2}\|_A^2=1}\left(\|X\|_{A-\mathrm{Ber}}^2\|\widehat{k}_{\mu_2}\|_A^2\|\widehat{k}_{\mu_1}\|_A^2 + \|X\|_{A-\mathrm{Ber}}^4\|\widehat{k}_{\mu_2}\|_A^4\right) \\ &= \sup_{0\in[0,\frac{\pi}{2}]}\|X\|_{A-\mathrm{Ber}}^2\sin^2\theta\left(\cos^2\theta + \|X\|_{A-\mathrm{Ber}}^2\sin^2\theta\right).\end{aligned}$$

First, we consider the case $\|X\|_{A-\mathrm{Ber}} = 0$. Then $ber_{A-dw}(Y) = 0$.
Now, we consider the case $0 < \|X\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}}$. Then

$$\sup_{0\in[0,\frac{\pi}{2}]}\|X\|_{A-\mathrm{Ber}}^2\sin^2\theta\left(\cos^2\theta + \|X\|_{A-\mathrm{Ber}}^2\sin^2\theta\right) = \frac{\|X\|_{A-\mathrm{Ber}}^2}{4(1-\|X\|_{A-\mathrm{Ber}}^2)}.$$

Therefore, $ber_{A-dw}(Y) \leq \frac{\|X\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|X\|_{A-\mathrm{Ber}}^2}}$.
Now, we show that there exists a sequence $\{\widehat{\mathbf{z}}_\mu^{(n)}\}$ of A-normalized reproducing kernels in $\bigoplus_{i=1}^2\mathcal{H}(\Omega_i)$ such that

$$\lim_{n\to\infty}\left\{|\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, \widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2 + |\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, Y\widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2\right\}^{\frac{1}{2}} = \frac{\|X\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|X\|_{A-\mathrm{Ber}}^2}}.$$

Since $X \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, there exists a sequence $\{\widehat{\mathbf{y}}_\mu^{(n)}\}$ of A-normalized reproducing kernels in $\mathcal{H}(\Omega_i)$ such that $\lim_{n\to\infty}\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A = \|X\|_{A-\mathrm{Ber}}$.
Let $\widehat{\mathbf{z}}_\mu^{(n)} = \frac{1}{\sqrt{\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2+k^2}}\begin{pmatrix} X\widehat{\mathbf{y}}_\mu^{(n)} \\ k\widehat{\mathbf{y}}_\mu^{(n)}\end{pmatrix}$, where $k = \frac{\|X\|_{A-\mathrm{Ber}}}{\sqrt{1-2\|X\|_{A-\mathrm{Ber}}^2}}$. Then

$$\lim_{n\to\infty}\left\{|\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, \widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2 + |\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, Y\widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2\right\}^{\frac{1}{2}} = \frac{\|X\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|X\|_{A-\mathrm{Ber}}^2}}.$$

Thus, $ber_{A-dw}(Y) = \frac{\|X\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|X\|_{A-\mathrm{Ber}}^2}}$

Now, we consider the case $\|X\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}}$. Then

$$\sup_{0\in[0,\frac{\pi}{2}]}\|X\|_{A-\mathrm{Ber}}^2\sin^2\theta\left(\cos^2\theta + \|X\|_{A-\mathrm{Ber}}^2\sin^2\theta\right) = \|X\|_{A-\mathrm{Ber}}^4.$$

Hence , $ber_{A-dw}(Y) \leq \|X\|_{A-\mathrm{Ber}}^2$.
Now, we show that there exists a sequence $\{\widehat{\mathbf{z}}_\mu^{(n)}\}$ of A-normalized reproducing kernels in $\bigoplus_{i=1}^2\mathcal{H}(\Omega_i)$ such that

$$\lim_{n\to\infty}\left\{|\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, \widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2 + |\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, Y\widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2\right\}^{\frac{1}{2}} = \|X\|_{A-\mathrm{Ber}}^2.$$

Since $X \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, there exists a sequence $\{\widehat{\mathbf{y}}_\mu^{(n)}\}$ of A-normalized reproducing kernels in $\mathcal{H}(\Omega_i)$ such that $\lim_{n\to\infty}\|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A = \|X\|_{A-\mathrm{Ber}}$.
Let $\widehat{\mathbf{z}}_\mu^{(n)} = \begin{pmatrix} 0 \\ \widehat{\mathbf{y}}_{\mu^{(n)}}\end{pmatrix}$. Then $\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, \widehat{\mathbf{z}}_\mu^{(n)}\rangle_A = 0$, and $\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, Y\widehat{\mathbf{z}}_\lambda^{(n)}\rangle_A = \|X\widehat{\mathbf{y}}_\mu^{(n)}\|_A^2$. Thus

$$\lim_{n\to\infty}\left\{|\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, \widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2 + |\langle Y\widehat{\mathbf{z}}_\mu^{(n)}, Y\widehat{\mathbf{z}}_\mu^{(n)}\rangle_A|^2\right\}^{\frac{1}{2}} = \|X\|_{A-\mathrm{Ber}}^2.$$

This completes the proof. □

Now, we give an upper bound for the A-Davis-Wielandt Berezin of sum of two operators in $\mathcal{B}_A(\mathcal{H}(\Omega))$.

