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The existence of a solution for nonlinear fractional differential equations where nonlinear term depends on the fractional and first order derivative of an unknown function

Suzana Aleksić[a], Alberto Cabada[b,c], Slaana Dimitrijević[a], Tatjana V. Tomović Mladenović[a]

[a]*University of Kragujevac, Faculty of Science, Radoja Domanovića 12, 34000 Kragujevac, Serbia*
[b]*CITMAga, 15782 Santiago de Compostela, Galicia, Spain*
[c]*Departamento de Estatística, Análise Matemática e Optimización, Facultade de Matemáticas, Universidade de Santiago de Compostela, Spain*

Abstract. In this paper, we consider the existence of solutions of the nonlinear fractional differential equation boundary-value problem

$$\mathrm{D}_*^\alpha u(t) = f(t, u(t), u'(t), {}^{C}\mathrm{D}^\beta u(t)), \quad 0 < t < 1,\ 1 < \alpha < 2,\ 0 < \beta \leqslant 1,$$
$$u(0) = A, \quad u(1) = Bu(\eta),$$

where $0 < \eta < 1$, $A \geqslant 0$, $B\eta > 1$, D_*^α is the modified Caputo fractional derivative of order α, ${}^{C}\mathrm{D}^\beta$ is the Caputo fractional derivative of order β, and f is a function in $C([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$. Existence results for a solution are obtained. Two examples are presented to illustrate the results.

1. Introduction

Fractional differential equations have been of great interest recently because of the fact that fractional models are more realistic than the classical ones for the application in many fields of science and engineering. For a deeper discussion of fractional differential equations and their applications, we refer the reader to [16, 18, 21].

For the convenience of the reader we compile the relevant material from [28] and [30], making our exposition self-contained.

Definition 1.1 ([28]). *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u \in L^1[0,T]$ is defined for almost every (a.e.) t by*

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\mathrm{d}s,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\mathrm{d}t.$

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Email addresses: suzana.aleksic@pmf.kg.ac.rs (Suzana Aleksić), alberto.cabada@usc.gal (Alberto Cabada), sladjana.dimitrijevic@pmf.kg.ac.rs (Slaana Dimitrijević), tatjana.tomovic@pmf.kg.ac.rs (Tatjana V. Tomović Mladenović)

We will denote by $AC[0,T]$ the set of absolutely continuous functions on $[0,T]$. For $n \in \mathbb{N}$, $AC^n[0,T]$ stands for the functions f whose n-th derivative $f^{(n)}$ is in $AC[0,T]$, hence $f^{(n+1)}(t)$ exists for a.e. t and is an L^1 function.

Let D^k denote the ordinary derivative operator of order $k \in \mathbb{N}$. Let a function u be such that $\mathrm{D}^k u(0)$ exists for $k = \overline{0,n}$. Then the Taylor polynomial of degree n of function u is defined by $T_n u(t) = \sum\limits_{k=0}^{n} \frac{\mathrm{D}^k u(0)}{k!} t^k$, where $T_0 u(t) = u(0)$.

Definition 1.2 ([28]). *Let $\alpha \in \mathbb{R}^+$ and let n be the smallest integer greater than or equal to α. The Riemann-Liouville fractional differential operator of order α is defined when $\mathrm{D}^{n-1}(I^{n-\alpha}u) \in AC[0,T]$, that is $I^{n-\alpha} \in AC^{n-1}[0,T]$, by*

$$\mathrm{D}^\alpha u = \mathrm{D}^n I^{n-\alpha} u.$$

The Caputo derivative is defined for $u \in AC^{n-1}[0,T]$, by

$${}^C\mathrm{D}^\alpha u = I^{n-\alpha}\mathrm{D}^n u.$$

The modified Caputo derivative is defined when $I^{n-\alpha}u \in AC^{n-1}[0,T]$ and $T_{n-1}u$ exists by

$$\mathrm{D}_*^\alpha u = \mathrm{D}^\alpha(u - T_{n-1}u),$$

where $T_{n-1}u$ is the Taylor polynomial of degree $n-1$.

Under the given conditions each fractional derivative exists for a.e. t.

Remark 1.3. *It is worth pointing out that in* [28, Lemma 4.12] *for $n \in \mathbb{N}$ and $0 < \alpha < 1$ is proved that*

$$\mathrm{D}_*^{n+\alpha} u = {}^C\mathrm{D}^{n+\alpha} u,$$

when $\mathrm{D}^n u \in AC[0,T]$.

The following theorem concerning initial value problems for modified Caputo derivatives is direct corollary of [28, Theorem 5.1, Remark 5.2]. Here we have conditions under which initial value problems for modified Caputo derivatives are equivalent with appropriate Volterra integral equations. This is of the great importance for solving fractional differential equations with modified Caputo derivatives.

