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Geometries and topologies of a manifold with π-quarter-symmetric projective conformal and mutual connections

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Abstract. We propose a new π-quarter-symmetric projective conformal connection family and its mutual connection family and study its geometrical properties. We also arrive at the Schur's theorem based on a symmetric-type π-quarter-symmetric non-metric connection and investigate its geometrical property. This paper will pose a new grid computing method on manifolds, which will be done in our next topic.

1. Introduction

K. Yano in [20] considered a semi-symmetric metric connection and studied some of its properties. In [3] a semi-symmetric non-metric connection was studied. And a semi-symmetric non-metric connection that is a geometrical model for scalar-tensor theories of gravitation was studied([5]). Also other types of semi-symmetric non-metric connection were studied([1, 4, 10, 11, 22–25]). On one hand, in [2] the Schur's theorem of the Levi-Civita connection is well known based only on the second Bianchi identity([10, 12, 13]). In 1975, S. Golab [8] defined and studied quarter-symmetric lineal connections in differentiable manifolds. A lineal connection is said to be a quarter-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \pi(Y)\varphi X - \pi(X)\varphi Y \tag{1.1}$$

where φ is a tensor of type (1.1) and π is a 1-form. In [9], a projective invariant of a quarter-symmetric metric connections was obtained. Afterwards, several types of a quarter-symmetric metric connection were studied ([6, 21]) and several types of a quarter-symmetric non-metric connection were studied ([12, 13]). In a statistical manifold a conjugate symmetry condition of the non-metric connection was studied ([14, 15]). In [7] the curvature copy problem of a symmetric connection studied. In [16] the concept of a new semi-symmetric connection was introduced and its physical model studied. In [9] a quarter-symmetric metric connection was studied. The quarter-symmetric metric connection $\overset{q}{\nabla}$ satisfies the relation

$$(\overset{q}{\nabla}_z g)(X,Y) = 0,\ \overset{q}{T}(X,Y) = \pi(Y)\varphi X - \pi(X)\varphi Y \tag{1.2}$$

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The local expression of (1.2) is

$$\overset{q}{\nabla}_k g_{ji} = 0, \overset{q}{T}{}^k_{ji} = \pi_i \varphi^k_j - \pi_j \varphi^k_i \tag{1.3}$$

and the connection coefficient is

$$\overset{q}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \pi_j U^k_i - \pi_i V^k_j - U_{ij}\pi^k \tag{1.4}$$

where $\{^k_{ji}\}$ is the coefficient of the Levi-Civita connection and φ^k_j and π_j are components of (1.1)-type tensor field φ and 1-form π respectively and $U_{ij} = \frac{1}{2}(\varphi_{ij} + \varphi_{ji})$, $V_{ij} = \frac{1}{2}(\varphi_{ij} - \varphi_{ji})$ and $\pi^k = g^{ki}\pi_i$ ([9]).

In [14] the connection homotopy based on geodesic of the Levi-Civita connection was proposed. In this paper we newly defined a π-quarter-symmetric connection family based on a quarter-symmetric connection using result of the research of a quarter-symmetric connection and studied its geometric properties. And we will propose a new π-quarter-symmetric projective conformal connection family and symmetric-type π-quarter-symmetric non-metric connection satisfying the Schur's theorem and study its properties.

The paper is organized as follows. Section 1 introduced previous works. Section 2 studied the property of the π-quarter-symmetric projective conformal connection family. Section 3 studied symmetric-type π-quarter-symmetric non-metric connection satisfying the Schur's theorem.

2. A π-quarter-symmetric projective conformal connection

Let (M, g) be a Riemannian manifold ($dimM > 2$), g be the Riemannian metric on M and $\overset{0}{\nabla}$ be the Levi-Civita connection with respect to g. Let TM denote the collection of all vector fields on M.

Definition 2.1. *A connection $\overset{\pi}{\nabla}$ is called a π-quarter-symmetric non-metric connection if it satisfies*

$$\overset{\pi}{\nabla}_Z g(X, Y) = 2\pi(Z)U(X, Y), \overset{\pi}{T}(X, Y) = \pi(Y)\varphi X - \pi(X)\varphi Y$$

The local expression of this relation is

$$\overset{\pi}{\nabla}_k g_{ji} = 2\pi_k U_{ji}, \overset{\pi}{T}{}^k_{ji} = \pi_i \varphi^k_j - \pi_j \varphi^k_i$$

and the connection coefficient is

$$\overset{\pi}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \pi_i U^k_j - \pi_j U^k_i$$

Definition 2.2. *A connection family $\overset{t}{\nabla}$ is called a π-quarter-symmetric connection family if it satisfies the relation*

$$\overset{t}{\nabla}_Z g(X, Y) = 2t\pi(Z)U(X, Y), \overset{t}{T}(X, Y) = \pi(Y)\varphi X - \pi(X)\varphi Y$$

where t is a family parameter ($t \in \mathbb{R}$).

The local expression of this connection is

$$\overset{t}{\nabla}_k g_{ji} = 2t\pi_k g_{ji}, \overset{t}{T}{}^k_{ji} = \pi_i \varphi^k_j - \pi_j \varphi^k_i$$

The connection coefficient is given as

$$\overset{t}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - t\pi_j U^k_i - \pi_i V^k_j - (t-1)\pi_j U^k_i + (t-1)U_{ij}\pi^k$$

Remark 2.1. *The π-quarter-symmetric connection family $\overset{t}{\nabla}$ is a connection homotopy from connection $\overset{q}{\nabla}$ to the connection $\overset{\pi}{\nabla}$. Namely if $t = 0$, then $\overset{t}{\nabla} = \overset{q}{\nabla}$ and if $t = 1$, then $\overset{t}{\nabla} = \overset{\pi}{\nabla}$.*

The connection $\overset{c}{\nabla}$ is called a π-quarter-symmetric conformal connection family, if $\overset{c}{\nabla}$ is conformally equivalent to a π-quarter-symmetric connection family $\overset{t}{\nabla}$. The coefficient of the connection $\overset{c}{\nabla}$ is

$$\overset{c}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + \sigma_i\delta^k_j - \sigma_j\delta^k_i - g_{ij}\delta^k - t\pi_iU^k_j - \pi_iV^k_j - (t-1)\pi_jU^k_i + (t-1)U_{ij}\pi^k$$

where the conformal metric $\bar{g}_{ij} = e^{2\sigma}g_{ij}$ and $\sigma_i = \partial_i\sigma$.

The connection $\overset{p}{\nabla}$ is called a π-quarter-symmetric projective connection family, if $\overset{p}{\nabla}$ is projectively equivalent to a π-quarter-symmetric connection family $\overset{t}{\nabla}$. The coefficient of the connection $\overset{p}{\nabla}$ is

$$\overset{p}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + \psi_i\delta^k_j - \psi_j\delta^k_i - t\pi_iU^k_j - \pi_iV^k_j - (t-1)\pi_jU^k_i + (t-1)U_{ij}\pi^k$$

where ψ_i is a projective component.

