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On m-closure of ideals in a class of subrings of $C(X)$

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Abstract. A subalgebra $A(X)$ of $C(X)$ is said to be a β-subalgebra if it is closed under bounded inversion and the space of its maximal ideals equipped with the hull-kernel topology is homeomorphic to βX with a homeomorphism which leaves X pointwise fixed. Kharbhih and Dutta in [Closure formula for ideals in intermediate rings, *Appl. Gen. Topol.* **21** (2) (2020), 195-200] showed that the closure of every ideal I of an intermediate ring with the m-topology, briefly, the m-closure of I, equals the intersection of all maximal ideals in $A(X)$ containing I. In this paper, we extend this fact to the class of β-subalgebras which is shown to be a larger class than intermediate rings. We also study a more extended class of subrings than β-subalgebras, namely, *LBI*-subalgebras, and characterize the conditions under which an *LBI*-subalgebra is a β-subalgebra. Moreover, some known facts in the context of $C(X)$ and intermediate rings of $C(X)$ are generalized to β-subalgebras.

1. Introduction

Throughout this article all topological spaces are assumed to be completely regular and Hausdorff. For a given topological space X, $C(X)$ denotes the algebra of all real-valued continuous functions on X, $C^*(X)$ denotes the subalgebra of $C(X)$ consisting of all bounded elements. A subalgebra $A(X)$ of $C(X)$ is called intermediate, if $C^*(X) \subseteq A(X)$.

For a topological space X, βX denotes the Stone-Cˇech compactification of X. It is well-known that every $f \in C(X)$ has a continuous extension f^* from βX to $\mathbb{R}^*$ (the one-point compactification of $\mathbb{R}$). For an element f of a subalgebra $A(X)$, we denote by $S_A(f)$ the set $\{p \in \beta X : (fg)^*(p) = 0; \forall g \in A(X)\}$. This mapping was first introduced and applied to study subalgebras of $C(X)$ in [18], and then was applied in [12] to establish a topological characterization of z-ideals in a class of subalgebras of $C(X)$, namely, *LBI*-subalgebras. It also was used in [13], [14], [15], [16], [14], and [17] to study the intermediate rings of $C(X)$. It is easy to show that $S_A(f) \cap S_A(g) = S_A(f^2 + g^2)$, $S_A(f) \cup S_A(g) = S_A(fg)$, and $S_A(f^n) = S_A(f)$ for each $n \in \mathbb{N}$. Also, $S_C(f) = \mathrm{cl}_{\beta X} Z(f)$, $S_{C^*}(f) = Z(f^\beta)$, $\mathrm{cl}_{\beta X} Z(f) \subseteq S_A(f) \subseteq Z(f^*)$, $S_A(f) \cap X = Z(f)$, and $\mathrm{int}_{\beta X} S_A(f) = \mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} Z(f)$ for each $f \in A(X)$. For $p \in \beta X$, $\{f \in A(X) : p \in S_A(f)\}$ is denoted by M^p_A. Obviously, $M^x_A = \{f \in A(X) : x \in Z(f)\}$ for each $x \in X$. M^p_C and $M^p_{C^*}$ are simply denoted by M^p and M^{*p}, respectively.

A subalgebra $A(X)$ of $C(X)$ is called intermediate, if $C^*(X) \subseteq A(X)$. Following [18], a subalgebra $A(X)$ is said to be β-determining if $\{Z(f^*) : f \in A(X)\}$ separates points from closed sets in βX, and is said to be a

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β-subalgebra if the mapping $p \mapsto M_A^p$ is a homeomorphism from βX to $Max(A(X))$ (the space of maximal ideals of $A(X)$ endowed with the hull-kernel topology). Also, $A(X)$ is called closed under inversion if $f^{-1} \in A(X)$, whenever $f \in A(X)$ with $Z(f) = \emptyset$ and is called closed under bounded inversion if $f^{-1} \in A(X)$, whenever $f \in A(X)$ is bounded away from zero. Furthermore, $A(X)$ is said to be closed under local bounded inversion, briefly, an *LBI*-subalgebra, if there exists $g \in A(X)$ such that $fg|_{X-Z} = 1$, whenever $f \in A(X)$ with $f > \delta > 0$ on $X - Z$, where $Z \in Z(X)$ is a zero-set. It is proved in [18, Theorem 2.8] that $A(X)$ is a β-subalgebra if and only if $A(X)$ is β-determining and closed under bounded inversion. Also, it is shown in Theorem 2.1 of [5] that in the class of uniformly closed subalgebras, bounded invertibility is equivalent to locally bounded invertibility, and both are equivalent to the subalgebra to be a lattice ordered ring. We give more properties of these subalgebras in the next section.