Theorem 1.4. *Let f be continuous function on* $[0,1]$ *and* $1 < \alpha < 2$. *If $u \in C^1[0,1]$ and $I^{2-\alpha}(u - T_1(u)) \in AC^1[0,1]$, then u satisfies*

$$\mathrm{D}_*^\alpha u(t) = f(t), \quad \text{for a.e. } t, \tag{1}$$

$$\mathrm{D}^k u(0) = u_0^{(k)}, \quad k = 0,1, \tag{2}$$

if and only if u satisfies the Volterra integral equation

$$u(t) = u_0 + t u_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s, \quad t \in [0,1]. \tag{3}$$

Nowadays, existence, multiplicity and positivity of a solution for nonlinear fractional differential equations with boundary value problems have attracted the attention of many mathematicians. Some boundary conditions are classified in several types, the classical, periodic, anti-periodic, nonlocal, multipoint, and the integral boundary conditions (see [1, 2, 4, 5, 7–12, 14, 15, 19, 20, 23–25, 27, 31, 32]). For the study of the existence of positive solutions for various classes of Riemann-Liouville and Caputo fractional differential equations and systems, subject to nonlocal boundary conditions, we refer the reader to [3] and references therein.

In [26] it was discussed the existence of solutions of the following nonlinear fractional boundary value problem:

$$^C\mathrm{D}^\alpha u(t) = f(t, u(t), {}^C\mathrm{D}^\beta u(t)), \quad 0 < t < 1,$$

coupled to one of the following boundary conditions

$$u(0) = u'(1) = 0 \quad \text{or} \quad u'(0) = u(1) = 0 \quad \text{or} \quad u(0) = u(1) = 0,$$

where $1 < \alpha \leqslant 2$, $0 < \beta \leqslant 1$, $^C\mathrm{D}^\alpha$, $^C\mathrm{D}^\beta$ are the Caputo fractional derivatives and f is continuous on $[0,1] \times \mathbb{R} \times \mathbb{R}$.

The existence of positive solutions for the fractional three-point boundary-value problem

$$\begin{aligned} &^C\mathrm{D}^\alpha x(t) = f(t, x(t), x'(t)), \quad a < t < b,\ 1 < \alpha < 2, \\ &x(a) = 0, \quad x(b) = \mu x(\eta), \quad a < \eta < b,\ \mu > \lambda, \end{aligned}$$

where $\lambda = \frac{b-a}{\eta-a}$, $^C\mathrm{D}^\alpha$ is the Caputo fractional derivative and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ is a continuous function, was investigated in [22].

Motivated by the previous results, in this paper we analyze a boundary value problem for a fractional differential equation where nonlinear term depends on the fractional and first order derivative of an unknown function.

To be concise, we consider the existence of a solution for the nonlinear fractional differential equation boundary-value problem

$$\mathrm{D}_*^\alpha u(t) = f(t, u(t), u'(t), {}^C\mathrm{D}^\beta u(t)), \quad 0 < t < 1,\ 1 < \alpha < 2,\ 0 < \beta \leqslant 1, \tag{4}$$

$$u(0) = A, \quad u(1) = Bu(\eta), \tag{5}$$

where $A \geqslant 0$, $0 < \eta < 1$, $B > \frac{1}{\eta}$, D_*^α is the modified Caputo fractional derivative of order α and $^C\mathrm{D}^\beta$ is the Caputo fractional derivative of order β, and f is a function in $C([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$. We look for solutions in $C^1[0,1]$.

Remark 1.5. *We consider the equation* (4) *with the modified Caputo derivative in order to look for a solution in* $C^1[0,1]$ *and be able to apply Theorem* 1.4. *In the case of equation analog to* (4) *but with the standard Caputo derivatives we do not have equivalence with the Volterra integral equation. Also, note that for* $u \in C^1[0,1]$ *and* $\beta \in (0,1]$, *from Remark* 1.3, *the equality* $^C\mathrm{D}^\beta u = \mathrm{D}_*^\beta u$ *holds.*

2. Green function properties

In the following theorem we find an integral representation of the solution of the linear problem related to (4)–(5).