Theorem 3.7. *Suppose that $X, Y \in \mathcal{B}_A(\mathcal{H}(\Omega))$. Then*

$$ber_{A-dw}(X+Y) \leq ber_{A-dw}(X) + ber_{A-dw}(Y) + ber_A((X^{\sharp_A}Y + Y^{\sharp_A}X)).$$

In particular, if $A(X^{\sharp_A}Y + Y^{\sharp_A}X) = 0$, then

$$ber_{A-dw}(X+Y) \leq ber_{A-dw}(X) + ber_{A-dw}(Y).$$

Proof. Let $\widehat{k_\lambda}$ be the A-normalized reproducing kernel of the space $\mathcal{H}$. By definition of the A-Davis-Wielandt Berezin shell, we have

$$\begin{aligned}
Ber_{A-dw}(X+Y) &= \{(\langle (X+Y)\widehat{k_\mu}, \widehat{k_\mu}\rangle_A, \langle (X+Y)\widehat{k_\mu}, (X+Y)\widehat{k_\mu}\rangle_A), \mu \in \Omega\} \\
&= \{(\langle X\widehat{k_\mu}, \widehat{k_\mu}\rangle_A, \langle X\widehat{k_\mu}, X\widehat{k_\mu}\rangle_A) + (\langle Y\widehat{k_\mu}, \widehat{k_\mu}\rangle_A, \langle Y\widehat{k_\mu}, Y\widehat{k_\mu}\rangle_A) \\
&+ (0, \langle (X^{\sharp_A}Y + Y^{\sharp_A}X)\widehat{k_\mu}, \widehat{k_\mu}\rangle_A), \mu \in \Omega\}.
\end{aligned}$$

Then, $Ber_{A-dw}(X+Y) \leq Ber_{A-dw}(X) + Ber_{A-dw}(Y) + \{(0, \langle (X^{\sharp_A}Y + Y^{\sharp_A}X)\widehat{k_\mu}, \widehat{k_\mu}\rangle_A), \mu \in \Omega\}$.
Also by considering $A\left(X^{\sharp_A}Y + Y^{\sharp_A}X\right) = 0$, we get the favorable inequality. □

In [9], the author showed that, if $S_{ij} \in \mathcal{B}_A(\mathcal{H})$ for $i, j = 1, 2$, then $(S_{ij})_{2\times 2} \in \mathcal{B}_A\left(\mathcal{H} \bigoplus \mathcal{H}\right)$ and

$$\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}^{\sharp_A} = \begin{pmatrix} S_{11}^{\sharp_A} & S_{21}^{\sharp_A} \\ S_{12}^{\sharp_A} & S_{22}^{\sharp_A} \end{pmatrix}. \tag{15}$$

Corollary 3.8. *Assume that $X, Y \in \mathcal{B}_A(\mathcal{H}(\Omega))$. Then*

$$ber_{A-dw}\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \leq \sqrt{\frac{1}{4}\|X\|^2_{A-\mathrm{Ber}} + \|X\|^4_{A-\mathrm{Ber}}} + \sqrt{\frac{1}{4}\|Y\|^2_{A-\mathrm{Ber}} + \|Y\|^4_{A-\mathrm{Ber}}}.$$

Proof. Since

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}^{\sharp_A}\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}^{\sharp_A}\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

applying theorem 3.7, we have

$$\begin{aligned}
& ber_{A-dw}\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \\
& \leq ber_{A-dw}\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} + ber_{A-dw}\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} \\
& \leq \sqrt{ber_A^2\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} + \left\|\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right\|^4_{A-\mathrm{Ber}}} + \sqrt{ber_A^2\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix} + \left\|\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}\right\|^4_{A-\mathrm{Ber}}} \\
& \leq \sqrt{\frac{1}{4}\left\|\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right\|^2_{A-\mathrm{Ber}} + \left\|\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}\right\|^4_{A-\mathrm{Ber}}} + \sqrt{\frac{1}{4}\left\|\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}\right\|^2_{A-\mathrm{Ber}} + \left\|\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}\right\|_{A-\mathrm{Ber}}} \\
& = \sqrt{\frac{1}{4}\|X\|^2_{A-\mathrm{Ber}} + \|X\|^4_{A-\mathrm{Ber}}} + \sqrt{\frac{1}{4}\|Y\|^2_{A-\mathrm{Ber}} + \|Y\|^4_{A-\mathrm{Ber}}}.
\end{aligned}$$