Edwin Hewitt in [9], first introduced and studied the notion of m-topology on $C(X)$. The basic open sets for the neighborhood system of every $f \in C(X)$ in this topology are the sets of the form $B(f, u) = \{g \in C(X) : |f - g| \leq u\}$ where u is a positive unit in $C(X)$. Hewitt in the same paper proved that every maximal ideal in $C(X)$ is closed under m-topology, briefly, is m-closed. Gillman et al. in [8] generalized this fact and showed that every m-closed ideal in $C(X)$ is an intersection of maximal ideals. This fact was also shown by Shirota in [23]. It was shown that $C(X)$ with the m-topology is a topological ring. It is also known that in any topological ring, the closure of a proper ideal is either a proper ideal or the whole ring, see [7, 2M.1]. The m-topology on subrings of $C(X)$ has been studied by several authors. Gillman et al in [8] studied $C^*(X)$ as a subspace of $C(X)$ with the m-topology and showed that the closed ideals in $C^*(X)$ coincide with the intersections of maximal ideals in this ring. Acharyya et al and Veisi independently studied the m-topology on the functionally countable subalgebra of $C(X)$; $C_c(X)$, and introduced m_c-topology in [2] and [24], respectively. They showed that m_c-closed ideals in $C_c(X)$ are precisely the intersections of maximal ideals in $C_c(X)$. Kharbhih and Dutta in [10] studied this topology on intermediate rings of $C(X)$ and proved that in such rings, m-closed ideals are precisely the intersections of maximal ideals. They applied the mapping $\mathcal{Z}_A$ in intermediate rings previously defined and studied in [20], to achieve their main result.

In this paper, we first study *LBI* and β-subrings of $C(X)$ and give some properties of these rings. It is shown that every β-subalgebra is an *LBI*-subalgebra, however, the converse of this fact does not necessarily hold; a class of counterexamples is provided. Moreover, the conditions under which these two classes of subrings coincide are characterized. We then extend the topological description of the mapping $\mathcal{Z}_A$, which was previously established in [12, Theorem 2.1] for intermediate rings, to the β-subalgebras. Using this, it is shown that, similar to intermediate rings, β-ideals (the ideals I for which $\mathcal{Z}_A^{-1}\mathcal{Z}_A[I] = I$) are intersections of maximal ideals in β-subalgebras. We then study the m-topology on β-subrings and show that the m-closed ideals in such rings are precisely the β-ideals, and hence are the intersections of maximal ideals. We note that, in this paper, subrings are assumed to be unitary and hence, all subalgebras contain the real numbers as constant functions.

2. More on *LBI* and β-subalgebras

It is easy to see that M_A^p, for each $p \in \beta X$, is a prime ideal in an arbitrary subalgebra $A(X)$ of $C(X)$ which is not necessarily maximal. In fact, every maximal ideal in $A(X)$ is of the form M_A^p for some $p \in \beta X$ if and only if $A(X)$ is closed under bounded inversion ([18, Proposition 2.7]). Hence, in such subalgebras, an element f is invertible if and only if $S_A(f) = \emptyset$. Consequently, the ideals M_A^p are neither necessarily maximal nor necessarily distinct. It was established in [12, Lemma 2.1] that in the class of *LBI*-subalgebras, each ideal M_A^p would be maximal. Also, according to [18, Proposition 2.6], the ideals M_A^p are all distinct if and only if $A(X)$ is a β-subalgebra. The next proposition investigates some equivalent conditions on an *LBI*-subalgebra to be a β-subalgebra.

Proposition 2.1. *Let $A(X)$ be an LBI-subalgbera of $C(X)$. The following statements are equivalent:*

1. *$A(X)$ is a β-subalgebra.*
2. *The collection $\{S_A(f) : f \in A(X)\}$ constitutes a base for the closed subsets in βX.*
3. *$A(X)$ separates zero-sets in X.*

Proof. 1⇔2) An easy consequence of [18, Theorem 2.8].