Definition 2.3. *A connection is ∇ called a π-quarter-symmetric projective conformal connection family, if ∇ is projectively and conformally equivalent to a π-quarter-symmetric connection family $\overset{t}{\nabla}$.*

In a Riemannian manifold, a π-quarter-symmetric projective conformal connection family ∇ satisfies the relation

$$\begin{cases} \nabla_Z g(X,Y) = -2[\Psi(Z) + Z\sigma]g(X,Y) - \psi(X)g(Y,Z) - \psi(Y)g(X,Z) + 2t\pi(Z)U(X,Y), \\ T(X,Y) = \pi(Y)\varphi X - \pi(X)\varphi Y \end{cases} \tag{2.1}$$

The local expression of this relation is

$$\begin{cases} \nabla_k g_{ji} = -2(\psi_k + \sigma_k)g_{ij} - \psi_i g_{jk} - \Psi_j g_{ik} + 2t\pi_k U_{ij}, \\ T^{\ k}_{ji} = \pi_i\varphi^k_j - \pi_j\varphi^k_i \end{cases} \tag{2.2}$$

and its coefficient is

$$\Gamma^{\ k}_{ij} = \{^{k}_{ij}\} + (\psi_i + \sigma_i)\delta^k_j + (\psi_j + \sigma_j)\delta^k_i - g_{ij}\sigma^k - t\pi_iU^k_j - \pi_iV^k_j - (t-1)\pi_jU^k_i + (t-1)U_{ij}\pi^k \tag{2.3}$$

Remark 2.2. *If $\Psi = 0$, then $\nabla = \overset{0}{\nabla}$ and if $\sigma_i = 0$, then $\nabla = \overset{p}{\nabla}$. If $\Psi_i = \sigma_i = 0$, then $\nabla = \overset{t}{\nabla}$. And if $t = 1$ and $\psi_i = \sigma_i = 0$, then $\nabla = \overset{\pi}{\nabla}$ and if $t = 0$ and $\psi_i = \sigma_i = 0$, then $\nabla = \overset{q}{\nabla}$. And if $t = 0$, then the π-quarter-symmetric projective conformal connection family ∇ is a quarter-symmetric projective conformal connection ([18]). And if $\varphi X = X$, then the π-quarter-symmetric projective conformal connection family ∇ is a projective conformal semi-symmetric connection family.*

From the expression (2.3) we find that the curvature tensor of ∇ is

$$\begin{aligned} R_{ijk}{}^{l} &= K_{ijk}{}^{l} + \delta^l_j\alpha_{ik} - \delta^l_i\alpha_{jk} + g_{ik}\beta^l_j - g_{jk}\beta^l_i + (t-1)(U^l_i\gamma_{jk} - U^l_j\gamma_{ik} + U_{jk}\gamma^l_i - U_{ik}\gamma^l_j) \\ &+ t(\pi_iU^l_{jk} - \pi_jU^l_{ik}) + (t-1)(\pi_iU^l_j\Psi_k - \pi_jU^l_i\Psi_k) + \delta^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) + \pi_iV^l_{jk} - \pi_jV^l_{ik} \\ &+ (t-1)[(\overset{0}{\nabla}_iU_{jk} - \overset{0}{\nabla}_jU_{ik})\pi^l - (\overset{0}{\nabla}_iU^l_j - \overset{0}{\nabla}_jU^l_i)\pi_k] - tU^l_k(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i) - V^l_k(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i) \end{aligned} \tag{2.4}$$

where ${K_{ijk}}^l$ is the curvature tensor of $\overset{0}{\nabla}$ of g_{ij}, and

$$\begin{cases} \alpha_{ik} &= \overset{0}{\nabla}_i(\psi_k+\sigma_k)-(\psi_i+\sigma_i)(\psi_k+\sigma_k)+(t-1)U_i^p(\psi_p+\sigma_p)\pi_k-(t-1)U_{ik}(\psi_p+\sigma_p)\pi^p+g_{ik}(\psi_p+\sigma_p)\sigma^p \\ \beta_{ik} &= \overset{0}{\nabla}_i\sigma_k-\sigma_i\sigma_k+(t-1)U_{ip}\sigma^p\pi_k-(t-1)U_{ik}\pi_p\sigma^p \\ \gamma_{ik} &= \overset{0}{\nabla}_i\pi_k-\pi_i\sigma_k+(t-1)U_{ip}\sigma^p\pi_k-\frac{1}{2}(t-1)U_{ik}\pi_p\pi^p \\ U_{ik}^l &= \overset{0}{\nabla}_iU_k^l-U_i^l(\psi_k+\sigma_k)+(t-1)U_i^pU_p^l\pi_k-(t-1)U_i^lU_k^p\pi_p+(t-1)U_{ip}U_k^p\pi^l-(t-1)U_{ik}U_p^l\pi^p \\ &\quad -U_{ik}\sigma^l+g_{ik}U_p^l\sigma^p+\delta_i^lU_k^p(\Psi_p+\sigma_p) \\ V_{ik}^l &= \overset{0}{\nabla}_iV_k^l-V_i^l(\psi_k+\sigma_k)+(t-1)U_i^pV_p^l\pi_k-(t-1)U_i^lV_k^p\pi_p+(t-1)U_{ip}V_k^p\pi^l-(t-1)U_{ik}V_p^l\pi^p \\ &\quad -V_{ik}\sigma^l+g_{ik}V_p^l\sigma^p+\delta_i^lV_k^p(\psi_p+\sigma_p) \end{cases} \tag{2.5}$$

From (2.2), the mutual connection $\overset{m}{\nabla}$ of the π-quarter-symmetric projective conformal connection family ∇ satisfies the relation

$$\begin{cases} \overset{m}{\nabla}_k g_{ji} = -2(\psi_k+\sigma_k)g_{ij}-\psi_ig_{jk}-\psi_jg_{ik}+2(t-1)\pi_kU_{ij}+\pi_i\varphi_{jk}+\pi_j\varphi_{ik}, \\ \overset{m}{T}{}^k_{ji} = \pi_i\varphi_j^k-\pi_j\varphi_i^k \end{cases} \tag{2.6}$$

and its coefficient is

$$\overset{m}{\Gamma}{}^k_{ij} = \left\{ {k \atop ij} \right\}+(\psi_i+\sigma_i)\delta_j^k+(\psi_j+\sigma_j)\delta_i^k-g_{ij}\sigma^k-(t-1)\pi_iU_j^k-\pi_jV_i^k+(t-1)U_{ij}\pi^k \tag{2.7}$$