□

For every $S \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, and $U \in \mathcal{B}_A(\mathcal{H})$, $ber_{A-dw}(U^{\sharp_A}SU) = ber_{A-dw}(S)$, and so by choosing $U = \begin{pmatrix} I & 0 \\ 0 & e^{i\frac{\theta}{2}}I \end{pmatrix}$, and $U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, respectively, we have the following inequalites:

$$ber_{A-dw}\begin{pmatrix} 0 & D \\ e^{i\theta}C & 0 \end{pmatrix} = ber_{A-dw}\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix}, \tag{16}$$

$$ber_{A-dw}\begin{pmatrix} 0 & D \\ C & 0 \end{pmatrix} = ber_{A-dw}\begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \tag{17}$$

for every $D, C \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, and $\theta \in \mathbb{R}$.

Theorem 3.9. *Let $K, H \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$, then*

$$ber_{A-dw}\begin{pmatrix} 0 & K \\ H & 0 \end{pmatrix} \leq \begin{cases} \frac{\|K\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|K\|^2_{A-\mathrm{Ber}}}} + \frac{\|H\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|H\|^2_{A-\mathrm{Ber}}}}, & \|K\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}}, \|H\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}} \\ \frac{\|K\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|K\|^2_{A-\mathrm{Ber}}}} + \|H\|^2_{A-\mathrm{Ber}}, & \|K\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}}, \|H\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}} \\ \|K\|^2_{A-\mathrm{Ber}} + \frac{\|H\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|H\|^2_{A-\mathrm{Ber}}}}, & \|K\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}}, \|H\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}} \\ \|K\|^2_{A-\mathrm{Ber}} + \|H\|^2_{A-\mathrm{Ber}}, & \|K\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}}, \|H\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}} \end{cases}$$

Proof. Since, $\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}^{\sharp_A}\begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}^{\sharp_A}\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then, by applying Theorem 3.6 and Theorem 3.7, we have

$$\begin{aligned} ber_{A-dw}(\begin{pmatrix} 0 & K \\ H & 0 \end{pmatrix}) &\leq ber_{A-dw}(\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}) + ber_{A-dw}(\begin{pmatrix} 0 & 0 \\ H & 0 \end{pmatrix}) \\ &= ber_{A-dw}(\begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}) + ber_{A-dw}(\begin{pmatrix} 0 & H \\ 0 & 0 \end{pmatrix}) \\ &\qquad \text{(by inequality 17)} \\ &= \begin{cases} \frac{\|K\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|K\|^2_{A-\mathrm{Ber}}}} + \frac{\|H\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|H\|^2_{A-\mathrm{Ber}}}}, & \|K\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}}, \|H\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}} \\ \frac{\|K\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|K\|^2_{A-\mathrm{Ber}}}} + \|H\|^2_{A-\mathrm{Ber}}, & \|K\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}}, \|H\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}} \\ \|K\|^2_{A-\mathrm{Ber}} + \frac{\|H\|_{A-\mathrm{Ber}}}{2\sqrt{1-\|H\|^2_{A-\mathrm{Ber}}}}, & \|K\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}}, \|H\|_{A-\mathrm{Ber}} < \frac{1}{\sqrt{2}} \\ \|K\|^2_{A-\mathrm{Ber}} + \|H\|^2_{A-\mathrm{Ber}}, & \|K\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}}, \|H\|_{A-\mathrm{Ber}} \geq \frac{1}{\sqrt{2}}. \end{cases} \end{aligned}$$

□

Now, we want to give an upper bound for the A-Davis-Wielandt Berezin number of sum of product operators in $\mathcal{B}_A(\mathcal{H})$.

Theorem 3.10. *Suppose that $S, T, X, Y \in \mathcal{B}_A(\mathcal{H})$. So for every $t \in \mathbb{R} - \{0\}$, we have*

$$ber^2_{A-dw}\left(SXT^{\sharp_A} \pm TYS^{\sharp_A}\right) \leq \left(t^2\|S\|^2_{A-\mathrm{Ber}} + \frac{1}{t^2}\|T\|^2_{A-\mathrm{Ber}}\right)^2 \left\{\left(t^2\|SX\|^2_A + \frac{1}{t^2}\|TY\|^2_A\right)^2 + \alpha^2\right\},$$

where $\alpha = ber_A\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$.

