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Quarter-symmetric generalized metric connections on a generalized Riemannian manifold

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Abstract. We define and study the quarter-symmetric connection preserving the generalized metric G in the generalized Riemannian manifold. It is proved that skew-symmetric part F of the generalized metric G in the generalized Riemannian manifold with the quarter-symmetric generalized metric connection is closed and hence the even-dimensional manifold is a symplectic manifold. We also observed the properties of curvature tensors and connection transformations in which the Riemannian tensor of the Levi-Civita connection is invariant. Finally, we observed the quarter-symmetric connection with a special conditions.

1. Introduction

The *generalized Riemannian manifold* is an n-dimensional differentiable manifold $\mathcal{M}$ with a non-symmetric basic (0,2) tensor G. The tensor G can be decomposed into symmetric part g and skew-symmetric part F, as follows

$$G(X,Y) = g(X,Y) + F(X,Y), \tag{1.1}$$

where

$$g(X,Y) = \frac{1}{2}(G(X,Y) + G(Y,X)), \quad F(X,Y) = \frac{1}{2}(G(X,Y) - G(Y,X)).$$

We assume that the symmetric part g is non-degenerate of arbitrary signature. The relation between symmetric part g and skew-symmetric part F is given with the equation

$$F(X,Y) = g(AX,Y), \tag{1.2}$$

where A is the (1,1) tensor field associated with tensor F.

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In this paper, we will study the non-symmetric linear connection $\overset{1}{\nabla}$ on a generalized Riemannian manifold. Torsion tensor $\overset{1}{T}$ and dual connection $\overset{2}{\nabla}$ of linear connection $\overset{1}{\nabla}$ are defined with the equations

$$\overset{1}{T}(X,Y) = \overset{1}{\nabla}_X Y - \overset{1}{\nabla}_Y X - [X,Y],$$
$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_Y X + [X,Y],$$

where X, Y are smooth vector fields on a differentiable manifold $\mathcal{M}$. According to the previous equations, the relation between connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ can be expressed via torsion tensor in the following way

$$\overset{2}{\nabla}_X Y = \overset{1}{\nabla}_X Y - \overset{1}{T}(X,Y). \tag{1.3}$$

By virtue of the connection $\overset{1}{\nabla}$ and its dual connection $\overset{2}{\nabla}$, one can define the symmetric connection $\overset{0}{\nabla}$ given with

$$\overset{0}{\nabla}_X Y = \frac{1}{2}(\overset{1}{\nabla}_X Y + \overset{2}{\nabla}_X Y).$$

In view of equation (1.3), the symmetric connection $\overset{0}{\nabla}$ can be expressed in terms of the torsion tensor and connections $\overset{1}{\nabla}, \overset{2}{\nabla}$, respectively, with equations

$$\overset{0}{\nabla}_X Y = \overset{1}{\nabla}_X Y - \frac{1}{2}\overset{1}{T}(X,Y), \quad \overset{0}{\nabla}_X Y = \overset{2}{\nabla}_X Y + \frac{1}{2}\overset{1}{T}(X,Y).$$

The curvature tensors of the linear connections $\overset{\theta}{\nabla}$, $\theta = 0,1,2$, are defined with equations

$$\overset{\theta}{R}(X,Y)Z = \overset{\theta}{\nabla}_X \overset{\theta}{\nabla}_Y Z - \overset{\theta}{\nabla}_Y \overset{\theta}{\nabla}_X Z - \overset{\theta}{\nabla}_{[X,Y]} Z, \; \theta = 0,1,2.$$

These three curvature tensors, together with the curvature tensors $\overset{3}{R}$, $\overset{4}{R}$ and $\overset{5}{R}$ form a base of linearly independent curvature tensors in the generalized Riemannian manifold [16], where

$$\overset{3}{R}(X,Y)Z = \overset{2}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z + \overset{2}{\nabla}_{\overset{1}{\nabla}_Y X} Z - \overset{1}{\nabla}_{\overset{2}{\nabla}_X Y} Z,$$
$$\overset{4}{R}(X,Y)Z = \overset{2}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z + \overset{2}{\nabla}_{\overset{2}{\nabla}_Y X} Z - \overset{1}{\nabla}_{\overset{1}{\nabla}_X Y} Z,$$
$$\overset{5}{R}(X,Y)Z = \frac{1}{2}(\overset{1}{\nabla}_X \overset{1}{\nabla}_Y Z - \overset{2}{\nabla}_Y \overset{1}{\nabla}_X Z + \overset{2}{\nabla}_X \overset{2}{\nabla}_Y Z - \overset{1}{\nabla}_Y \overset{2}{\nabla}_X Z - \overset{1}{\nabla}_{[X,Y]} Z - \overset{2}{\nabla}_{[X,Y]} Z).$$