2⇒3) Let Z_1 and Z_2 be two disjoint zero-sets in X. It follows that $\mathrm{cl}_{\beta X}Z_1 \cap \mathrm{cl}_{\beta X}Z_2 = \emptyset$ and hence, by the hypothesis, there exists $f \in A(X)$ such that $\mathrm{cl}_{\beta X}Z_1 \subseteq S_A(f)$ and $\mathrm{cl}_{\beta X}Z_2 \cap S_A(f) = \emptyset$. By the properties of βX, there exists $Z \in Z(X)$, such that $S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z$ and $\mathrm{cl}_{\beta X}Z_2 \cap \mathrm{cl}_{\beta X}Z = \emptyset$. Hence, $S_A(f) \cap (\beta X - \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z) = \emptyset$. We denote the set $\beta X - \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z$ briefly by F. It follows that for each $p \in F$, there exits $g_p \in A(X)$ such that $(fg_p)^*(p) \neq 0$ which means there exists a neighborhood U_p of p in βX such that, without lose of generality, $fg_p \geq 1$ on $U_p \cap X$. Evidently, $F \subseteq \cup_{p \in F} U_p$. Hence, as F is compact, there exist $p_1, ..., p_n \in F$ such that $F \subseteq \cup_{i=1}^n U_{p_i}$. We set $g = g_{p_1}^2 + ... + g_{p_n}^2$ and $U = U_{p_1} \cup ... \cup U_{p_n}$. It follows that $f^2 g \geq 1$ on $U \cap X$ and hence on $X - Z$, since $X - Z \subseteq U \cap X$. Now, as $A(X)$ is an *LBI*-subalgebra, there exists $h \in A(X)$ such that $(f^2 gh))|_{X-Z} = 1$. Now, if we set $k = f^2.g.h$, then $k \in A(X)$, $k(Z_1) = \{0\}$, since $Z_1 \subseteq Z(f)$, and $k(Z_2) = \{1\}$, since $Z_2 \subseteq X - Z$.

3⇒2) Let F be a closed set in βX and $p \notin F$. Hence, there exist $f, g \in C(X)$ such that $p \in \mathrm{cl}_{\beta X}Z(f)$, $F \subseteq \mathrm{cl}_{\beta X}Z(g)$, and $\mathrm{cl}_{\beta X}Z(f) \cap \mathrm{cl}_{\beta X}Z(g) = \emptyset$. It follows that $Z(f) \cap Z(g) = \emptyset$, and hence, by the hypothesis, there exists $h \in A(X)$ such that $h(Z(f)) = \{1\}$ and $h(Z(g)) = \{0\}$. We set $k = \frac{2h}{1+h^2}$. As $A(X)$ is an *LBI*-subalgebra, we clearly have $k \in A^*(X)$, $k(Z(f)) = \{1\}$, and $k(Z(g)) = \{0\}$. Obviously it is inferred that $\mathrm{cl}_{\beta X}Z(g) \subseteq \mathrm{cl}_{\beta X}Z(k) \subseteq S_A(k)$, and $\mathrm{cl}_{\beta X}Z(f) \subseteq \mathrm{cl}_{\beta X}k^{-1}(\{1\}) \subseteq \beta X - S_A(k)$. Therefore, $k \in A(X)$, $p \notin S_A(k)$, and $F \subseteq S_A(k)$. Note that, for each $p \in \mathrm{cl}_{\beta X}k^{-1}(\{1\})$, we have $h^\beta(p) = 1$ which implies $p \notin S_A(k)$. □

We next show that every β-subalgebra is an *LBI*-subalgebra. To this aim, we need the next lemma which investigates a characterization of the subalgebras closed under bounded inversion in terms of the mapping S_A.

Lemma 2.2. *The following statements are equivalent for a subalgebra $A(X)$:*

1. *f is invertible in $A(X)$ if and only if $S_A(f) = \emptyset$.*
2. *$A(X)$ is closed under bounded inversion.*

Proof. 1⇒2) Let M be a maximal ideal in $A(X)$ and $f \in M$. Assume on the contrary that $M \neq M_A^p$ for each $p \in \beta X$. Hence, for each $p \in \beta X$, there exists $f_p \in M$ such that $f_p \notin M_A^p$; i.e., $p \notin S_A(f_p)$. It follows that $\{\beta X - S_A(f_p)\}_{p \in \beta X}$ constitutes an open cover for βX. Thus, there exist $p_1, ..., p_n$ in βX such that $\{\beta X - S_A(f_{p_i})\}_{i=1}^n$ covers βX. This implies that $f = f_{p_1}^2 + ... + f_{p_n}^2 \in M$ and $S_A(f) = \emptyset$ which leads to a contradiction with regard to the hypothesis.

2⇒1) If $S_A(f) = \emptyset$, then $f \notin M_A^p$ for each $p \in \beta X$ which by [18, Proposition 2.7], means that f misses every maximal ideal in $A(X)$, and thus $f^{-1} \in A(X)$. Moreover, it is obvious that whenever $f^{-1} \in A(X)$, then $S_A(f) = \emptyset$. □

Proposition 2.3. *Every β-subalgebra is an LBI-subalgebra.*

Proof. If $f \in A(X)$ with $f \geq 1$ on $X - Z$ for some $Z \in Z(X)$, then $Z(f^*) \cap \mathrm{cl}_{\beta X}(X - Z) = \emptyset$, and hence, by Proposition 2.1, there exists $g \in A(X)$ such that $\mathrm{cl}_{\beta X}(X - E) \subseteq S_A(g)$ and $Z(f^*) \cap S_A(g) = \emptyset$. Thus, $S_A(f^2 + g^2) = S_A(f) \cap S_A(g) = \emptyset$, which by the fact that every β-subalgebra is closed under bounded inversion (see [18, Theorem 2.8]), and Lemma 2.2, imply that $f^2 + g^2$ is invertible in $A(X)$. Hence, there exists $h \in A(X)$ such that $(f^2 + g^2)h = 1$. Obviously, $(f^2 + g^2)h|_{X-E} = f(fh)|_{X-E} = 1$. □

It should be emphasized that the converse of Proposition 2.3 does not necessarily hold, see Remark 2.4.