And the curvature tensor of $\overset{m}{\nabla}$ is

$$\begin{aligned} \overset{m}{R}{}_{ijk}{}^l \;=\;& {K_{ijk}}^l+\delta_j^l\overset{m}{\alpha}_{ik}-\delta_i^l\overset{m}{\alpha}_{jk}+g_{ik}\overset{ml}{\beta}_j-g_{jk}\overset{ml}{\beta}_i+(t-1)(U_{jk}\overset{ml}{\pi}_i-U_{ik}\overset{ml}{\pi}_j+\overset{m}{U}{}_{jk}^{\,l}\pi_i-\overset{m}{U}{}_{ik}^{\,l}\pi_j) \\ +\;& t(\overset{m}{\rho}_{jk}U_i^l-\overset{m}{\rho}_{ik}U_j^l)-(t-1)U_k^l(\overset{0}{\nabla}_i\pi_j-\overset{0}{\nabla}_j\pi_i)+\delta_k^l(\overset{0}{\nabla}_i\Psi_j-\overset{0}{\nabla}_j\Psi_i)+(\overset{m}{V}{}_{ij}^{\,l}-\overset{m}{V}{}_{ji}^{\,l})\pi_k \\ +\;& (t-1)(\overset{0}{\nabla}_iU_{jk}-\overset{0}{\nabla}_jU_{ik})\pi^l-t(\overset{0}{\nabla}_iU_j^l-\overset{0}{\nabla}_jU_i^l)\pi_k+V_i^l\overset{m}{\gamma}_{jk}-V_j^l\overset{m}{\gamma}_{ik} \end{aligned} \tag{2.8}$$

where

$$\begin{cases} \overset{m}{\alpha}_{ik} &= \overset{0}{\nabla}_i(\psi_k+\sigma_k)-(\psi_i+\sigma_i)(\psi_k+\sigma_k)+(t-1)U_k^p(\psi_p+\sigma_p)\pi_i-(t-1)U_{ik}(\psi_p+\sigma_p)\pi^p \\ &\quad +\; tU_i^p(\psi_p+\sigma_p)\pi_k \\ \overset{m}{\beta}_{ik} &= \overset{0}{\nabla}_i\sigma_k-\sigma_i\sigma_k+(t-1)U_{ip}\sigma^p\pi_k-(t-1)U_{kp}\pi_i\sigma^p-tU_{ik}\pi_p\sigma^p+g_{ik}(\psi_p+\sigma_p)\sigma^p \\ \overset{m}{\rho}_{ik} &= \overset{0}{\nabla}_i\sigma_k-\pi_i(\psi_k+\sigma_k)+tU_i^p\pi_p\sigma_k \\ \overset{m}{\pi}_{ik} &= \overset{0}{\nabla}_i\sigma_k-\pi_i\sigma_k+(t-1)U_{ip}\pi^p\pi_k-tU_{ik}\pi_p\pi^p \\ \overset{m}{\gamma}_{ik} &= \overset{0}{\nabla}_i\pi_k-\pi_i(\psi_k+\sigma_k)+(t-1)U_k^p\pi_i\pi_p+g_{ik}\pi_p\sigma^p+tU_i^p\pi_p\pi_k+V_i^p\pi_p\pi_k-(t-1)U_{ik}\pi_p\pi^p \\ \overset{m}{U}{}_{ik}{}^l &= \overset{0}{\nabla}_iU_k^l-U_i^l(\psi_k+\sigma_k)+tU_i^pU_p^l\pi_k-tU_i^lU_k^p\pi_p+(t-1)U_{ip}U_k^p\pi^l-(t-1)U_{ik}U_p^l\pi^p-U_{ik}\sigma^l \\ \overset{m}{V}{}^l_{ij} &= \overset{0}{\nabla}_iV_j^l+\delta_i^lV_j^p(\psi_p+\sigma_p)+V_{ij}\sigma^l+(t-1)U_p^lV_i^p\pi_j-tU_i^lV_j^p\pi_p+(t-1)U_{ip}V_j^p\pi^l \end{cases} \tag{2.9}$$

Let f and s be two 1-form with their components

$$f_i = \pi_i^k\pi_k,\;\; s_i = \phi_k^k\pi_i \tag{2.10}$$

Theorem 2.1. *In a Riemannian manifold, if a 1-form ψ and s are closed, then a volume curvature tensor of the π-quarter-symmetric projective conformal family ∇ is zero, namely*

$$P_{ij} = 0 \tag{2.11}$$

where $P_{ij} \hat{=} R_{ijkl}g^{kl}$ is a volume curvature tensor of ∇.

Proof. Contracting the indices k and l of (2.4), then we obtain

$$\begin{aligned} P_{ij} &= \overset{0}{P}_{ij} + \alpha_{ij} - \alpha_{ji} + \beta_{ji} - \beta_{ij} + (t-1)(U_i^k\gamma_{jk} - U_j^k\gamma_{ik} + U_{jk}\gamma_i^k - U_{ik}\gamma_j^k) + t(\pi_j U_{jk}^k - \pi_j U_{ik}^k) \\ &+ (t-1)(\pi_i U_j^k\psi_k - \pi_j U_i^k\psi_k) + n(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) - tU_k^k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) + \pi_i V_{jk}^k - \pi_j V_{ik}^k \\ &+ (t-1)[(\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^k - (\overset{0}{\nabla}_i U_j^k - \overset{0}{\nabla}_j U_i^k)\pi_k] - V_k^k(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i) \end{aligned} \tag{2.12}$$

where $\overset{0}{P}_{ij}$ is the volume curvature tensor of the Levi-Civita connection $\overset{0}{\nabla}$ of g_{ij}.

Since $\overset{0}{P}_{ij} = 0, V_k^k = 0$, by using the expression (2.5), we have

$$\begin{aligned} &\alpha_{ij} - \alpha_{ji} + \beta_{ji} - \beta_{ij} = (\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) - (t-1)(\pi_i U_j^p\psi_p - \pi_j U_i^p\psi_p), \\ &U_i^k\gamma_{jk} - U_j^k\gamma_{ik} + U_{jk}\gamma_i^k - U_{ik}\gamma_j^k = 0, \\ &\pi_i U_{jk}^k - \pi_j U_{ik}^k = \pi_i\overset{0}{\nabla}_j U_k^k - \pi_j\overset{0}{\nabla}_i U_k^k, \\ &(\pi_i\overset{0}{\nabla}_i U_{jk} - \pi_i\overset{0}{\nabla}_j U_{ik})\pi^k - (\nabla_i U_j^k - \nabla_j U_i^k)\pi_k = 0, \\ &V_{ik}^k = V_{jk}^k = 0. \end{aligned}$$

Substituting these expressions above into (2.12) and using the expression (2.10) we have

$$P_{ij} = (n+1)(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) - t(\overset{0}{\nabla}_i s_j - \overset{0}{\nabla}_j s_i) \tag{2.13}$$

If a 1-form ψ and a 1-form s are of closed, then $\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i = 0$ and $\overset{0}{\nabla}_i s_j - \overset{0}{\nabla}_j s_i = 0$. Hence from (2.13), we obtain the expression (2.11). This ends the proof of Theorem 2.1. □

Remark 2.3. *If $\psi = 0$, then $\nabla = \overset{0}{\nabla}$. From (2.12) if $t = 0$, then $\overset{c}{P}_{ij} = 0$ and $\overset{q}{P}_{ij} = 0$. If $t = 1$ and a 1-form s is a closed form, then $\overset{\pi}{P}_{ij} = 0$.*

Theorem 2.2. *In a Riemannian manifold, if a 1-form ψ, s and f are closed forms, then a volume curvature tensor of the mutual connection $\overset{m}{\nabla}$ of the π-quarter-symmetric projective conformal connection family ∇ is zero, namely*

$$\overset{m}{P}_{ij} = 0, \tag{2.14}$$

where $\overset{m}{P}_{ij} = 0$ is a volume curvature tensor of $\overset{m}{\nabla}$.