Theorem 2.1. *Let* $1 < \alpha < 2$, $y \in C[0,1]$ *and* $u \in C^1[0,1]$ *such that* $I^{2-\alpha}(u - T_1(u)) \in AC^1[0,1]$. *Then fractional differential equation boundary-value problem*

$$\mathrm{D}_*^\alpha u(t) = y(t), \quad \text{a.e. } t, \tag{6}$$

$$u(0) = A, \quad u(1) = Bu(\eta), \tag{7}$$

where $A \geqslant 0$, $0 < \eta < 1$ *and* $B\eta \neq 1$, *has a unique solution* u *given by*

$$u(t) = A + \frac{A(1-B)}{B\eta - 1}t + \int_0^1 G(t,s)y(s)\mathrm{d}s,$$

where $G(t,s) = H(t,s) + \dfrac{t}{B\eta - 1}\,(H(1,s) - B\,H(\eta,s))$ *for* $H : [0,1]\times[0,1] \to \mathbb{R}$ *defined by*

$$H(\xi,s) = \begin{cases} \dfrac{(\xi - s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant \xi \leqslant 1, \\ 0, & 0 \leqslant \xi < s \leqslant 1. \end{cases} \tag{8}$$

Here G *is called the Green function of boundary-value Problem* (6)–(7).

Proof. From Theorem 1.4 it follows that equation (6) is equivalent to the integral equation

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + c_0 + c_1 t.$$

The condition $u(0) = A$ implies that $c_0 = A$, and $u(1) = Bu(\eta)$ gives

$$B\left(\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + A + c_1\eta\right) = \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + A + c_1.$$

An easy computation shows that

$$c_1 = \frac{1}{B\eta - 1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s - B\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + A(1-B)\right).$$

From the above it follows that

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + A + \frac{t}{B\eta-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s - B\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\mathrm{d}s + A(1-B)\right).$$

Therefore, we have

$$u(t) = A + \frac{A(1-B)}{B\eta - 1} t + \int_0^1 G(t,s) y(s)\mathrm{d}s,$$

where

$$G(t,s) = \frac{t}{B\eta-1} H(1,s) + H(t,s) - \frac{tB}{B\eta - 1} H(\eta,s), \tag{9}$$

for $H : [0,1]\times[0,1] \to \mathbb{R}$ defined by

$$H(\xi,s) = \begin{cases} \dfrac{(\xi - s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant \xi \leqslant 1, \\ 0, & 0 \leqslant \xi < s \leqslant 1, \end{cases} \tag{10}$$

and we conclude the proof. □

In the following lemma we derive some properties of Green function G that will be used for proving the existence of a solution of Problem (4)–(5).

Lemma 2.2. *Let* $\eta \in (0,1)$, $A \geqslant 0$, $B > \frac{1}{\eta}$ *be given. Then the Green function* G*, defined by* (9)*, satisfies the following properties:*

1) *function* G *is bounded from above, i.e.,* $\max\limits_{t,s\in[0,1]} G(t,s) \leqslant \dfrac{B\eta}{(B\eta - 1)\Gamma(\alpha)}$;

2) $\max\limits_{t\in[0,1]} \int\limits_0^1 |G(t,s)|\,\mathrm{d}s \leqslant \dfrac{B\eta(1+\eta^{\alpha-1})}{(B\eta-1)\Gamma(\alpha+1)}$;

3) $\max\limits_{t\in[0,1]} \int\limits_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right|\mathrm{d}s \leqslant \dfrac{\alpha(B\eta-1)+B\eta^{\alpha}+1}{\Gamma(\alpha+1)(B\eta-1)}$.

Proof. 1) Since

$$\max_{t\in[0,1]} G(t,s) \leqslant \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \max_{t\in[0,1]} \frac{t(1-s)^{\alpha-1}}{(B\eta-1)\Gamma(\alpha)} = \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} \cdot \frac{B\eta}{B\eta-1},$$

then $\max\limits_{t,s\in[0,1]} G(t,s) \leqslant \dfrac{B\eta}{(B\eta-1)\Gamma(\alpha)}$.

2) Since for all t and s holds

$$|G(t,s)| \leqslant \frac{t}{B\eta-1}H(1,s) + H(t,s) + \frac{tB}{B\eta-1}H(\eta,s),$$

then

$$\begin{aligned}\max_{t\in[0,1]} \int_0^1 |G(t,s)|\,\mathrm{d}s &\leqslant \max_{t\in[0,1]}\left(\frac{t}{(B\eta-1)\Gamma(\alpha+1)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{tB\eta^{\alpha}}{(B\eta-1)\Gamma(\alpha+1)}\right)\\ &= \frac{1}{(B\eta-1)\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)} + \frac{B\eta^{\alpha}}{(B\eta-1)\Gamma(\alpha+1)}\\ &= \frac{B\eta(1+\eta^{\alpha-1})}{(B\eta-1)\Gamma(\alpha+1)}.\end{aligned}$$