Remark 2.4. The subrings of the form $I + \mathbb{R}$ and $I^u + \mathbb{R}$, where I is an ideal in $C(X)$ and I^u denotes the closure of I under the uniform topology on $C(X)$, were first introduced and studied by Rudd in [22]. These subalgebras then were again extensively studied by Azarpanah et al. in [4]. It was asserted in [22, Remark 4.1] that $I + \mathbb{R}$, for each ideal $I \in C(X)$, is closed under inversion. However, the same fact does not hold for $I^u + \mathbb{R}$, in general, see [22, Example 4.2]. According to [22, Lemma 2.2], the subalgebras $I^u + \mathbb{R}$ are closed under bounded inversion. It was also proved in [5, Theorem 2.1] that a uniformly closed subalgebra of $C(X)$ is an *LBI*-subalgebra if and only if it is closed under bounded inversion. Hence, the subalgebras $I^u + \mathbb{R}$ are all *LBI*-subalgebras of $C(X)$ as they are uniformly closed; in fact, $(I + \mathbb{R})^u = I^u + \mathbb{R}$ according to [22, Remark 2.13]. A straightforward proof shows that the subalgebras $I + \mathbb{R}$ and $I^u + \mathbb{R}$ are β-subalgebras if and only if $I = M^p$ for some $p \in \beta X$. Consequently, whenever X is a non-pseudocompact space (i.e., $C(X) \neq C^*(X)$ or equivalently $\beta X \neq \upsilon X$ where υX denotes the Hewitt realcompactification of X), then $(M^p \cap M^q)^u + \mathbb{R}$, for each $p, q \in \beta X - \upsilon X$, is an *LBI*-subalgebra which is not a β-subalgebra. Also, $M^p + \mathbb{R}$, for each $p \in \beta X - \upsilon X$, is a β-subalgebra which is not an intermediate ring.

In the final result of this section, we provide a generalization of [13, Theorem 2.3] to the β-subalgebras. It was shown in the mentioned theorem that the equality $S_A(f) = \mathrm{cl}_{\beta X} Z(f)$, for each $f \in A(X)$, characterizes $C(X)$ among its intermediate rings. We note that an intermediate ring $A(X)$ is closed under inversion if and only if it coincides with $C(X)$.

Proposition 2.5. *Let $A(X)$ be a β-subalgebra of $C(X)$. Then $A(X)$ is closed under inversion if and only if $S_A(f) = cl_{\beta X} Z(f)$ for each $f \in A(X)$.*

Proof. The sufficiency is obvious. For the necessity, assume on the contrary that there exists $f \in A(X)$ such that $\mathrm{cl}_{\beta X} Z(f) \neq S_A(f)$. Then there exists $p \in S_A(f) - \mathrm{cl}_{\beta X} Z(f)$, and thus, by the hypothesis and Proposition 2.1, there exists $g \in A(X)$ such that $p \in S_A(g)$ and $S_A(g) \cap \mathrm{cl}_{\beta X} Z(f) = \emptyset$. It follows that $Z(f^2 + g^2) = Z(f) \cap Z(g) = \emptyset$, and hence, by the hypothesis, $f^2 + g^2$ is a unit of $A(X)$ which by Proposition 2.3 implies that $S_A(f) \cap S_A(g) = S_A(f^2 + g^2) = \emptyset$; a contradiction. □

3. An *m*-closure formula for ideals of β-subalgebras

In [10], the mapping $\mathcal{Z}_A$ was applied to obtain the *m*-closure formula for ideals of intermediate rings. This mapping was first introduced in [20] to characterize maximal ideals of intermediate rings, as $\mathcal{Z}_A(f) = \{E \in Z(X) : \exists g \in A(X), fg|_{X-E} = 1\}$. This mapping has been extensively used in the context of subrings of $C(X)$, see for example [19] and [21]. The notion of β-ideals in intermediate rings introduced and studied in [6] as the ideals I for which $\mathcal{Z}_A^{-1}\mathcal{Z}_A(I) = I$, where $\mathcal{Z}_A^{-1}(\mathcal{F})$ means $\{f \in A(X) : \mathcal{Z}_A(f) \subseteq \mathcal{F}\}$ for a z-filter $\mathcal{F}$. It was established in [10] that *m*-closed ideals in intermediate rings are precisely β-ideals.