Proof. Contracting the indices k and l to (2.8), then we have

$$\begin{aligned} \overset{m}{P}_{ij} &= \overset{0}{P}_{ij} + \overset{m}{\alpha}_{ij} - \overset{m}{\alpha}_{ji} + \overset{m}{\beta}_{ji} - \overset{m}{\beta}_{ij} + t(U_i^k\overset{m}{\rho}_{jk} - U_j^k\overset{m}{\rho}_{ik}) + (t-1)(U_{jk}\overset{mk}{\pi_i} - U_{ik}\overset{mk}{\pi_j}) + (t-1)(\pi_i\overset{m}{U}{}_{jk}^{k} - \pi_j\overset{m}{U}{}_{ik}^{k}) \\ &+ n(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) - (t-1)U_k^k(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i) + (t-1)(\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^k - t(\overset{0}{\nabla}_i U_j^k - \overset{0}{\nabla}_j U_i^k)\pi_k \\ &+ V_i^k\overset{0}{\gamma}_{jk} - V_j^k\overset{0}{\gamma}_{ik} + (\overset{m}{V}{}_{ij}^{k} - \overset{m}{V}{}_{ji}^{k})\pi_k \end{aligned}$$

From (2.9) we arrive at

$$\begin{aligned} \overset{m}{\alpha}_{ij} - \overset{m}{\alpha}_{ji} + \overset{m}{\beta}_{ji} - \overset{m}{\beta}_{ij} &= (\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) - [\pi_i U_j^p(\psi_p + \sigma_p) - \pi_j U_i^p(\psi_p + \sigma_p)] + 2(t-1)(\pi_i U_{jp}\sigma^p - \pi_j U_{ip}\sigma^p) \\ t(U_i^k\overset{m}{\rho}_{jk} - U_j^k\overset{m}{\rho}_{ik}) &= t(U_i^k\overset{0}{\nabla}_j\pi_k - U_j^k\overset{0}{\nabla}_i\pi_k) + t[\pi_i U_j^k(\psi_k + \sigma_k) - \pi_j U_i^k(\psi_k + \sigma_k)] \end{aligned}$$

$$(t-1)(U_{jk}\overset{m}{\pi}{}^{k}_{i} - U_{ik}\overset{m}{\pi}{}^{k}_{j}) = (t-1)(U_{jk}\overset{0}{\nabla}_i\pi^k - U_{ik}\overset{0}{\nabla}_j\pi^k) - (t-1)(\pi_i U_{jk}\sigma^k - \pi_j U_{ik}\sigma^k)$$

$$\begin{aligned}(t-1)(\pi_i\overset{m}{U}{}^{k}_{jk} - \pi_j\overset{m}{U}{}^{k}_{ik}) &= (t-1)(\pi_i\overset{0}{\nabla}_j U^k_k - \pi_j\overset{0}{\nabla}_i U^k_k) + (t-1)[\pi_i U^p_j(\psi_p+\sigma_p) - \pi_j U^p_i(\psi_p+\sigma_p)]\\ &\quad - (t-1)(\pi_i U_{jk}\sigma^k - \pi_j U_{ik}\sigma^k)\end{aligned}$$

$$V^k_i\overset{0}{\gamma}_{jk} - V^k_j\overset{0}{\gamma}_{ik} + (\overset{m}{V}{}^{k}_{ij} - \overset{m}{V}{}^{k}_{ji})\pi_k = V^k_i\overset{0}{\nabla}_j\pi_k - V^k_j\overset{0}{\nabla}_i\pi_k + \pi_k(\overset{0}{\nabla}_i V^k_j - \overset{0}{\nabla}_j V^k_i).$$

On the other hand, there holds the following

$$\overset{0}{p}_{ij} = 0,\ V^k_k = 0, f_i = (U^k_i + V^k_i)\pi_k, s_i = (U^k_k + V^k_k)\pi_i.$$

Hence

$$\overset{m}{P}_{ij} = (n+1)(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) - (t-1)(\overset{0}{\nabla}_i s_j - \overset{0}{\nabla}_j s_i) - (\overset{0}{\nabla}_i f_j - \overset{0}{\nabla}_j f_i). \tag{2.15}$$

If a 1-form ψ and s are of closed, then $(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) = 0$, $(\overset{0}{\nabla}_i s_j - \overset{0}{\nabla}_j s_i) = 0$ and $(\overset{0}{\nabla}_i f_j - \overset{0}{\nabla}_j f_i) = 0$. Hence from the expression (2.15), we obtain that (2.14) is tenable. □

Denote by

$$\begin{aligned}A_{ijk}{}^{l} \ &\hat{=}\ \delta^l_j\alpha_{ik} + g_{ik}\beta^l_j + (t-1)(U^l_i\gamma_{jk} + U_{jk}\gamma^l_i) + t\pi_i U^l_{jk} + (t-1)\pi_i U^l_j\psi_k + \delta^l_k\overset{0}{\nabla}_i\psi_j - tU^l_k\overset{0}{\nabla}_i\pi_j\\ &+\ \pi_i V^l_{jk} - V^l_k\overset{0}{\nabla}_i\pi_j + (t-1)(\overset{0}{\nabla}_i U_{jk}\pi^l - \overset{0}{\nabla}_i U^l_j\pi_k)\end{aligned}$$

and

$$\begin{aligned}\overset{m}{A}_{ijk}{}^{l} \ &\hat{=}\ \delta^l_j\overset{m}{\alpha}_{ik} + g_{ik}\overset{m}{\beta}{}^{l}_{j} + tU^l_i\overset{m}{\beta}_{jk} + (t-1)U_{jk}\overset{m}{\pi}{}^{l}_{i} + (t-1)\pi_i\overset{m}{U}{}^{l}_{jk} + \delta^l_k\overset{0}{\nabla}_i\psi_j - (t-1)U^l_k\overset{0}{\nabla}_i\pi_j\\ &+\ (t-1)\overset{0}{\nabla}_i U_{jk}\pi^l - t\pi_i U^l_j\pi_k + V^l_i\overset{m}{\gamma}_{jk} + \overset{m}{V}{}^{l}_{ij}\pi_k.\end{aligned}$$

From these expressions, the expressions (2.4) and (2.8) are

$$R_{ijk}{}^{l} = K_{ijk}{}^{l} + A_{ijk}{}^{l} - A_{jik}{}^{l},$$

and

$$\overset{m}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + \overset{m}{A}_{ijk}{}^{l} - \overset{m}{A}_{jik}{}^{l},$$

respectively. So there exists the following.