Proof. Let $M, N \in \mathcal{B}_A\left(\mathcal{H} \bigoplus \mathcal{H}\right)$, such that $M = \begin{pmatrix} S & T \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$. Then $MNM^{\sharp_A} = \begin{pmatrix} SXT^{\sharp_A} + TYS^{\sharp_A} & 0 \\ 0 & 0 \end{pmatrix}$. Therefore

$$\begin{aligned} ber^2_{A-dw}\left(SXT^{\sharp_A} \pm TYS^{\sharp_A}\right) &\leq ber^2_{A-dw}\begin{pmatrix} SXT^{\sharp_A} + TYS^{\sharp_A} & 0 \\ 0 & 0 \end{pmatrix} \\ &= ber^2_{A-dw}(MNM^{\sharp_A}) \\ &= \sup_{\mu\in\Omega}\left\{|\langle MNM^{\sharp_A}\widehat{k_\mu}, \widehat{k_\mu}\rangle_A|^2 + \|MNM^{\sharp_A}\widehat{k_\mu}\|_A^4\right\} \\ &= \sup_{\mu\in\Omega}\left\{|\langle NM^{\sharp_A}\widehat{k_\mu}, M^{\sharp_A}\widehat{k_\mu}\rangle_A|^2 + \|MNM^{\sharp_A}\widehat{k_\mu}\|_A^4\right\} \\ &\leq \sup_{\mu\in\Omega}\left\{ber^2_A(N)\|M^{\sharp_A}\widehat{k_\mu}\|_A^4 + \|M\widehat{k_\mu}\|_A^4\|M^{\sharp_A}\widehat{k_\mu}\|_A^4\right\} \\ &= \left(ber^2_A(N) + \|M\widehat{k_\mu}\|_A^4\right)\|M\|^4_{A-\text{Ber}}. \end{aligned}$$

Since $\|M\|^2_{A-\text{Ber}} = \|SS^{\sharp_A} + TT^{\sharp_A}\|_A$ and $\|MN\|_A^2 = \|(TY)(TY)^{\sharp_A} + (SX)(SX)^{\sharp_A}\|_A$, applying the above inequality, we deduce that

$$ber^2_{A-dw}\left(SXT^{\sharp_A} + TYS^{\sharp_A}\right) \leq \left(\|S\|^2_{A-\text{Ber}} + \|T\|^2_{A-\text{Ber}}\right)^2\left\{\left(\|TY\|_A^2 + \|SX\|_A^2\right)^2 + ber_A^2\right\}. \tag{18}$$

Replacing Y by $-Y$ in (18) and applying (16), we have

$$ber^2_{A-dw}\left(SXT^{\sharp_A} - TYS^{\sharp_A}\right) \leq \left(\|S\|^2_{A-\text{Ber}} + \|T\|^2_{A-\text{Ber}}\right)^2\left\{\left(\|TY\|_A^2 + \|SX\|_A^2\right)^2 + ber_A^2\right\}. \tag{19}$$

Obviously, inequalities (18) and (19) hold for all $S, T \in \mathcal{B}_A(\mathcal{H})$. Then, replacing S by tS and T by $\frac{1}{t}T$, we have the required inequality of the theorem. □

Corollary 3.11. *Assume that $S, T, X, Y \in \mathcal{B}_A(\mathcal{H})$ with $\|SX\|_A, \|TY\|_A \neq 0$. Then*
(1) $ber^2_{A-dw}\left(SXT^{\sharp_A} \pm TYS^{\sharp_A}\right) \leq \left(\frac{\|TY\|_A}{\|SX\|_A}\|S\|^2_{A-\text{Ber}} + \frac{\|SX\|_A}{\|TY\|_A}\|T\|^2_{A-\text{Ber}}\right)^2\left\{4\|SX\|_A^2\|TY\|_A^2 + \alpha^2\right\}$,
where $\alpha = ber_A\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$;
(2) $ber^2_{A-dw}(X \pm Y) \leq \left(\frac{\|Y\|_{A-\text{Ber}}}{\|X\|_{A-\text{Ber}}} + \frac{\|X\|_{A-\text{Ber}}}{\|Y\|_{A-\text{Ber}}}\right)^2\left\{(2\|X\|_{A-\text{Ber}}\|Y\|_{A-\text{Ber}})^2 + ber_A^2\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}\right\}$.

Proof. By taking $t = \sqrt{\frac{\|TY\|_A}{\|SX\|_A}}$, and $S = T = I$ in Theorem 3.10, respectively we get the inequalities (1) *and* (2). □

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