A topological description of the mapping $\mathcal{Z}_A$ in terms of the mapping S_A was investigated in Theorem 2.1 of [15]; $\mathcal{Z}_A(f) = \{Z \in Z(X) : S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\}$ for each $f \in A(X)$. Using the mentioned description, it is easy to see that β-ideals in intermediate rings are simply intersections of maximal ideals. Hence, a β-ideal in an intermediate ring $A(X)$ is a z-ideal in $A(X)$. We next extend the mentioned topological description of $\mathcal{Z}_A$ to the β-subalgebras and then show that in such subalgebras β-ideals are also precisely the intersections of maximal ideals; we need the following lemma.

Lemma 3.1. *The following statements hold for a β-subalgebra $A(X)$.*

1. *For any two disjoint closed sets F_1 and F_2 in X, there exists $f \in A(X)$, such that $F_1 \cap S_A(f) = \emptyset$ and $F_2 \subseteq S_A(f)$.*
2. *For each two disjoint closed sets F_1, F_2 in βX, there exists $f \in A(X)$ such that $F_1 \subseteq int_{\beta X} S_A(f)$ and $F_2 \cap S_A(f) = \emptyset$.*

Proof. (1) Straightforward.

(2) For each $p \in F_1$, we have $p \notin F_2$, and hence, there exists $g_p \in C^*(X)$ such that $p \in \mathrm{int}_{\beta X} Z(g_p^\beta)$ and $Z(g_p^\beta) \cap F_2 = \emptyset$. Thus, there exists $f_p \in A(X)$ such that $Z(g_p^\beta) \subseteq S_A(f_p)$ and $S_A(f_p) \cap F_2 = \emptyset$. It follows that $p \in \mathrm{int}_{\beta X} Z(g_p^\beta) \subseteq S_A(f_p)$, and thus $F_1 \subseteq \bigcup_{p \in F_1} \mathrm{int}_{\beta X} S_A(f_p)$. Hence, there exist $p_1, .., p_n \in F_1$ such that $F_1 \subseteq \bigcup_{i=1}^n \mathrm{int}_{\beta X} S_A(f_{p_i})$. We set $f = f_1 ... f_n$. It follows that $f \in A(X)$, $F_1 \subseteq \mathrm{int}_{\beta X} S_A(f)$ and $S_A(f) \cap F_2 = \emptyset$. □

Theorem 3.2. *For each element f of a β-subalgebra $A(X)$, we have*

$$\mathcal{Z}_A(f) = \{Z \in Z(X) : S_A(f) \subseteq int_{\beta X} cl_{\beta X} Z\}.$$

Proof. Similar to the proof of [15, Theorem 2.1], we can easily prove that for each $E \in \mathcal{Z}_A(f)$, $S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z$; i.e., $\mathcal{Z}_A(f) \subseteq \{Z \in Z(X) : S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z\}$. For the reverse inclusion, let $Z \in Z(X)$ and $S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z$. Hence, $S_A(f) \cap (\beta X - \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z) = \emptyset$. By Lemma 3.1, there exists $g \in A(X)$ such that $S_A(f) \cap S_A(g) = \emptyset$ and $(\beta X - \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z) \subseteq \mathrm{int}_{\beta X} S_A(g)$. As by Proposition 2.1, every β-subalgebra is an *LBI*-subalgebra, $f^2 + g^2$ is a unit in $A(X)$, and hence, $(f^2 + g^2)h = 1$ for some $h \in A(X)$. Therefore, $(f(fh))|_{X-Z} = ((f^2 + g^2)h)|_{X-Z} = 1$; i.e., $Z \in \mathcal{Z}_A(f)$. □

Using Theorem 3.2, we can easily observe that $\mathcal{Z}_A^{-1}(\mathcal{U}_p) = M_A^p$ for each $p \in \beta X$, which is clearly a maximal ideal in $A(X)$. Indeed, if $\mathcal{Z}_A(f) \subseteq \mathcal{U}_p$, and $p \notin S_A(f)$, then there exists $Z \in Z(X)$ such that $p \notin \mathrm{cl}_{\beta X} Z$ and $S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} Z$ which implies $Z \in \mathcal{Z}_A(f)$, but $Z \notin \mathcal{U}_p$; a contradiction. Hence, $\mathcal{Z}_A^{-1}(\mathcal{U}_p) \subseteq M_A^p$, and the reverse inclusion is evident. Note that $\mathcal{U}_p$ denotes the z-ultrafilter $\{Z \in Z(X) : p \in \mathrm{cl}_{\beta X} Z\}$ on X. This fact can be extended to arbitrary z-filters as it is shown in the next proposition. Note that $\bigcap_{Z\in\mathcal{F}} \mathrm{cl}_{\beta X} Z$ is designated by $\overline{\mathcal{F}}$ for a z-filter $\mathcal{F}$ on X.