Theorem 2.3. *When $A_{ijk}{}^{l} = A_{jik}{}^{l}$, then the curvature tensor will keep unchanged under the connection transformation $\overset{0}{\nabla} \to \nabla$ and when $\overset{m}{A}_{ijk}{}^{l} = \overset{m}{A}_{jik}{}^{l}$, then the curvature tensor will keep unchanged under the connection transformation $\overset{0}{\nabla} \to \overset{m}{\nabla}$.*

From (2.2) and (2.3), the connection coefficient of dual connection $\overset{*}{\nabla}$ of the π-quarter-symmetric projective conformal connection family ∇ is

$$\overset{*}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} - (\psi_i+\sigma_i)\delta^k_j + \sigma_j\delta^k_i - g_{ij}(\psi^k+\sigma^k) - (t-1)\pi_j U^k_i + t\pi_i U^k_j - \pi_i V^k_j + (t-1)U_{ij}\pi^k$$

and the curvature tensor of $\overset{*}{\nabla}$ is

$$\begin{aligned}\overset{*}{R}_{ijk}{}^{l} \ &=\ K_{ijk}{}^{l} + \delta^l_j\beta_{ik} - \delta^l_i\beta_{jk} + g_{ik}\alpha^l_j - g_{jk}\beta^l_i + (t-1)(U^l_i\gamma_{jk} - U^l_j\gamma_{ik} + U_{jk}\gamma^l_i - U_{ik}\gamma^l_j)\\ &-\ t(\overset{*}{U}{}^{l}_{jk}\pi_i - \overset{*}{U}{}^{l}_{ik}\pi_j) - (t-1)(\pi_i U_{jk}\psi^l - \pi_j U_{ik}\psi^l) - \delta^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) + tU^l_k(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i)\\ &+\ (t-1)[(\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^l - (\overset{0}{\nabla}_i U^l_j - \overset{0}{\nabla}_j U^l_i)\pi_k] + \pi_i\overset{*}{V}{}^{l}_{jk} - \pi_j\overset{*}{V}{}^{l}_{ik} - V^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i),\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
\overset{*}{U}{}^{l}_{ik} &= \overset{0}{\nabla}_i U^l_k + \delta^l_i U^p_k \sigma_p - (t-1)U^l_i U^p_k \pi_p + (t-1)U_{ip}U^p_k \pi^l - U_{ik}(\psi^l + \sigma^l) - U^l_i \sigma_k - (t-1)U_{ik}U^l_p \pi^p \\
&+ (t-1)U^p_i U^l_k \pi_k + g_{ik}U^l_p(\psi^p + \sigma^p), \\
\overset{*}{V}{}^{l}_{ik} &= \overset{0}{\nabla}_i V^l_k + \delta^l_i V^p_k \sigma_p - (t-1)U^l_i V^p_k \pi_p + (t-1)U_{ip}V^p_k \pi^l - V_{ik}(\psi^l + \sigma^l) - V^l_i \sigma_k - (t-1)U_{ik}V^l_p \pi^p \\
&+ (t-1)U^p_i V^l_k \pi_k + g_{ik}V^l_p(\psi^p + \sigma^p),
\end{aligned}$$

From the expressions (2.4) and (2.16), we obtain

$$\begin{aligned}
\overset{*}{R}_{ijk}{}^{l} &= K_{ijk}{}^{l} + \delta^l_i(\alpha_{jk} - \beta_{jk}) - \delta^l_j(\alpha_{ik} - \beta_{ik}) + g_{ik}(\alpha^l_j - \beta^l_j) - g_{jk}(\alpha^l_i - \beta^l_i) + t[\pi_i(\overset{*}{U}{}^{l}_{jk} + U^{\ l}_{jk}) - \pi_j(\overset{*}{U}{}^{l}_{ik} + U^{\ l}_{ik})] \\
&- (t-1)[\pi_i(U_{jk}\psi^l + U^l_j\psi_k) - \pi_j(U_{ik}\psi^l + U^l_i\psi_k)] - 2\delta^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) + 2tU^l_k(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i) \qquad (2.17) \\
&+ \pi_i(\overset{*}{V}{}^{l}_{jk} - V^{\ l}_{jk}) - \pi_j(\overset{*}{V}{}^{l}_{ik} - V^{\ l}_{ik}),
\end{aligned}$$

From (2.6) and (2.16), the connection coefficient of dual connection $\overset{m*}{\nabla}$ of the mutual connection $\overset{m}{\nabla}$ of the π-quarter-symmetric projective conformal connection family ∇ is

$$\overset{m*}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} - (\psi_i + \sigma_i)\delta^k_j + \sigma_j\delta^k_i - g_{ij}(\psi^k + \sigma^k) + (t-1)\pi_i U^k_j - (t-1)\pi_j U^k_i + tU_{ij}\pi^k + V_{ij}\pi^k$$

and the curvature tensor of $\overset{m}{\nabla}$ is

$$\begin{aligned}
\overset{m*}{R}_{ijk}{}^{l} &= K_{ijk}{}^{l} + \delta^l_j\overset{m}{\beta}_{ik} - \delta^l_i\overset{m}{\beta}_{jk} + g_{ik}\overset{ml}{\alpha}_j - g_{jk}\overset{ml}{\alpha}_i + (t-1)(U^l_i\overset{m}{\pi}_{jk} - U^l_j\overset{m}{\pi}_{ik} + \overset{m*}{U}{}^{l}_{ik}\pi_j - \overset{m*}{U}{}^{l}_{jk}\pi_i) \\
&+ t(\overset{m}{\rho}{}^{l}_{i}U_{jk} - \overset{m}{\rho}{}^{l}_{j}U_{ik}) - (t-1)(\overset{0}{\nabla}_i U^l_j - \overset{0}{\nabla}_j U^l_i)\pi_k - \delta^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) + t(\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^l \qquad (2.18) \\
&+ tU^l_k(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i) + (V_{jk}\overset{m}{\gamma}{}^{l}_{i} - V_{ik}\overset{m}{\gamma}{}^{l}_{j}) + (\overset{m*}{V}_{ijk} - \overset{m*}{V}_{jik})\pi^l,
\end{aligned}$$

where

$$\begin{aligned}
\overset{m*}{U}{}^{l}_{jk} &= \overset{0}{\nabla}_i U^l_k - U^l_i\sigma_k - (t-1)U^l_i U^p_k\pi_p - U_{ik}(\psi^l + \sigma^l) + tU_{ip}U^p_k\pi^l - tU_{ik}U^l_p\pi^p + (t-1)U^p_i U^l_p\pi_k, \\
\overset{m*}{V}{}^{l}_{jk} &= \overset{0}{\nabla}_i V^l_k + V_{ij}\sigma_k + g_{ik}V_{jp}(\psi^p + \sigma^p) - (t-1)\pi_i V_{jp}U^p_k + (t-1)U^p_i V_{jp}\pi_k - tU_{ik}V_{jp}\pi^p.
\end{aligned}$$