The Riemannian curvature tensor $\overset{g}{R}$ with respect to the Levi-Civita connection $\overset{g}{\nabla}$ is defined with equation

$$\overset{g}{R}(X,Y)Z = \overset{g}{\nabla}_X \overset{g}{\nabla}_Y Z - \overset{g}{\nabla}_Y \overset{g}{\nabla}_X Z - \overset{g}{\nabla}_{[X,Y]} Z.$$

In [12], S. Ivanov and M. Zlatanović gave significant results on linear connections in the generalized Riemannian manifold. We now present a known theorem that determines linear connections in the generalized Riemannian manifold, where we will use a (0,3) torsion tensor defined by the equation

$$\overset{1}{T}(X,Y,Z) = g(\overset{1}{T}(X,Y),Z).$$

Theorem 1.1. *[12] Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold and $\overset{g}{\nabla}$ be a Levi-Civita connection of g.*

(1) A linear connection $\overset{1}{\nabla}$ preserves the generalized Riemannian metric G if and only if it preserves its symmetric part g and its skew-symmetric part F, i.e. $\overset{1}{\nabla}G = 0 \Leftrightarrow \overset{1}{\nabla}g = \overset{1}{\nabla}F = 0 \Leftrightarrow \overset{1}{\nabla}g = \overset{1}{\nabla}A = 0$.

(2) If there exists a linear connection $\overset{1}{\nabla}$ preserving generalized Riemannian metric G, $\overset{1}{\nabla}G = 0$, with torsion $\overset{1}{T}$ then the following condition holds

$$\begin{aligned}(\overset{g}{\nabla}_X F)(Y,Z) = &-\frac{1}{2}(\overset{1}{T}(X,Y,AZ) + \overset{1}{T}(Z,X,AY))\\ &-\frac{1}{2}(\overset{1}{T}(AZ,X,Y) + \overset{1}{T}(AZ,Y,X) + \overset{1}{T}(X,AY,Z) + \overset{1}{T}(Z,AY,X)).\end{aligned} \tag{1.4}$$

In particular, the exterior derivative of F satisfies the following equality

$$\begin{aligned}&\mathrm{d}F(X,Y,Z) = F(\overset{1}{T}(X,Y),Z) + F(\overset{1}{T}(Y,Z),X) + F(\overset{1}{T}(Z,X),Y), \quad \textit{equivalently}\\ &\mathrm{d}F(X,Y,Z) = -\overset{1}{T}(X,Y,AZ) - \overset{1}{T}(Y,Z,AX) - \overset{1}{T}(Z,X,AY).\end{aligned} \tag{1.5}$$

Conversely, if the condition (1.4) is valid then there exists a unique linear connection $\overset{1}{\nabla}$ with torsion $\overset{1}{T}$ preserving the generalized Riemannian metric G determined by the torsion $\overset{1}{T}$ with the formula

$$g(\overset{1}{\nabla}_X Y,Z) = g(\overset{g}{\nabla}_X Y,Z) + \frac{1}{2}(\overset{1}{T}(X,Y,Z) + \overset{1}{T}(Z,X,Y) - \overset{1}{T}(Y,Z,X))$$

In the next section we will study the quarter-symmetric connection with a torsion tensor containing tensor A. The quarter-symmetric connection in differentiable manifolds was introduced by S. Golab in the paper [10] and later studied by many authors in different manifolds. For example, in the paper [17], R. S. Mishra and S. Pandey studied several types of the quarter-symmetric connections in Riemannian, Einstein, Kähler, Grayan and Sasakian manifolds. K. Yano and T. Imai [25] studied the quarter-symmetric connection in Riemannian, Hermitian and Kähler manifolds. They generalized the result in the Kähler manifold obtained by R. S. Mishra and S. Pandey in [17]. The natural quarter-symmetric connection on a hyperbolic Kähler manifold was studied by N. Pušić [21] and B. B. Chaturvedi and B. K. Gupta in [5]. More information on quarter-symmetric connections can be found in the papers [1–4, 6–9, 11, 13–15, 19, 22–24, 26].