3) A simple computation gives for all t

$$\begin{aligned}\int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right|\mathrm{d}s &= \int_0^1 \left|\frac{1}{B\eta-1}H(1,s) + \frac{\partial}{\partial t}H(t,s) - \frac{B}{B\eta-1}H(\eta,s)\right|\mathrm{d}s\\ &\leqslant \frac{1}{(B\eta-1)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\mathrm{d}s + \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2}\mathrm{d}s + \frac{B}{(B\eta-1)\Gamma(\alpha)}\int_0^{\eta} (\eta-s)^{\alpha-1}\mathrm{d}s\\ &= \frac{t+\alpha(B\eta-1)t^{\alpha}+Bt\eta^{\alpha}}{t\Gamma(\alpha+1)(B\eta-1)} \leqslant \frac{\alpha(B\eta-1)+B\eta^{\alpha}+1}{\Gamma(\alpha+1)(B\eta-1)}.\end{aligned}$$

□

3. Main results

In this section we will prove existence of the solution of Problem (4)–(5) under some additional conditions on function f.

Let $X = C^1[0,1]$ be the class of all continuous functions having continuous first order derivatives on $[0,1]$ and $\|u\| = \max\limits_{t\in[0,1]} |u(t)|$. Equipped with the norm

$$\|u\|_* = \begin{cases} \|u\| + \|u'\| + \|{}^{C}\mathrm{D}^{\beta}u\|, & 0<\beta<1,\\ \|u\| + \|u'\|, & \beta = 1,\end{cases}$$

the space $(X, \|\cdot\|_*)$ is a Banach space (see [26, Lemma 3.2]).

In order to solve Problem (4)–(5) we define the operator $T : X \to C[0,1]$ as

$$Tu(t) := A + \frac{A(1-B)}{B\eta - 1}t + \int_0^1 G(t,s) f(s, u(s), u'(s), {}^C\mathrm{D}^\beta u(s))\mathrm{d}s, \tag{11}$$

where G is defined by formula (9).

Remark 3.1. *Note that if $u \in C^1[0,1]$, then ${}^C\mathrm{D}^\beta u = I^{1-\beta}u' \in AC[0,1]$ (see* [28, Proposition 3.2 (6)]). *So, for $f \in C([0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$ and $u \in C^1[0,1]$, we have that $f(t, u(t), u'(t), {}^C\mathrm{D}^\beta u(t))$ is continuous function.*

To simplify notation, from now on, $\gamma_u(\cdot)$ stands for $f(\cdot, u(\cdot), u'(\cdot), {}^C\mathrm{D}^\beta u(\cdot))$.

For further on we will use the following notations:

$$N_1 := \frac{B\eta(1+\eta^{\alpha-1})}{\Gamma(\alpha+1)(B\eta-1)} \quad \text{and} \quad N_2 := \frac{\alpha(B\eta-1) + B\eta^\alpha + 1}{\Gamma(\alpha+1)(B\eta-1)}. \tag{12}$$

From Lemma 2.2 we have the following inequalities:

$$N_1 \geqslant \max_{t\in[0,1]} \int_0^1 |G(t,s)|\,\mathrm{d}s \quad \text{and} \quad N_2 \geqslant \max_{t\in[0,1]} \int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right| \mathrm{d}s.$$

Lemma 3.2. *The operator T, defined by* (11)*, is continuous.*

Proof. Let $u_n \in X$, $n \in \mathbb{N}$, and $u \in X$ be such that $\|u_n - u\|_* \to 0$ as $n \to +\infty$. First observe that

$$|Tu_n(t) - Tu(t)| = \left|\int_0^1 G(t,s)\,(\gamma_{u_n}(s) - \gamma_u(s))\,\mathrm{d}s\right| \leqslant N_1 \max_{s\in[0,1]} \left|\gamma_{u_n}(s) - \gamma_u(s)\right|,$$

for all $t \in [0,1]$.

Using Lemma 2.2 we conclude that

$$|(Tu_n)'(t) - (Tu)'(t)| = \left|\int_0^1 \frac{\partial}{\partial t}G(t,s)\,(\gamma_{u_n}(s) - \gamma_u(s))\,\mathrm{d}s\right| \leqslant N_2 \max_{s\in[0,1]} \left|\gamma_{u_n}(s) - \gamma_u(s)\right|.$$

Repeating the previous argument leads to

$$\begin{aligned}\left|{}^C\mathrm{D}^\beta Tu_n(t) - {}^C\mathrm{D}^\beta Tu(t)\right| &= \left|\frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}\left((Tu_n)'(s) - (Tu)'(s)\right)\mathrm{d}s\right| \\ &\leqslant \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}\,|(Tu_n)'(s) - (Tu)'(s)|\,\mathrm{d}s \\ &\leqslant \frac{N_2}{\Gamma(1-\beta)} \max_{\tau\in[0,1]} \left|\gamma_{u_n}(\tau) - \gamma_u(\tau)\right| \cdot \int_0^t (t-s)^{-\beta}\mathrm{d}s \\ &\leqslant \frac{N_2}{\Gamma(2-\beta)} \max_{\tau\in[0,1]} \left|\gamma_{u_n}(\tau) - \gamma_u(\tau)\right|.\end{aligned}$$

From what has already been proved, we deduce that operator T is continuous. □

In order to get our main result of existence of solutions, it is necessary to put some restrictions on function f. Thus, we impose the following assumptions.