From (2.8) and (2.18), we obtain

$$\begin{aligned}
\overset{m*}{R}_{ijk}{}^{l} &= \overset{m}{R}_{ijk}{}^{l} + \delta^l_i(\overset{m}{\alpha}_{jk} - \overset{m}{\beta}_{jk}) - \delta^l_j(\overset{m}{\alpha}_{ik} - \overset{m}{\beta}_{ik}) + (t-1)(U^l_i\overset{m}{\pi}_{jk} - U^l_j\overset{m}{\pi}_{ik} + \overset{m}{\pi}{}^{l}_{j}U_{ik} - \overset{m}{\pi}{}^{l}_{i}U_{jk}) + g_{ik}(\overset{ml}{\alpha}_j - \overset{ml}{\beta}_j) \\
&+ t(\overset{m}{\rho}_{ik}U^l_j - \overset{m}{\rho}_{jk}U^l_i + U_{jk}\overset{m}{\rho}{}^{l}_{i} - U_{ik}\overset{m}{\rho}{}^{l}_{j}) + (t-1)[(\overset{m*}{U}_{ik}{}^{l} + \overset{m}{U}_{ik}{}^{l})\pi_j - (\overset{m*}{U}_{jk}{}^{l} + \overset{m}{U}_{jk}{}^{l})\pi_i] \qquad (2.19) \\
&- 2\delta^l_k(\overset{0}{\nabla}_i\psi_j - \overset{0}{\nabla}_j\psi_i) - g_{jk}(\overset{ml}{\alpha}_i - \overset{ml}{\beta}_i) + 2(t-1)U^l_k(\overset{0}{\nabla}_i\pi_j - \overset{0}{\nabla}_j\pi_i) + (\overset{0}{\nabla}_i U^l_j - \overset{0}{\nabla}_j U^l_i)\pi_k \\
&+ (\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^l + (V_{jk}\overset{m}{\gamma}{}^{l}_{i} - V_{ik}\overset{m}{\gamma}{}^{l}_{j} + V^l_j\overset{m}{\gamma}_{ik} - V^l_i\overset{m}{\gamma}_{jk}) - (\overset{m}{V}{}^{l}_{ij} - \overset{m}{V}{}^{l}_{ji})\pi_k + (\overset{m*}{V}_{ijk} - \overset{m*}{V}_{jik})\pi^l,
\end{aligned}$$

Denote by

$$\begin{aligned}
B_{ijk}{}^{l} &= \delta^l_i(\alpha_{jk} - \beta_{jk}) + g_{ik}(\alpha^l_j - \beta^l_j) - t\pi_i(\overset{m*}{U}{}^{l}_{jk} + \overset{m}{U}{}^{l}_{jk}) - (t-1)\pi_i(U_{jk}\psi^l + U^l_j\psi_k) - 2\delta^l_k\overset{0}{\nabla}_i\psi_j \\
&+ 2tU^l_k\overset{0}{\nabla}_i\pi_j + \pi_i(\overset{m*}{U}{}^{l}_{jk} - U^{\ l}_{jk})
\end{aligned}$$

and

$$\begin{aligned}\overset{m}{B}_{ijk}{}^{l} &= \delta_i^l(\overset{m}{\alpha}_{jk} - \overset{m}{\beta}_{jk}) + g_{ik}(\overset{m}{\alpha}{}^{l}_{j} - \overset{m}{\beta}{}^{l}_{j}) + (t-1)(U_i^l\overset{m}{\pi}_{jk} + U_{ik}\overset{m}{\pi}{}^{l}_{j}) + t(U_j^l\overset{m}{\rho}_{ik} + U_{jk}\overset{m}{\rho}{}^{l}_{i}) - 2\delta_k^l\overset{0}{\nabla}_i\pi_j + \overset{0}{\nabla}_iU_j^l\pi_k \\ &+ (t-1)(\overset{m*}{U}{}^{l}_{ik} + \overset{m}{U}{}^{l}_{ik})\pi_j + 2(t-1)U_k^l\overset{0}{\nabla}_i\pi_j + (\overset{m*}{V}_{ijk}\pi^l - \overset{m*}{V}_{ij}{}^{l}\pi_k + V_{jk}\overset{m}{\gamma}{}^{l}_{i} + V_j^l\overset{m}{\gamma}_{ik}) + \overset{0}{\nabla}_iU_{jk}\pi^k\end{aligned}$$

Then we get by (2.17) and (2.19) there holds

$$\overset{m}{R}_{ijk}{}^{l} = R_{ijk}{}^{l} + B_{ijk}{}^{l} - mB_{jik}{}^{l},$$

and

$$\overset{m*}{B}_{ijk}{}^{l} = \overset{m}{B}_{ijk}{}^{l} + \overset{m}{B}_{ijk}{}^{l} - \overset{m}{B}_{jik}{}^{l},$$

respectively. Thus we arrive at the following

Theorem 2.4. *When $B_{ijk}{}^{l} = B_{jik}{}^{l}$, then the π-quarter-symmetric projective conformal connection family ∇ is a conjugate symmetry and when $\overset{m}{B}_{ijk}{}^{l} = \overset{m}{B}_{jik}{}^{l}$, then the mutual connection $\overset{m}{\nabla}$ of ∇ is a conjugate symmetric connection.*

3. Symmetric-type π-quarter-symmetric non-metric connection

Now, let ψ_{ij} be symmetric, that is, $\psi_{ij} = U_{ij}$.

Definition 3.1. *In a Riemannian manifold, a connection D is called a symmetric-type π-quarter-symmetric non-metric connection, if the connection D satisfies*

$$D_Zg(X,Y) = 2\pi(Z)U(X,Y),\ T(X,Y) = \pi(Y)UX - \pi(X)UY.$$

The local expression of this expression is

$$D_kg_{ij} = 2\pi_kU_{ij},\ \ T_{ij}^k = \pi_jU_i^k - \pi_iU_j^k, \tag{3.1}$$

and the connection coefficient is

$$\Gamma^{k}_{ij} = \{^{k}_{ij}\} - \pi_iU_j^k \tag{3.2}$$

and the curvature tensor is

$$R_{ijk}{}^{l} = K_{ijk}{}^{l} - \overset{0}{\nabla}_i(\pi_jU_k^l) + \overset{0}{\nabla}_j(\pi_iU_k^l) \tag{3.3}$$

From (3.1) and (3.2), the connection coefficient of dual connection $\overset{*}{D}$ of the symmetric-type π-quarter-symmetric non-metric connection D is

$$\overset{*}{\Gamma}{}^{k}_{ij} = \{^{k}_{ij}\} + \pi_iU_j^k$$

and the curvature tensor is

$$\overset{*}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + \overset{0}{\nabla}_i(\pi_jU_k^l) - \overset{0}{\nabla}_j(\pi_iU_k^l) \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$\overset{*}{R}_{ijk}{}^{l} = K_{ijk}{}^{l} + 2[\overset{0}{\nabla}_i(\pi_jU_k^l) - \overset{0}{\nabla}_j(\pi_iU_k^l)] \tag{3.5}$$

Thus we have the following

Theorem 3.1. *In a Riemannian manifold (M, g), if there holds*

$$\overset{0}{\nabla}_i(\pi_j U^l_k) = \overset{0}{\nabla}_j(\pi_i U^l_k) \tag{3.6}$$

then the curvature tensor will keep unchanged under the connection transformation $\overset{0}{\nabla} \to D$ and the symmetric-type π-quarter-symmetric non-metric connection D is a conjugate symmetric connection.