2. Quarter-symmetric generalized metric connection

The linear connection preserving the generalized metric G of the generalized Riemannian manifold is the *generalized metric* connection. In this paper, we will observe linear connection $\overset{1}{\nabla}$ preserving the generalized Riemannian metric G, $\overset{1}{\nabla}G = 0$, with torsion tensor

$$\overset{1}{T}(X,Y) = \pi(Y)AX - \pi(X)AY, \tag{2.1}$$

where π is a 1-form associated with vector field P, i.e. $\pi(X) = g(X,P)$ and A is a (1,1) type tensor field associated with skew-symmetric part F of generalized Riemannian metric G, i.e. $F(X,Y) = g(AX,Y)$. Such a connection is called a *quarter-symmetric generalized metric connection*. A 1-form π is a *generator* of that connection. From (2.1), it follows

$$\overset{1}{T}(X,Y,Z) = \pi(Y)F(X,Z) - \pi(X)F(Y,Z). \tag{2.2}$$

From Theorem 1.1 we see that the linear connection $\overset{1}{\nabla}$ preserving generalized Riemannian metric G is entirely determined by its torsion tensor $\overset{1}{T}$ and the Levi-Civita connection $\overset{g}{\nabla}$ of symmetric part g. Accordingly, we have the following statement for the quarter-symmetric generalized metric connection with torsion tensor given with (2.1), i.e. (2.2).

Corollary 2.1. *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold and $\overset{g}{\nabla}$ be the Levi-Civita connection. Quarter-symmetric generalized metric connection $\overset{1}{\nabla}$, with torsion tensor (2.1), is determined with equation*

$$\overset{1}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \pi(X)AY. \tag{2.3}$$

For covariant derivative of 1-form π, with respect to quarter-symmetric generalized metric connection (2.3), the following equation holds

$$(\overset{1}{\nabla}_X \pi)(Y) = (\overset{g}{\nabla}_X \pi)(Y) + \pi(X)\pi(AY). \tag{2.4}$$

For covariant derivative of tensor A we have $(\overset{1}{\nabla}_X A)Y = (\overset{g}{\nabla}_X A)Y$ and based on the fact that the quarter-symmetric connection $\overset{1}{\nabla}$ preserves the generalized Riemannian metric G, $\overset{1}{\nabla} G = 0$, we have (according to Theorem 1.1 (1))

$$(\overset{1}{\nabla}_X A)Y = (\overset{g}{\nabla}_X A)Y = 0. \tag{2.5}$$

Theorem 2.2. *In the generalized Riemannian manifold $(\mathcal{M}, G = g + F)$ with quarter-symmetric generalized metric connection (2.3), tensor A associated with skew-symmetric part F is parallel with respect to the Levi-Civita connection.*

Curvature tensor $\overset{1}{R}(X,Y)Z$ of quarter-symmetric generalized metric connection $\overset{1}{\nabla}$ and Riemannian curvature tensor $\overset{g}{R}(X,Y)Z$ of Levi-Civita connection $\overset{g}{\nabla}$ satisfy the following relation

$$\overset{1}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z - ((\overset{g}{\nabla}_X \pi)(Y) - (\overset{g}{\nabla}_Y \pi)(X))AZ. \tag{2.6}$$

Directly from the previous equation, we can draw certain conclusions.

Theorem 2.3. *The curvature tensor of quarter-symmetric generalized metric connection $\overset{1}{\nabla}$, given with (2.3), is equal to the Riemannian curvature tensor of Levi-Civita connection if and only if π is closed.*

Corollary 2.4. *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold, $\overset{g}{\nabla}$ be a Levi-Civita connection and $\overset{1}{\nabla}$ be a quarter-symmetric generalized metric connection (2.3). Riemannian curvature tensor $\overset{g}{R}(X,Y)Z$ is invariant under connection transformation $\overset{g}{\nabla} \to \overset{1}{\nabla}$ if and only if 1-form π is closed.*