Proposition 3.3. *Let $A(X)$ be a β-subalgebra and $\mathcal{F}$ be a z-filter on X. Then $\mathcal{Z}_A^{-1}(\mathcal{F}) = \bigcap_{p\in\overline{\mathcal{F}}} M_A^p$.*

Proof. From Theorem 3.2, it easily follows that $\overline{\mathcal{Z}_A(f)} = S_A(f)$ for each $f \in A(X)$. Now, if $f \in \mathcal{Z}_A^{-1}(\mathcal{F})$, then $\mathcal{Z}_A(f) \subseteq \mathcal{F}$, and hence, $\overline{\mathcal{F}} \subseteq \overline{\mathcal{Z}_A(f)} = S_A(f)$. This means that $f \in M_A^p$ for each $p \in \overline{\mathcal{F}}$ which proves the left to right inclusion. For the reverse one, let $f \in \bigcap_{p\in\overline{\mathcal{F}}} M_A^p$. It follows that $\overline{\mathcal{F}} \subseteq S_A(f)$. Therefore, if $E \in \mathcal{Z}_A(f)$, then, by Theorem 3.2, $\overline{\mathcal{F}} \subseteq S_A(f) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E$. This implies that $\beta X - \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X} E \subseteq \beta X - \overline{\mathcal{F}} = \bigcup_{Z\in\mathcal{F}}(\beta X - \mathrm{cl}_{\beta X} Z)$. Thus, there exists $Z_0 \in \mathcal{F}$, such that $\mathrm{cl}_{\beta X} Z_0 \subseteq \mathrm{cl}_{\beta X} E$ which clearly implies $E \in \mathcal{F}$. This means $\mathcal{Z}_A(f) \subseteq \mathcal{F}$; i.e., $f \in \mathcal{Z}_A^{-1}(\mathcal{F})$. □

From Theorem 3.2 and Proposition 3.3, the next statement follows. Note that, by a straightforward way, we can observe that $\overline{\mathcal{Z}_A[I]} = \theta_A(I)$ where $\mathcal{Z}_A[I] = \bigcup_{f\in I} \mathcal{Z}_A(f)$ and $\theta_A(I) = \bigcap_{f\in I} S_A(f)$.

Theorem 3.4. *Let I be an ideal in a β-subalgebra $A(X)$. Then $\mathcal{Z}_A^{-1}\mathcal{Z}_A[I] = \bigcap_{p\in\theta_A(I)} M_A^p$.*

We infer that, not only in intermediate rings but also in β-subalgebras, β-ideals are precisely the intersections of maximal ideals.

It is a known fact in $C(X)$ that whenever $Z(g) \subseteq \mathrm{int}_X Z(f)$, then $f = gh$ for some $h \in C(X)$ (see [7, 1D.1]). The next theorem extends this fact to the β-subalgebras of $C(X)$ using the mapping S_A.

Theorem 3.5. *Let $A(X)$ be a β-subalgebra of $C(X)$ and $f, g \in A(X)$. If $S_A(g) \subseteq \mathrm{int}_{\beta X} S_A(f)$, then $f = gh$, for some $h \in A(X)$.*

Proof. The proof is similar to Theorem 3.2. As $S_A(g) \subseteq \mathrm{int}_{\beta X} S_A(f)$, we have $S_A(g) \cap (\beta X - \mathrm{int}_{\beta X} S_A(f)) = \emptyset$. Thus, there exists $k \in A(X)$ such that $S_A(k) \cap S_A(g) = \emptyset$ and $(\beta X - \mathrm{int}_{\beta X} S_A(f)) \subseteq S_A(k)$. It follows that $k^2 + g^2$ is a unit of $A(X)$, and hence, there exists $l \in A(X)$ such that $(g^2 + k^2)l = 1$. If we set $h = f.g.l$, then, as $X - Z(f) \subseteq Z(k)$, we have $gh = fg^2 l = f(g^2 + k^2)l = f$. □

It is asserted in [7, 7O] that whenever I is an ideal in $C(X)$, and $f \in C(X)$ is such that $\mathrm{cl}_{\beta X} Z(f)$ is a neighborhood of $\theta(I)$ $(= \bigcap_{f\in I} \mathrm{cl}_{\beta X} Z(f))$, then $f \in I$. A countable analogue of this fact was proved in Theorem 3.4 of [2] for ideals of $C_c(X)$; i.e, whenever I is an ideal in $C_c(X)$ and $\theta_{C_c}(I) \subseteq \mathrm{int}_{\beta_\circ X}\mathrm{cl}_{\beta_\circ X} Z(f)$ where $f \in C_c(X)$, then $f \in I$. We next investigate a generalized version of this fact to the β-subalgebras of $C(X)$.