(H_1) There exist a nonnegative function $h \in C([0,1])$, $h \not\equiv 0$ on $[0,1]$, and a nonnegative continuous function $g \in C(\mathbb{R}^3)$ such that

$$|f(t,x,y,z)| \leqslant h(t) + g(x,y,z), \quad t \in [0,1],\ (x,y,z) \in \mathbb{R}^3.$$

(H_2) $\displaystyle\lim_{(|x|+|y|+|z|)\to+\infty} \frac{g(x,y,z)}{|x|+|y|+|z|} < K := \frac{1}{N_1+N_2(1+1/\Gamma(2-\beta))}$.

Let $\varepsilon = \frac{1}{3}\left(K - \displaystyle\lim_{(|x|+|y|+|z|)\to+\infty} \frac{g(x,y,z)}{|x|+|y|+|z|}\right) > 0$. From ($H_2$), we know that there exists $\bar{r}_1 > 0$, such that

$$0 \leqslant g(x,y,z) \leqslant (K-\varepsilon)(|x|+|y|+|z|), \quad \text{whenever} \quad |x|+|y|+|z| \geqslant \bar{r}_1.$$

Let $\overline{M} = \max\{g(x,y,z),\ |x|+|y|+|z| \leqslant \bar{r}_1\}$ and choose $\bar{r}_2 > \bar{r}_1$ such that $\overline{M} \leqslant \bar{r}_2(K-\varepsilon)$. Therefore, we have

$$0 \leqslant g(x,y,z) \leqslant (K-\varepsilon)\bar{r}_2, \quad \text{whenever} \quad |x|+|y|+|z| \leqslant \bar{r}_2.$$

Let $k_1 = \max\limits_{t\in[0,1]} \int_0^1 |G(t,s)|\, h(s)\mathrm{d}s$ and $k_2 = \max\limits_{t\in[0,1]} \int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right| h(s)\mathrm{d}s$.

Set

$$k = \max\left\{k_1, k_2, \frac{k_2}{\Gamma(2-\beta)}\right\} = \max\left\{k_1, \frac{k_2}{\Gamma(2-\beta)}\right\},$$
$$L = \max\left\{A, \frac{AB(1-\eta)}{B\eta-1}, \frac{A(B-1)}{(B\eta-1)\Gamma(2-\beta)}\right\},$$
$$p = \max\left\{\bar{r}_2, \frac{3K}{\varepsilon}(k+L)\right\}.$$

We will look for solutions in the ball

$$U = \{u \in X \mid \|u\|_* \leqslant p\}. \tag{13}$$

Clearly, the set U is closed, bounded and convex in $(X, \|\cdot\|_*)$.

Note that, in particular, the following inequalities are fulfilled:

$$0 \leqslant g(x,y,z) \leqslant (K-\varepsilon)p, \quad \text{whenever} \quad |x|+|y|+|z| \leqslant p. \tag{14}$$

Let us prove that $T : U \to U$.

Lemma 3.3. *If f from Problem* (4)–(5) *satisfies the assumptions* (H_1) *and* (H_2) *and T is operator defined by* (11), *then $T : U \to U$ and T is completely continuous operator.*

Proof. Using the assumption (H_1) and the inequality (14), for all $t \in [0,1]$ we have

$$\begin{aligned}|Tu(t)| &= \left|A + \frac{A(1-B)}{B\eta-1}t + \int_0^1 G(t,s) f(s, u_n(s), u_n'(s), {}^{C}\mathrm{D}^{\beta}u_n(s))\mathrm{d}s\right| \\ &\leqslant \left|A + \frac{A(1-B)}{B\eta-1}t\right| + \int_0^1 |G(t,s)|\, h(s)\mathrm{d}s + \int_0^1 |G(t,s)| \left|g\left(u(s), u'(s), {}^{C}\mathrm{D}^{\beta}u_n(s)\right)\right| \mathrm{d}s\end{aligned}$$

$$\leqslant \max\left\{A, \frac{AB(1-\eta)}{B\eta-1}\right\} + k_1 + p(K-\varepsilon)N_1$$

and

$$\begin{aligned}
|(Tu)'(t)| &= \left|\frac{A(1-B)}{B\eta-1} + \int_0^1 \frac{\partial}{\partial t}G(t,s) f(s,u_n(s),u_n'(s),{}^C\mathrm{D}^\beta u_n(s))\mathrm{d}s\right| \\
&\leqslant \left|\frac{A(1-B)}{B\eta-1}\right| + \int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right| h(s)\mathrm{d}s + \int_0^1 \left|\frac{\partial}{\partial t}G(t,s)\right| \left|g\left(u(s),u'(s),{}^C\mathrm{D}^\beta u_n(s)\right)\right| \mathrm{d}s \\
&\leqslant \frac{A(B-1)}{B\eta-1} + k_2 + p(K-\varepsilon)N_2 := \overline{K}.
\end{aligned}$$