The symmetric connection $\overset{0}{\nabla}$ and the dual connection $\overset{2}{\nabla}$ of quarter-symmetric generalized metric connection $\overset{1}{\nabla}$ are determined with equations

$$\overset{0}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \frac{1}{2}\pi(X)AY - \frac{1}{2}\pi(Y)AX, \tag{2.7}$$

$$\overset{2}{\nabla}_X Y = \overset{g}{\nabla}_X Y - \pi(Y)AX. \tag{2.8}$$

It is easy to show that connections $\overset{0}{\nabla}$ and $\overset{2}{\nabla}$ satisfy the following relations (using (1.4))

$$(\overset{0}{\nabla}_X g)(Y,Z) = \frac{1}{2}\pi(Y)F(X,Z) + \frac{1}{2}\pi(Z)F(X,Y),\ (\overset{0}{\nabla}_X F)(Y,Z) = -\frac{1}{2}\pi(Y)g(AX,AZ) + \frac{1}{2}\pi(Z)g(AX,AY),$$
$$(\overset{2}{\nabla}_X g)(Y,Z) = \pi(Y)F(X,Z) + \pi(Z)F(X,Y), \qquad (\overset{2}{\nabla}_X F)(Y,Z) = -\pi(Y)g(AX,AZ) + \pi(Z)g(AX,AY).$$

The relations between curvature tensors $\overset{\theta}{R}(X,Y)Z$, $\theta = 0,2,...,5$, and Riemannian curvature tensor $\overset{g}{R}(X,Y)Z$ are given in Theorem 2.5.

Theorem 2.5. *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold with the quarter-symmetric generalized metric connection (2.3). The curvature tensors $\overset{\theta}{R}(X,Y)Z$, $\theta = 0,2,...,5$ and Riemannian curvature tensor $\overset{g}{R}(X,Y)Z$ satisfy the following relations*

$$\begin{aligned}\overset{0}{R}(X,Y)Z =& \overset{g}{R}(X,Y)Z - \frac{1}{2}(\overset{0}{D}(X,Y) - \overset{0}{D}(Y,X))AZ - \frac{1}{2}\overset{0}{D}(X,Z)AY + \frac{1}{2}\overset{0}{D}(Y,Z)AX \\ &- \frac{1}{4}\pi(Z)(\pi(Y)A^2X - \pi(X)A^2Y),\end{aligned} \tag{2.9}$$

$$\overset{2}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z - \overset{2}{D}(X,Z)AY + \overset{2}{D}(Y,Z)AX, \tag{2.10}$$

$$\overset{3}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z - \overset{2}{D}(X,Y)AZ + \overset{3}{D}(Y,Z)AX, \tag{2.11}$$

$$\overset{4}{R}(X,Y)Z = \overset{g}{R}(X,Y)Z - \overset{3}{D}(X,Y)AZ + \overset{3}{D}(Y,Z)AX - \pi(Z)(\pi(Y)A^2X - \pi(X)A^2Y), \tag{2.12}$$

$$\begin{aligned}\overset{5}{R}(X,Y)Z =& \overset{g}{R}(X,Y)Z - \frac{1}{2}(\overset{2}{D}(X,Y) - \overset{3}{D}(Y,X))AZ - \frac{1}{2}\overset{3}{D}(X,Z)AY + \frac{1}{2}\overset{2}{D}(Y,Z)AX \\ &+ \frac{1}{2}\pi(Y)(\pi(X)A^2Z - \pi(Z)A^2X),\end{aligned} \tag{2.13}$$

where

$$\overset{0}{D}(X,Y) = (\overset{g}{\nabla}_X\pi)(Y) + \frac{1}{2}\pi(X)\pi(AY) + \frac{1}{2}\pi(Y)\pi(AX), \tag{2.14}$$

$$\overset{2}{D}(X,Y) = (\overset{g}{\nabla}_X\pi)(Y) + \pi(Y)\pi(AX), \tag{2.15}$$

$$\overset{3}{D}(X,Y) = (\overset{g}{\nabla}_X\pi)(Y) + \pi(X)\pi(AY) = (\overset{1}{\nabla}_X\pi)(Y). \tag{2.16}$$

Proof. For the quarter-symmetric generalized metric connection (2.3) and torsion tensor $\overset{1}{T}$ the following relations hold

$$(\overset{1}{\nabla}_X\overset{1}{T})(Y,Z) = AY(\overset{1}{\nabla}_X\pi)(Z) - AZ(\overset{1}{\nabla}_X\pi)(Y), \tag{2.17}$$

$$\overset{1}{T}(\overset{1}{T}(X,Y),Z) = \pi(Y)(\pi(Z)A^2X - \pi(AX)AZ) - \pi(X)(\pi(Z)A^2Y - \pi(AY)AZ), \tag{2.18}$$

$$\underset{XYZ}{\sigma}\overset{1}{T}(\overset{1}{T}(X,Y),Z) = \underset{XYZ}{\sigma}\pi(X)(\pi(AY)AZ - \pi(AZ)AY). \tag{2.19}$$

Using Theorem 2.1 in [20] and equations (2.4), (2.6), (2.17)-(2.19), after a simple computation, we obtain relations (2.9-2.13). □

The previous theorem, or equation (2.10) more precisely, implies the following corollary.