Theorem 3.6. *Let I be an ideal in a β-subalgebra $A(X)$. If $f \in A(X)$ and $\theta_A(I) \subseteq \mathrm{int}_{\beta X} S_A(f)$, then $f \in I$.*

Proof. As $\theta_A(I) \subseteq \mathrm{int}_{\beta X} S_A(f)$, $\beta X - \mathrm{int}_{\beta X} S_A(f) \subseteq \beta X - \theta_A(I) = \bigcup_{g\in I}(\beta X - S_A(g))$, and hence, there exist $g_1, ..., g_n \in I$ such that $\beta X - \mathrm{int}_{\beta X} S_A(f) \subseteq \bigcup_{i=1}^n (\beta X - S_A(g_n)) = \beta X - \bigcap_{i=1}^n S_A(g_i) = \beta X - S_A(g_1^2 + ... + g_n^2)$. By setting $g = g_1^2 + ... + g_n^2$, we will have $g \in I$ and $S_A(g) \subseteq \mathrm{int}_{\beta X} S_A(f)$. Therefore, by Theorem 3.5, there exists $h \in A(X)$ such that $f = gh$ which means $f \in I$. □

Theorem 3.7. *Let $A(X)$ be a β-subalgebra. For each $g \in A(X)$ and each positive unit $u \in A(X)$, there exists $f \in A(X)$ such that $|g - f| \leq u$ and $S_A(g) \subseteq \mathrm{int}_{\beta X} S_A(f)$.*

Proof. We first show that there exists $f \in A(X)$ such that $|g - f| \leq 1$ and $S_A(g) \subseteq \mathrm{int}_{\beta X} S_A(f)$. We set $A = g^{-1}([-\frac{1}{4}, \frac{1}{4}])$ and $B = g^{-1}((-\infty, -\frac{1}{3}] \cup [\frac{1}{3}, \infty))$. It is obvious that A and B are two disjoint zero-sets in X and hence $\mathrm{cl}_{\beta X} A \cap \mathrm{cl}_{\beta X} B = \emptyset$. As $A(X)$ is a β-ring, by Lemma 3.1, there exists $h \in A(X)$ such that $\mathrm{cl}_{\beta X} A \subseteq \mathrm{int}_{\beta X} S_A(h)$ and $S_A(h) \cap \mathrm{cl}_{\beta X} B = \emptyset$. It follows that B and $Z(h)$ are two disjoint zero-sets in X, and thus by Proposition 2.1, there exists $k \in A^*(X)$, such that $|k| \leq 1$, $k(Z(h)) = \{0\}$ and $k(B) = \{1\}$. It follows that $S_A(g) \subseteq \mathrm{cl}_{\beta X}A \subseteq \mathrm{int}_{\beta X}S_A(h) = \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(h) \subseteq \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(k) = \mathrm{int}_{\beta X}S_A(k)$. We set $f = gk$. It follows that $S_A(g) \subseteq \mathrm{int}_{\beta X}S_A(f)$ and $|g - f| \leq 1$, since, for each $x \in B$, $|g(x) - f(x)| = |g(x)(1 - k(x))| = 0$, and for each $x \notin B$, $|g(x) - f(x)| = |g(x)||1 - k(x)| \leq \frac{2}{3} \leq 1$. Now, let u be a positive unit in $A(X)$. From the above discussion, it follows that there exists $h \in A(X)$ such that $|\frac{g}{u} - h| \leq 1$. Hence, by setting $f = \frac{h}{u}$, we have $|\frac{g}{u} - f| = \frac{1}{u}|g - h| \leq 1$ and hence $|g - f| \leq u$. □

As stated in [7, 2M.1], in a topological ring, the closure of an ideal is either an ideal or the whole ring. Hence, for each $p \in \beta X$, every maximal ideal in a β-subalgebra $A(X)$ is m-closed, since, otherwise, we will have $(M^p)^m = A(X)$, which means $1 \in (M^p)^m$. Hence, there exists some $f \in B_A(1, \frac{1}{2}) \cap M^p_A$ and this means $|f - 1| < \frac{1}{2}$ which implies $f > \frac{1}{2}$, and hence, $Z(f^*) = \emptyset$. This is a contradiction as we have $f \in M^p_A$; i.e., $p \in S_A(f) \subseteq Z(f^*)$. The next theorem characterizes m-closed ideals in β-rings.