Theorem 3.2. *If a Riemannian metric g admits a symmetric-type π-quarter-symmetric non-metric connection D with a zero curvature on a Riemannian manifold (M, g), then the Riemannian metric is flat.*

Proof. Adding the expressions (3.3) and (3.4), we obtain

$$\overset{*}{R}_{ijk}{}^l + R_{ijk}{}^l = 2K_{ijk}{}^l. \tag{3.7}$$

If $R_{ijk}{}^l = 0$, then $\overset{*}{R}_{ijk}{}^l = 0$. From the expression (3.7) we have $K_{ijk}{}^l = 0$. Hence the Riemannian metric is flat. □

From (3.1) and (3.2), the connection coefficient of mutual connection $\overset{*}{D}$ of the symmetric-type π-quarter-symmetric non-metric connection D is

$$\overset{*}{\Gamma}{}^k_{ij} = \{^k_{ij}\} + \pi_j U^k_i \tag{3.8}$$

and the mutual connection $\overset{m}{D}$ satisfies the relation

$$\overset{m}{D}_k g_{ij} = -\pi_i U_{jk} - \pi_j U_{ik}, \ \ \overset{m}{\Gamma}{}^k_{ij} = \pi_i U^k_j - \pi_j U^k_i, \tag{3.9}$$

and the curvature tensor of $\overset{m}{D}$ is

$$\overset{m}{R}_{ijk}{}^l = K_{ijk}{}^l + U^l_j \pi_{ik} - U^l_i \pi_{jk} + (\overset{0}{\nabla}_i U^l_j - \overset{0}{\nabla}_j U^l_i), \tag{3.10}$$

where $\pi_{ik} = \overset{0}{\nabla}_i \pi_k - U_{ip}\pi^p\pi_k$.

From (3.8) and (3.9), the connection coefficient of dual connection $\overset{m*}{D}$ of the mutual connection $\overset{m}{D}$ is

$$\overset{m*}{\Gamma}{}^k_{ij} = \{^k_{ij}\} - U_{ij}\pi^k, \tag{3.11}$$

and the curvature tensor of is

$$\overset{m*}{R}_{ijk}{}^l = K_{ijk}{}^l + U_{ik}\pi^l_j - U_{jk}\pi^l_i - (\overset{0}{\nabla}_i U_{jk} - \overset{0}{\nabla}_j U_{ik})\pi^l, \tag{3.12}$$

From the expression (3.11), the dual connection $\overset{m*}{D}$ of the mutual connection $\overset{m}{D}$ satisfies the relation

$$\overset{m*}{D}_k g_{ij} = \pi_i U_{jk} + \pi_j U_{ik}, \ \ \overset{m*}{T}_{ij}{}^k = 0,$$

So there exists the following

Theorem 3.3. *In a Riemannian manifold (M, g), the first Bianchi identity and the second Bianchi identity of the curvature tensor of dual connection $\overset{m*}{D}$ of mutual connection $\overset{m}{D}$ of the symmetric-type π-quarter-symmetric non-metric connection D are*

$$\begin{cases} \overset{m*}{R}_{ijk}{}^l + \overset{m*}{R}_{jik}{}^l + \overset{m*}{R}_{kij}{}^l = 0, \\ \overset{m*}{D}_h \overset{m*}{R}_{ijk}{}^l + \overset{m*}{D}_i \overset{m*}{R}_{jhk}{}^l + \overset{m*}{D}_j \overset{m*}{R}_{hik}{}^l = 0. \end{cases} \tag{3.13}$$

It is well known that if a sectional curvature at a point p in a Riemannian manifold (M, g) is independent of E (an 2-dimensional subspace of T_pM), the curvature tensor is

$$R_{ijk}{}^{l} = k(p)(\delta_i^l g_{jk} - \delta_j^l g_{ik}), \tag{3.14}$$

In this case, if $k(p) = const, \forall p \in M$, then the Riemannian manifold (M, g) is a constant curvature manifold.

Theorem 3.4. *Suppose a connected Riemannian manifold (M, g) associated with a symmetric-type π-quarter-symmetric non-metric connection D is isotropic. If $dimM \geq 3$, then the Riemannian manifold (M, g) is of constant curvature.*

Proof. Proof. Substituting the expression (3.13) for the second Bianchi identity of the curvature tensor of the symmetric-type π-quarter-symmetric non-metric connection, we get

$$D_h R_{ijk}{}^{l} + D_i R_{jhk}{}^{l} + D_j R_{hik}{}^{l} = T_{hi}^{p} R_{jpk}{}^{l} + T_{ij}^{p} R_{hpk}{}^{l} + T_{jh}^{p} R_{ipk}{}^{l},$$

then we have

$$\begin{aligned} & D_h k(p)(\delta_i^l g_{jk} - \delta_j^l g_{ik}) + D_i k(\delta_j^l g_{hk} - \delta_h^l g_{jk}) + D_j k(p)(\delta_h^l g_{ik} - \delta_i^l g_{hk}) \\ & +2k(p)[\pi_h(\delta_i^l U_{jk} - \delta_j^l U_{ik}) + \pi_i(\delta_j^l U_{hk} - \delta_h^l U_{jk}) + \pi_j(\delta_h^l U_{ik} - \delta_i^l U_{hk})] \\ = \; & -k(p)[U_{hk}(\delta_i^l \pi_j - \delta_j^l \pi_i) + U_{ik}(\delta_j^l \pi_h - \delta_h^l \pi_j) + U_{jk}(\delta_h^l \pi_i - \delta_i^l \pi_h) \\ & +g_{hk}(U_i^l \pi_j - U_j^l \pi_i) + g_{ik}(U_j^l \pi_h - U_h^l \pi_j) + g_{jk}(U_h^l \pi_i - U_i^l \pi_h)]. \end{aligned}$$