Corollary 2.6. *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold, $\overset{g}{\nabla}$ be a Levi-Civita connection and $\overset{2}{\nabla}$ be a dual connection of quarter-symmetric generalized metric connection (2.3), given with (2.8). Riemannian curvature tensor $\overset{g}{R}(X, Y)Z$ is invariant under connection transformation $\overset{g}{\nabla} \to \overset{2}{\nabla}$ if and only if $\overset{2}{D}(X, Z)AY = \overset{2}{D}(Y, Z)AX$, where $\overset{2}{D}$ is given by (2.15).*

The equation (2.9) of the curvature tensor of the zero kind can be rewritten as

$$\overset{0}{R}(X, Y)Z = \overset{g}{R}(X, Y)Z + \overset{0}{M}(X, Y)Z - \overset{0}{M}(Y, X)Z,$$

where

$$\overset{0}{M}(X, Y)Z = -\frac{1}{2}\overset{0}{D}(X, Y)AZ - \frac{1}{2}\overset{0}{D}(X, Z)AY + \frac{1}{4}\pi(X)\pi(Z)A^2Y, \tag{2.20}$$

from which we have the following corollary.

Corollary 2.7. *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold, $\overset{g}{\nabla}$ be a Levi-Civita connection and $\overset{0}{\nabla}$ be a symmetric connection given with (2.7). Riemannian curvature tensor $\overset{g}{R}(X, Y)Z$ is invariant under connection transformation $\overset{g}{\nabla} \to \overset{0}{\nabla}$ if and only if the (1,3) tensor $\overset{0}{M}(X, Y)Z$ is symmetric with respect to X and Y, where $\overset{0}{M}$ is given by (2.20).*

We will give some identities for the torsion tensor $\overset{1}{T}$.

Theorem 2.8. *The torsion tensor of quarter-symmetric generalized metric connection (2.3) in the generalized Riemannian manifold satisfies the following relations*

$$\underset{XYZ}{\sigma}\overset{1}{T}(X, Y, Z) = -2\underset{XYZ}{\sigma}\pi(X)F(Y, Z), \tag{2.21}$$

$$2\underset{XYZ}{\sigma}\pi(X)F(AY, AZ) = -\underset{XYZ}{\sigma}(\overset{1}{T}(AX, Y, AZ) + \overset{1}{T}(X, AY, AZ)), \tag{2.22}$$

$$\underset{XYZ}{\sigma}\overset{1}{T}(X, Y, AZ) = 0, \tag{2.23}$$

$$\underset{XYZ}{\sigma}\overset{1}{T}(AX, AY, Z) = 0, \tag{2.24}$$

where $\underset{XYZ}{\sigma}$ denote the cyclic sum with respect to the vector fields X, Y, Z.

Proof. All relations are proved very easily. For example, we will prove relation (2.23). Using equation for torsion tensor (2.2) and using equation (1.2), we obtain

$$\begin{aligned}\underset{XYZ}{\sigma}\overset{1}{T}(X, Y, AZ) =&\overset{1}{T}(X, Y, AZ) + \overset{1}{T}(Y, Z, AX) + \overset{1}{T}(Z, X, AY)\\ =&\pi(Y)F(X, AZ) - \pi(X)F(Y, AZ) + \pi(Z)F(Y, AX) - \pi(Y)F(Z, AX)\\ &+ \pi(X)F(Z, AY) - \pi(Z)F(X, AY)\\ =&\pi(Y)g(AX, AZ) - \pi(X)g(AY, AZ) + \pi(Z)g(AY, AX) - \pi(Y)g(AZ, AX)\\ &+ \pi(X)g(AZ, AY) - \pi(Z)g(AX, AY) = 0.\end{aligned}$$

□

With the help of equation (1.5) for the exterior derivative of F and equation (2.23) we get

$$\mathrm{d}F(X,Y,Z) = -\underset{XYZ}{\sigma} T(X,Y,AZ) = 0, \tag{2.25}$$

from which we see that tensor F is closed, and accordingly, we obtain the following claim.