Theorem 3.8. *Let I be an ideal in a β-subalgebra $A(X)$. Then $I^m = \bigcap_{p\in\theta_A(I)} M^p_A$.*

Proof. Since each M^p_A is m-closed in $A(X)$, it follows that $I^m \subseteq \bigcap_{p\in\theta_A(I)} M^p_A$. For the reverse inclusion, let $f \in \bigcap_{p\in\theta_A(I)} M^p_A$. It follows that $\theta_A(I) \subseteq S_A(f)$. We show that $B_A(f, u) \cap I \neq \emptyset$ for each positive unit $u \in A(X)$. Let u be a given positive unit in $A(X)$. By Theorem 3.7, there exists $g \in A(X)$ such that $|f - g| \leq u$ and $S_A(f) \subseteq \mathrm{int}_{\beta X}S_A(g)$ which implies $\theta_A(I) \subseteq \mathrm{int}_{\beta X}S_A(g)$. Using Theorem 3.6, we have $g \in I$, and hence $B_A(f, u) \cap I \neq \emptyset$. □

The following corollary evidently follows from Theorem 3.4 and 3.8.

Corollary 3.9. *The m-closed ideals in β-subalgebras are exactly β-ideals.*

In Proposition 3.2 of [11], it was proved that a proper intermediate ring could never be a regular ring (in the sense of Von-Neumann). We provide a generalized version of this fact by showing that a proper β-subalgebra not closed under inversion could never be regular. We need the following lemma, an extension of [7, Theorem 7.13] to β-rings with an analogous proof. We designate by O^p_A, the ideal $\{f \in A(X) : p \in \mathrm{int}_{\beta X}S_A(f)\}$ in $A(X)$ for each $p \in \beta X$. As $\mathrm{int}_{\beta X}S_A(f) = \mathrm{int}_{\beta X}\mathrm{cl}_{\beta X}Z(f)$ for each $f \in A(X)$, we have $O^p_A = O^p \cap A(X)$ where $O^p = O^p_{C(X)}$.

Lemma 3.10. *The following statements hold for a β-subalgebra $A(X)$.*

1. *$f \in O^p_A$ if and only if $fg = 0$ for some $g \notin M^p_A$.*
2. *An ideal I in $A(X)$ is contained in a unique maximal ideal M^p_A if and only if $O^p_A \subseteq I$.*
3. *For a prime ideal P in $A(X)$, there exists a unique $p \in \beta X$, such that $O^p_A \subseteq P \subseteq M^p_A$.*

Theorem 3.11. *Let $A(X)$ be a β-subalgebra in which every ideal is m-closed. Then $A(X)$ is closed under inversion.*

Proof. Assume on the contrary that $A(X)$ is not closed under inversion. Hence, by Corollary 2.5, there exists $f \in A(X)$ such that $S_A(f) \neq \mathrm{cl}_{\beta X}Z(f)$. For $p \in S_A(f) - \mathrm{cl}_{\beta X}Z(f)$, we have $f \in M^p_A$, however, $f \notin O^p_A$; i.e., $O^p_A \neq M^p_A$. But, by Lemma 3.10, M^p_A is the only maximal ideal in $A(X)$ containing O^p_A, which, by the hypothesis and Theorem 3.8, implies that $O^p_A = M^p_A$. This contradiction shows that $A(X)$ must be closed under inversion. □

It immediately follows from Theorem 3.11 that whenever $A(X)$ is an intermediate ring in which every ideal is m-closed, or equivalently, whenever$A(X)$ is a regular ring, then $A(X) = C(X)$ [10, Remark 2.12].

Remark 3.12. We note that from regularity of a β-subalgebra $A(X)$ we can not necessarily infer that X is a P-space. There may exist a space X and a β-subalgebra $A(X)$ such that X is not a P-space, however, every ideal in $A(X)$ is m-closed. For example, consider the space Σ constructed in [7, 4M]; $\Sigma = \mathbb{N} \cup \{\sigma\}$ where $\sigma \notin \mathbb{N}$, all points of $\mathbb{N}$ are isolated, and neighborhoods of σ are the sets $U \cup \{\sigma\}$ where U is an element of a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. It is easy to see that Σ is a realcompact space, $\beta\Sigma = \beta\mathbb{N}$, and according to [1, Proposition 1.2], every point of $\beta\Sigma - \Sigma$ is a P-point of Σ. i.e., $O^p = M^p$. Let $A_p(\Sigma) = M^p + \mathbb{R}$. Then, as stated in Remark 2.4, $A_p(\Sigma)$ is a β-subalgebra of $C(\Sigma)$. Also, by Theorem 1.10 in [3], M^p, for each $p \in \beta\Sigma - \Sigma$, is a P-ideal in $C(\Sigma)$, and hence, by Proposition 2.10 in [4], $A_p(\Sigma)$ is a regular ring. It obviousely follows that every ideal in $A_p(\Sigma)$ is m-closed. However, as stated in [7, 4M.4], Σ is not a P-space.

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