We also have

$$\left|{}^C\mathrm{D}^\beta Tu(t)\right| = \left|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(Tu)'(s)\mathrm{d}s\right| \leqslant \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\,|(Tu)'(s)|\,\mathrm{d}s \leqslant \frac{1}{\Gamma(2-\beta)}\overline{K}.$$

Combining previous inequalities, yields

$$\begin{aligned}
\|Tu\|_* &\leqslant \max\left\{A, \frac{AB(1-\eta)}{B\eta-1}\right\} + k_1 + p(K-\varepsilon)N_1 + \frac{A(B-1)}{B\eta-1} + k_2 + p(K-\varepsilon)N_2 \\
&\quad + \frac{1}{\Gamma(2-\beta)}\left(\frac{A(B-1)}{B\eta-1} + k_2 + p(K-\varepsilon)N_2\right) \\
&\leqslant 3L + 3k + p(K-\varepsilon)\left(N_1 + N_2 + \frac{N_2}{\Gamma(2-\beta)}\right) \\
&= \frac{\varepsilon p}{K} + p(K-\varepsilon)\frac{1}{K} = p,
\end{aligned}$$

and we conclude that $T : U \to U$.

For $u \in U$ and $\tau, t \in [0,1]$ such that $\tau < t$, $D := \max\limits_{s\in[0,1]} |\gamma_u(s)|$ and $\overline{K} := \frac{A(B-1)}{B\eta-1} + k_2 + p(K-\varepsilon)N_2$ we have

$$\begin{aligned}
|Tu(t) - Tu(\tau)| &= \left|\frac{A(1-B)}{B\eta-1}(t-\tau) + \int_0^1 (G(t,s) - G(\tau,s))\,\gamma_u(s)\mathrm{d}s\right| \\
&\leqslant \frac{A(B-1)}{B\eta-1}\,|t-\tau| + D\int_0^1 |G(t,s) - G(\tau,s)|\,\mathrm{d}s \\
&\leqslant \frac{A(B-1)}{B\eta-1}\,|t-\tau| + D\Bigg(\frac{|t-\tau|}{B\eta-1}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s + \int_0^1 |H(t,s) - H(\tau,s)|\mathrm{d}s \\
&\quad + \frac{B|t-\tau|}{B\eta-1}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\Bigg) \\
&\leqslant |t-\tau|\left(\frac{A(B-1)}{B\eta-1} + \frac{D}{(B\eta-1)\Gamma(\alpha+1)}(B\eta^\alpha + 1)\right) + \frac{D}{\Gamma(\alpha+1)}|t^\alpha - \tau^\alpha|.
\end{aligned}$$

A trivial verification shows that

$$
\begin{aligned}
|(Tu)'(t)-(Tu)'(\tau)| &= \left|\int_0^1 \frac{\partial}{\partial t}G(t,s)\gamma_u(s)\mathrm{d}s - \int_0^1 \frac{\partial}{\partial t}G(\tau,s)\gamma_u(s)\mathrm{d}s\right| \\
&\leqslant D\int_0^1 \left|\frac{\partial}{\partial t}G(t,s)-\frac{\partial}{\partial t}G(\tau,s)\right|\mathrm{d}s \\
&\leqslant \frac{D}{\Gamma(\alpha)}|t^{\alpha-1}-\tau^{\alpha-1}|.
\end{aligned}
$$

Finally, assuming that $0 \leqslant t \leqslant \tau \leqslant 1$, we have

$$
\begin{aligned}
\left|{}^{C}\mathrm{D}^{\beta}Tu(t)-{}^{C}\mathrm{D}^{\beta}Tu(\tau)\right| &= \left|\frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}(Tu)'(s)\mathrm{d}s - \frac{1}{\Gamma(1-\beta)}\int_0^\tau (\tau-s)^{-\beta}(Tu)'(s)\mathrm{d}s\right| \\
&\leqslant \frac{1}{\Gamma(1-\beta)}\int_0^t \left|(t-s)^{-\beta}-(\tau-s)^{-\beta}\right|\cdot|(Tu)'(s)|\,\mathrm{d}s + \frac{1}{\Gamma(1-\beta)}\int_t^\tau (\tau-s)^{-\beta}\,|(Tu)'(s)|\,\mathrm{d}s \\
&\leqslant \frac{\overline{K}}{\Gamma(2-\beta)}\left(t^{1-\beta}-\tau^{1-\beta}+2\,(\tau-t)^{1-\beta}\right).
\end{aligned}
$$