Theorem 2.9. *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold with the quarter-symmetric generalized metric connection (2.3). If the manifold $\mathcal{M}$ is even-dimensional then the pair $(\mathcal{M}, F)$ is a symplectic manifold.*

For linear connection $\overset{1}{\nabla}$ that preserves tensor A, i.e. $\overset{1}{\nabla}A = 0$, the Nijenhuis tensor of tensor A can be expressed in terms of torsion tensor $\overset{1}{T}$ and tensor A with equation (see [12])

$$N(X,Y) = -\overset{1}{T}(AX,AY) - A^2\overset{1}{T}(X,Y) + A\overset{1}{T}(AX,Y) + A\overset{1}{T}(X,AY).$$

Considering the equation for torsion tensor (2.1) of quarter-symmetric generalized metric connection (2.3), we get the following statement.

Theorem 2.10. *In the generalized Riemannian manifold with quarter-symmetric generalized metric connection (2.3), the Nijenhuis tensor vanishes, i.e. $N(X,Y) = 0$.*

3. Special classes of quarter-symmetric generalized metric connection

3.1. Special quarter-symmetric generalized metric connection

In this part of the paper, we will observe a quarter-symmetric connection (2.3) that satisfies the condition

$$\pi(X) = \pi(AX) \tag{3.1}$$

and we will call such a connection a *special quarter-symmetric generalized metric connection*. A quarter-symmetric connection with this condition was observed in [18]. Based on equations (2.14)-(2.16), we conclude that the following relations hold for special quarter-symmetric generalized metric connection

$$\begin{aligned}
\overset{0}{D}(X,Y) &= (\overset{g}{\nabla}_X\pi)(Y) + \pi(X)\pi(Y),\\
\overset{2}{D}(X,Y) &= (\overset{g}{\nabla}_X\pi)(Y) + \pi(Y)\pi(X),\\
\overset{3}{D}(X,Y) &= (\overset{g}{\nabla}_X\pi)(Y) + \pi(X)\pi(Y) = (\overset{1}{\nabla}_X\pi)(Y),
\end{aligned}$$

i.e.

$$D(X,Y) = \overset{0}{D}(X,Y) = \overset{2}{D}(X,Y) = \overset{3}{D}(X,Y) = (\overset{1}{\nabla}_X\pi)(Y) = (\overset{g}{\nabla}_X\pi)(Y) + \pi(X)\pi(Y). \tag{3.2}$$

The previous equation implies the corollary of Theorem 2.5.

Corollary 3.1. *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold with the special quarter-symmetric generalized metric connection. The curvature tensors $\overset{\theta}{R}(X,Y)Z$, $\theta = 0,1,2,...,5$ and Riemannian curvature tensor*

$\overset{g}{R}(X,Y)Z$ satisfy the following relations

$$\begin{aligned}\overset{0}{R}(X,Y)Z =&\overset{g}{R}(X,Y)Z - \frac{1}{2}(D(X,Y) - D(Y,X))AZ - \frac{1}{2}D(X,Z)AY + \frac{1}{2}D(Y,Z)AX\\&- \frac{1}{4}\pi(Z)(\pi(Y)A^2X - \pi(X)A^2Y),\end{aligned} \tag{3.3}$$

$$\overset{1}{R}(X,Y)Z =\overset{g}{R}(X,Y)Z - (D(X,Y) - D(Y,X))AZ, \tag{3.4}$$

$$\overset{2}{R}(X,Y)Z =\overset{g}{R}(X,Y)Z - D(X,Z)AY + D(Y,Z)AX, \tag{3.5}$$

$$\overset{3}{R}(X,Y)Z =\overset{g}{R}(X,Y)Z - D(X,Y)AZ + D(Y,Z)AX, \tag{3.6}$$

$$\overset{4}{R}(X,Y)Z =\overset{g}{R}(X,Y)Z - D(X,Y)AZ + D(Y,Z)AX - \pi(Z)(\pi(Y)A^2X - \pi(X)A^2Y), \tag{3.7}$$

$$\begin{aligned}\overset{5}{R}(X,Y)Z =&\overset{g}{R}(X,Y)Z - \frac{1}{2}(D(X,Y) - D(Y,X))AZ - \frac{1}{2}D(X,Z)AY + \frac{1}{2}D(Y,Z)AX\\&+ \frac{1}{2}\pi(Y)(\pi(X)A^2Z - \pi(Z)A^2X),\end{aligned} \tag{3.8}$$

where D is given by (3.2).