Contracting the indices i, l on both sides of this expression above, then we have

$$\begin{aligned} & (n-2)[D_h k g_{jk} - D_j k g_{hk} + 2k(\pi_h U_{jk} - \pi_j U_{hk})] \\ & = k[(n-3)(\pi_h U_{jk} - \pi_j U_{hk}) + g_{hk}(U_j^i \pi_i - U_i^i \pi_j) - g_{jk}(U_h^i \pi_i - U_i^i \pi_h)] \end{aligned}$$

Multiplying both sides of this expression again by g^{jk}, then we obtain

$$(n-1)(n-2)D_h k + 2(n-2)k(\pi_h U_i^i - \pi_i U_h^i) = 2(n-2)k(\pi_h U_i^i - \pi_h U_h^i).$$

This expression implies that $k = const$. The ends the proof of Theorem 3.4. □

Definition 3.2. *In a Riemannian manifold a connection family $\overset{t}{D}$ is a symmetric-type π-quarter-symmetric projective conformal connection family, if $\overset{t}{D}$ satisfies the relation*

$$\begin{cases} \overset{t}{D}_Z g(X, Y) = -2[\psi(Z) + Z\sigma]g(X, Y) - \psi(X)g(Y, Z) - \psi(Y)g(X, Z) + 2t\pi(Z)U(X, Y) \\ \overset{t}{T}(X, Y) = \pi(Y)UX - \pi(X)UY \end{cases} \tag{3.15}$$

The local expression of the relation (3.15) is

$$\overset{t}{D}_k g_{ij} = -2(\psi_k + \sigma_k)g_{ij} - \psi_i g_{jk} - \psi_j g_{ik} + 2t\pi_k U_{ij}, \quad \overset{t}{T}_{ij}{}^{k} = \pi_j U_i^k - \pi_i U_j^k \tag{3.16}$$

and its coefficient is

$$\overset{t}{\Gamma}{}_{ij}^{k} = \left\{ {k \atop ij} \right\} + (\psi_i + \sigma_i)\delta_j^k + (\psi_j + \sigma_j)\delta_i^k - g_{ij}\sigma^k - t\pi_i U_j^k - (t-1)\pi_j U_i^k + (t-1)U_{ij}\pi^k. \tag{3.17}$$

Remark 3.1. *If $\varphi_i^j = U_i^j (V_i^j = 0)$, then the π-quarter-symmetric projective conformal connection family ∇ is $\nabla = \overset{t}{D}$.*

Theorem 3.5. *(***Generalized Schur's theorem***) Suppose a connected Riemannian manifold* (M, g) *associated with a symmetric-type* π*-quarter-symmetric projective conformal connection family* $\overset{t}{D}$ *is isotropic. If* $dimM \geq 3$ *and there holds*

$$\psi_h + 2\sigma_h = 2(t-1)T^i_{ih}, \tag{3.18}$$

then $(M, g, \overset{t}{D})$ *is a constant curvature manifold.*

Proof. Substituting the expression (3.13) for the second Bianchi identity of the curvature tensor of the symmetric-type π-quarter-symmetric projective conformal connection family $\overset{t}{D}$, we get

$$\overset{t}{D}_h\overset{t}{R}_{ijk}{}^l + \overset{t}{D}_i\overset{t}{R}_{jhk}{}^l + \overset{t}{D}_j\overset{t}{R}_{hik}{}^l = \overset{t}{T}_{hi}{}^p\overset{t}{R}_{jpk}{}^l + \overset{t}{T}_{ij}{}^p\overset{t}{R}_{hpk}{}^l + \overset{t}{T}_{jh}{}^p\overset{t}{R}_{ipk}{}^l$$

then we have

$$\begin{aligned}
&[\overset{t}{D}_h k(p) - k(p)(\psi_h + 2\sigma_h)](\delta^l_i g_{jk} - \delta^l_j g_{ik}) + [\overset{t}{D}_i k(p) + k(p)(\psi_i + 2\sigma_i)](\delta^l_j g_{hk} - \delta^l_h g_{jk})\\
&+[\overset{t}{D}_j k(p) - k(p)(\psi_j + 2\sigma_j)](\delta^l_i g_{hk} - \delta^l_h g_{ik}) + 2tk(p)[\pi_h(\delta^l_i U_{jk} - \delta^l_j U_{ik}) + \pi_i(\delta^l_j U_{hk} - \delta^l_h U_{jk}) + \pi_j(\delta^l_h U_{ik} - \delta^l_i U_{hk})]\\
&= -k(p)[U_{hk}(\delta^l_i \pi_j - \delta^l_j \pi_i) + U_{ik}(\delta^l_j \pi_h - \delta^l_h \pi_j) + U_{jk}(\delta^l_h \pi_i - \delta^l_i \pi_h) + g_{hk}(U^l_i \pi_j - U^l_j \pi_i) + g_{ik}(U^l_j \pi_h - U^l_h \pi_j)\\
&+g_{jk}(U^l_h \pi_i - U^l_i \pi_h)]
\end{aligned}$$

Contracting the indices i, l of both sides of this equation above, then we have

$$\begin{aligned}
&(n-2)\{[\overset{t}{D}_h k(p) - k(p)(\psi_h + 2\sigma_h)]g_{jk} - [\overset{t}{D}_j k(p) - k(p)(\psi_j + 2\sigma_j)]g_{hk} + 2tk(p)(\pi_h U_{jk} - \pi_j U_{hk})\}\\
&= k(p)[(n-3)(\pi_h U_{jk} - \pi_j U_{hk}) + g_{hk}(U^i_j \pi_i - U^i_i \pi_j) - g_{jk}(U^i_h \pi_i - U^i_i \pi_h)]
\end{aligned}$$

Multiplying both sides of this expression again by g^{jk}, then we have

$$(n-1)(n-2)[\overset{t}{D}_h k(p) - k(p)(\psi_h + 2\sigma_h)] + 2t(n-2)k(p)(\pi_h U^i_i - \pi_i U^i_h) = 2(n-2)k(p)(\pi_h U^i_i - \pi_i U^i_h).$$

This equation implies that there holds

$$\overset{t}{D}_h k(p) = k(p)[\psi_h + 2\sigma_h - \frac{2(t-1)}{n-1} T_{ih}{}^i].$$

One can see that consequently $k = const$ if and only if

$$\psi_h + 2\sigma_h = \frac{2(t-1)}{n-1} T_{ih}{}^i.$$

The completes the proof of Theorem 3.5. □

Remark 3.2. *For the constant curvature condition in (3.18) of the symmetric-type* π*-quarter-symmetric projective conformal connection family* $\overset{t}{D}$*, there holds the following conclusions. If* $\psi_h = 0$*, then* $\sigma_h = \frac{t-1}{n-1} T_{ih}{}^i$ *and if* $\sigma_h = 0$*, then* $\psi_h = \frac{2(t-1)}{n-1} T_{ih}{}^i$ *and if* $\psi_h = \sigma_h = 0$*, then* $T_{ih}{}^i = 0$*. And if* $t = 0$*, then* $\psi_h + 2\sigma_h = \frac{2}{n-1} T_{ih}{}^i$*.*

4. Ackonowedement

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