Analogously, if $0 \leqslant \tau \leqslant t \leqslant 1$, we deduce that

$$
\left|{}^{C}\mathrm{D}^{\beta}Tu(t)-{}^{C}\mathrm{D}^{\beta}Tu(\tau)\right| \leqslant \frac{\overline{K}}{\Gamma(2-\beta)}\left(\tau^{1-\beta}-t^{1-\beta}+2\,(t-\tau)^{1-\beta}\right).
$$

So, we conclude that TU is an equicontinuous set. Thus, T is completely continuous. □

Using Schauder fixed-point theorem and lemmas 3.2 and 3.3, we arrive at our main result, which is enunciated in the following theorem.

Theorem 3.4. *Under the assumptions of Lemma* 3.3, *there exists a non trivial solution* $u \in U$ *of Problem* (4)–(5) *if* $f(t,0,0,0) \not\equiv 0$.

Example 3.5. *Consider the boundary value problem*

$$
\mathrm{D}_*^{\frac{3}{2}}u(t) = f(t,u(t),u'(t),{}^{C}\mathrm{D}^{\frac{1}{2}}u(t)), \quad 0<t<1, \tag{15}
$$

$$
u(0)=A, \quad u(1)=Bu(\eta), \quad A\geqslant 0,\ \eta\in(0,1),\ B\eta>1, \tag{16}
$$

where $B=\ell\eta^{-r}$ *for some* $\ell>1$ *and* $r\in\left(0,\frac{3}{2}\right)$ *and the continuous function* $f:[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is given by*

$$
f(t,x,y,z) = a\,t+3+\sin x+\sin y+\sin z, \quad a\in\mathbb{R}\setminus\{0\}.
$$

For $h(t)=|a|\,t$ *and* $g(x,y,z)=3+\sin x+\sin y+\sin z$, *it is immediate to verify that all the assumptions of Theorem* 3.4 *are fulfilled and, as a consequence, Problem* (15)–(16) *has at least one solution.*

Remark 3.6. *When a solution of the Problem* (4)–(5) *is in* $AC^1[0,1]$, *then it is also a solution of the nonlinear fractional differential equation boundary-value problem*

$$
\begin{aligned}
&{}^{C}\mathrm{D}^{\alpha}u(t) = f(t,u(t),u'(t),{}^{C}\mathrm{D}^{\beta}u(t)), \quad 0<t<1,\ 1<\alpha<2,\ 0<\beta\leqslant 1, \\
&u(0)=A, \quad u(1)=Bu(\eta),
\end{aligned}
$$

where $A\geqslant 0$, $0<\eta<1$, $B>\frac{1}{\eta}$, ${}^{C}\mathrm{D}^{\alpha}$ *and* ${}^{C}\mathrm{D}^{\beta}$ *are the Caputo fractional derivative of order* α *and* β, *respectably, and* f *is a function in* $C([0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$.

Remark 3.7. *A sufficient condition to ensure that $u \in AC^1[0,1]$, is that $\gamma_u \in C^1(I)$. Indeed, by means of the integration by parts it is immediate to verify that the first derivative of the solution is given by the following expression:*

$$u'(t) = \frac{A(1-B)}{B\eta-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha-1)}\gamma_u'(s)\mathrm{d}s + \frac{1}{B\eta-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\gamma_u(s))\mathrm{d}s - B\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\gamma_u(s)\mathrm{d}s\right)$$

and, as a direct consequence:

$$u''(t) = (\alpha-1)\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\gamma_u'(s)\mathrm{d}s,$$

which, obviously, is a continuous function.

Notice that, in particular, if $y \in C^1(I)$, we have that u is a solution of the linear problem (6)-(7) *if and only if u is a solution of the following one*

$$^C\mathrm{D}^\alpha u(t)u(t) = y(t), \quad a.e.\ t \in I, \quad u(0) = A, \quad u(1) = Bu(\eta).$$

4. Existence results of a positive solution

Now we will focus on finding only positive solutions of Problem (4)–(5). In order to get a positive solution we consider the case $A = 0$, i.e., we consider the existence of positive solutions for the nonlinear fractional differential equation boundary-value problem

$$\mathrm{D}_*^\alpha u(t) = f(t, u(t), u'(t), {}^C\mathrm{D}^\beta u(t)), \quad 0 < t < 1,\ 1 < \alpha < 2,\ 0 < \beta \leqslant 1, \tag{17}$$

$$u(0) = 0, \quad u(1) = Bu(\eta), \tag{18}$$

where $0 < \eta < 1$, $B > \frac{1}{\eta}$, D_*^α is the modified Caputo fractional derivative of order α and $^C\mathrm{D}^\beta$ is the Caputo fractional derivative of order β, and f is a function in $C([0,1] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R})$. We look for solutions in $C^1[0,1]$.