If we replace (3.1) in (2.17), then for the covariant derivative of torsion tensor $\overset{1}{T}$ with respect to the special quarter-symmetric generalized metric connection, we have

$$\begin{aligned}&(\overset{1}{\nabla}_X\overset{1}{T})(Y,Z) = AY(\overset{g}{\nabla}_X\pi)(Z) - AZ(\overset{g}{\nabla}_X\pi)(Y) - \pi(X)(\pi(Y)AZ - \pi(Z)AY),\\&\underset{XYZ}{\sigma}(\overset{1}{\nabla}_X\overset{1}{T})(Y,Z) = -\underset{XYZ}{\sigma}((\overset{g}{\nabla}_X\pi)(Y)AZ - (\overset{g}{\nabla}_X\pi)(Z)AY).\end{aligned}$$

The (0,3) torsion tensor of special quarter-symmetric generalized metric connection satisfies the following relations

$$\begin{aligned}&\overset{1}{T}(\overset{1}{T}(X,Y),Z) = \pi(Z)(\pi(Y)A^2X - \pi(X)A^2Y),\\&\underset{XYZ}{\sigma}\overset{1}{T}(\overset{1}{T}(X,Y),Z) = 0.\end{aligned}$$

Based on the relations (3.3)-(3.8) we can easily examine the skew-symmetric properties of the curvature tensors.

Theorem 3.2. *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold with the special quarter-symmetric generalized metric connection. The curvature tensors $\overset{\theta}{R}(X,Y)Z$, $\theta = 0,1,2,...,5$ satisfy the following relations*

$$\begin{aligned}&\overset{\alpha}{R}(X,Y)Z = -\overset{\alpha}{R}(Y,X)Z,\ \alpha = 0,1,2,\\&\overset{\beta}{R}(X,Y)Z = -\overset{\beta}{R}(Y,X)Z - (D(X,Y) + D(Y,X))AZ + D(Y,Z)AX + D(X,Z)AY,\ \beta = 3,4,\\&\overset{5}{R}(X,Y)Z = -\overset{5}{R}(Y,X)Z + \pi(X)\pi(Y)A^2Z - \frac{1}{2}\pi(Z)(\pi(X)A^2Y + \pi(Y)A^2X).\end{aligned}$$

Proof. We will prove the relation for tensor $\overset{3}{R}$. By adding the equations (3.6) and

$$\overset{3}{R}(Y,X)Z = \overset{g}{R}(Y,X)Z - D(Y,X)AZ + D(X,Z)AY, \tag{3.9}$$

we obtain

$$\overset{3}{R}(X,Y)Z + \overset{3}{R}(Y,X)Z = -(D(X,Y) + D(Y,X))AZ + D(Y,Z)AX + D(X,Z)AY.$$

□

If we cyclically sum the equations (3.3)-(3.8) over X, Y, Z, then we obtain cyclic-symmetry identities for curvature tensors $\overset{\theta}{R}(X,Y)Z$, $\theta = 0, 1, 2, ..., 5$.

Theorem 3.3. *Let* $(\mathcal{M}, G = g + F)$ *be a generalized Riemannian manifold with the special quarter-symmetric generalized metric connection. The curvature tensors* $\overset{\theta}{R}(X,Y)Z$, $\theta = 0, 1, 2, ..., 5$ *satisfy the following relations*

$$\underset{XYZ}{\sigma}\overset{\theta}{R}(X,Y)Z = 0, \ \theta = 0, 3, 4, 5,$$
$$\underset{XYZ}{\sigma}\overset{1}{R}(X,Y)Z = -\underset{XYZ}{\sigma}((\overset{g}{\nabla}_X\pi)(Y)AZ - (\overset{g}{\nabla}_X\pi)(Z)AY),$$
$$\underset{XYZ}{\sigma}\overset{2}{R}(X,Y)Z = \underset{XYZ}{\sigma}((\overset{g}{\nabla}_X\pi)(Y)AZ - (\overset{g}{\nabla}_X\pi)(Z)AY).$$