In the following, having some additional restrictions on function f, we prove existence of positive solutions of Problem (17)–(18).

Theorem 4.1. *Assume that the assumptions (H_1) and (H_2) are fulfilled. Let $c(B,\eta,\alpha) := \frac{1-B^{\frac{1}{\alpha-1}}\eta}{1-B^{\frac{1}{\alpha-1}}}$ and let f be a continuous function on $[0,1] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ such that $f(t,x,y,z) = 0$ for $t \in [0, c(B,\eta,\alpha)]$ and $f(t,x,y,z) \geqslant 0$ for $t \in [c(B,\eta,\alpha), 1]$, with $f(t,0,0,0) > 0$ for some $t \in [c(B,\eta,\alpha), 1]$. Then there exists a positive solution $u \in U$ of Problem* (17)–(18).

Proof. Let us define the cone of nonnegative $C^1(I)$ functions $P := \{u \in X \mid u(t) \geqslant 0, t \in [0,1]\}$ and denote $U_P = P \cap U$, with U introduced in (13). We first prove that $T(U_P) \subseteq U_P$, where T is operator defined in (11). Having in mind conditions on A and f, and that

$$(1-s)^{\alpha-1} - B(\eta-s)^{\alpha-1} \leqslant 0, \quad \text{for } s \in [0, c(B,\eta,\alpha)],$$

$$(1-s)^{\alpha-1} - B(\eta-s)^{\alpha-1} \geqslant 0, \quad \text{for } s \in [c(B,\eta,\alpha), \eta],$$

we can conclude that for $u(t) \geqslant 0$, $t \in [0,1]$, for each t holds

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s) f(s, u(s), u'(s), {}^C\mathrm{D}^\beta u(s))\mathrm{d}s \\ &= \frac{t}{B\eta-1}\left(\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\gamma_u(s)\mathrm{d}s - B\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\gamma_u(s)\mathrm{d}s\right) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\gamma_u(s)\mathrm{d}s \\ &= \frac{t}{(B\eta-1)\Gamma(\alpha)}\left(\int_0^{c(B,\eta,\alpha)} ((1-s)^{\alpha-1} - B(\eta-s)^{\alpha-1})\gamma_u(s)\mathrm{d}s + \int_{c(B,\eta,\alpha)}^\eta ((1-s)^{\alpha-1} - B(\eta-s)^{\alpha-1})\gamma_u(s)\mathrm{d}s\right. \end{aligned}$$

$$+\int_\eta^1 (1-s)^{\alpha-1}\gamma_u(s)\mathrm{d}s\Big) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\gamma_u(s)\mathrm{d}s$$
$$\geqslant 0.$$

Together with the conclusions from Lemma 3.3, we prove $Tu \in U_P$, so $T(U_P) \subseteq U_P$, and that T is completely continuous operator. The set U_P is compact and $T(U_P)$ is compact. By the Schauder fixed point theorem, the operator T has a fixed point in the set U_P. Therefore, Problem (17)–(18), has a positive solution. □

Example 4.2. *Consider the boundary value problem*

$$D_*^{\frac{3}{2}}u(t) = f(t, u(t), u'(t), {}^C D^{\frac{1}{2}}u(t)), \quad 0 < t < 1, \tag{19}$$

$$u(0) = 0, \quad u(1) = Bu(\eta), \quad \eta \in (0,1), \ B\eta > 1, \tag{20}$$

where $B = \ell\eta^{-r}$ *for some* $\ell > 1$ *and* $r \in \left(0, \frac{3}{2}\right)$ *and the continuous function* $f : [0,1] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ *for some* $a \in \mathbb{R}^+$ *is given by*

$$f(t,x,y,z) = \begin{cases} 0, & 0 \leqslant t \leqslant c(B,\eta,\alpha), \\ a(t - c(B,\eta,\alpha))^2(3 + x + \sin y + \sin z), & c(B,\eta,\alpha) < t \leqslant 1. \end{cases}$$

For $h(t) = a(t - c(B,\eta,\alpha))^2$ *and* $g(x,y,z) = a(2 + x + \sin y + \sin z)$, *it is immediate to verify that all the assumptions of Theorem* 4.1 *are fulfilled and, as a consequence, in this case Problem* (19)–(20) *has at least one positive solution.*

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