Based on the previous theorem, we see that curvature tensors $\overset{\theta}{R}$, $\theta = 0, 3, 4, 5$, are cyclically symmetric, while curvature tensors $\overset{1}{R}$ and $\overset{2}{R}$ satisfy the following relation

$$\underset{XYZ}{\sigma}(\overset{1}{R} + \overset{2}{R})(X,Y)Z = 0.$$

3.2. *Special quarter-symmetric generalized metric connection with recurrent torsion tensor*

A linear connection $\overset{1}{\nabla}$ is said to be with *recurrent torsion tensor* if the covariant derivative $\overset{1}{\nabla}$ of torsion tensor $\overset{1}{T}$ is equal to the tensor product of an arbitrary 1-form ω and $\overset{1}{T}$ itself. This condition can be expressed as follows

$$(\overset{1}{\nabla}_X\overset{1}{T})(Y,Z) = \omega(X)\overset{1}{T}(Y,Z). \tag{3.10}$$

The 1-form ω is called recurrence 1-form of torsion tensor $\overset{1}{T}$.

Below we will study the special quarter-symmetric generalized metric connection with recurrent torsion tensor. If we contract torsion tensor (2.1) with respect to X, we get

$$\overset{1}{t}(Y) = Trace\{X \to \overset{1}{T}(X,Y)\} = -\pi(AY) = -\pi(Y), \tag{3.11}$$

where we used the fact that the tensor A is trace-free. Covariant derivative of equation (3.11) gives

$$(\overset{1}{\nabla}_X\overset{1}{t})(Y) = -(\overset{1}{\nabla}_X\pi)(Y).$$

By contracting equation (3.10) with respect to the vector Y we obtain

$$(\overset{1}{\nabla}_X\overset{1}{t})(Z) = \omega(X)\overset{1}{t}(Z) = -\omega(X)\pi(Z),$$

where we take into account equation (3.11). By comparing the last two equations we get

$$(\overset{1}{\nabla}_X\pi)(Y) = \omega(X)\pi(Y).$$

Theorem 3.4. *If the torsion tensor of special quarter-symmetric generalized metric connection is recurrent then the 1-form π is also recurrent.*

If we choice $\omega = \pi$ in the last equation then we have

$$(\overset{1}{\nabla}_X \pi)(Y) = \pi(X)\pi(Y)$$

and in view of equation (3.2) we obtain

$$(\overset{g}{\nabla}_X \pi)(Y) + \pi(X)\pi(Y) = \pi(X)\pi(Y)$$

i.e.

$$(\overset{g}{\nabla}_X \pi)(Y) = 0.$$

Theorem 3.5. *If the torsion tensor of special quarter-symmetric generalized metric connection is recurrent with a recurrence 1-form π then π is parallel with respect to Levi-Civita connection.*

By virtue of Theorems 3.3 and 3.5 we get the following claim.

Corollary 3.6. *Let $(\mathcal{M}, G = g + F)$ be a generalized Riemannian manifold with the special quarter-symmetric generalized metric connection. If the torsion tensor of special quarter-symmetric generalized metric connection is recurrent with a recurrence 1-form π then the curvature tensors $\overset{\theta}{R}(X,Y)Z$, $\theta = 0, 1, 2, ..., 5$ are cyclically-symmetric.*

4. Conclusion and further work

In the presented paper, we have defined quarter-symmetric connection $\overset{1}{\nabla}$ with torsion tensor containing tensor A, associated with skew-symmetric part F of generalized metric G in the generalized Riemannian manifold. We have determined symmetric connection $\overset{0}{\nabla}$ and dual connection $\overset{2}{\nabla}$ of this connection. Then we observe the transformations of the Levi-Civita connection to $\overset{0}{\nabla}$, $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ for which the Riemannian tensor of the Levi-Civita connection is invariant. It is shown that tensor F is closed, and we have proved that the even-dimensional manifold M with skew-symmetric part F is a symplectic manifold. We have proved that the Nijenhuis tensor vanishes in the generalized Riemannian manifold with a quarter-symmetric generalized metric connection. Finally, a special quarter-symmetric generalized metric connection has been introduced and relations for curvature tensors $\overset{\theta}{R}$, $\theta = 0, 1, \ldots, 5$, are given. Then the skew-symmetric and cyclic-symmetric properties of these curvature tensors are determined.

This research on the quarter-symmetric connection will be continued in the examples of the generalized Riemannian manifold.

Remark 4.1. *Theorem 2.3 is actually a generalization of Corollary 2.1. in [17] and of Theorem 6. in paper [21].*

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