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Comparison of F-contraction and φ-contractions

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Abstract. During the last two decades the concept of F-contraction, its modifications, extensions and applications, is in the spotlight of scientific research in the area of Metric Fixed Point Theory. It is natural to ask what is the relation between the newly introduced F-contraction and some well-known classes of contraction mappings. We will discuss the relation between F-contraction and some of its modification on one side and Boyd-Wong contraction or Matkowski contraction on the other. The main aim is to compare the conditions under which those classes of contractions obtain the unique fixed point on a complete metric space and to prove under which assumptions we assert that F-contraction is a Boyd-Wong or Matkowski contraction.

1. Preliminaries

In 2012. Wardowski [29] introduced the concept of F-contraction and presented the result that it possesses a unique fixed point in a complete metric space. Moreover, each sequence of successive approximations converges to the fixed point in the case of F-contraction. In the last couple of years, many papers are published on this topic among which some are in a new setting like b-matric space, cone metric space, partial metric space, fuzzy metric space, etc., and some of them include extended or just modified contractive condition. It is important to emphasize wide area of the applications of F-contraction in solving differential, integral, difference and functional equations, fractional calculus, homotopy theory. (see [1–9, 13, 16, 17, 19, 20, 23–26, 28, 30**? ?** , 31])

Naturally, we are wondering what is the place of F-contraction in the metric fixed point theory, meaning can we obtain some relations between some well-known types of contractions and F-contraction. What has already been discussed is the relation between F-contraction and θ-contraction introduced by M. Jleli and B. Samet [18], but in this paper we will focus on the comparison of F-contraction and its modification on one side and Boyd-Wong [10] and Matkowski [21] contraction on the other. Some remarks are also made regarding Meir-Keeler contraction mappings [22] and F-contraction.

The main aim is to answer the question: Under which conditions F-contraction is/is not Boyd-Wong contraction or Matkowski-contraction?

What is common for all mentioned types of contractions is that they all are the extensions of Banach contraction, so certainly their classes intersect. In order to compare them with more details, we collect some

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basic definitions and results.
Presented are two approaches to the concept of F-contraction, one originally introduced by Wardowski in [29] and second one is a modification made by Piri and Kumam in [24].

Definition 1.1. *[29] Let $F : (0,\infty) \to \mathbb{R}$ be a function fulfilling the following conditions:*

(F_1) F is strictly increasing, i.e., $0 < x < y \implies F(x) < F(y)$;

(F_2) For any sequence $(x_n) \subseteq (0,\infty)$,
$$\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} F(x_n) = -\infty;$$

(F_3) There exists $k \in (0,1)$, such that $\lim_{x\to 0^+} x^k F(x) = 0$.

Denote with $\mathcal{F}$ the set of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying $(F_1) - (F_3)$.
Accordingly, we introduce the concept of F-contraction as follows:

Definition 1.2. *Let (X,d) be a metric space and $T : X \to X$ a mapping. If there exist $F \in \mathcal{F}$ and $\tau > 0$ such that:*

$$(\forall x, y \in X)\, d(Tx,Ty) > 0 \implies \tau + F(d(Tx,Ty)) \le F(d(x,y)), \tag{1}$$

then the mapping T is called a F-contraction.

Theorem 1.3. *[29] Let (X,d) be a complete metric space and $T : X \mapsto X$ a F-contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ converges to x^*.*

In [24], the authors presented a modification of the concept of F-contraction by replacing (F_3) with the continuity condition.

Definition 1.4. *[24] Let $F : (0,\infty) \to \mathbb{R}$ be a function fulfilling the following conditions:*

(F_1) F is strictly increasing, i.e., $0 < x < y \implies F(x) < F(y)$;

(F_2^*) $\inf F = -\infty$;

(F_3^*) F is a continuous function.

We denote by $\mathcal{F}^*$ the set of all functions $F : (0,\infty) \to \mathbb{R}$ satisfying $(F_1), (F_2^*)$ and (F_3^*). Remark that this notation differs form [24] since the authors kept the notation $\mathcal{F}$, but we will change it in order to distinguish those two types of contractions.
In [25] it was proven that (F_2) and (F_2^*) are equivalent thanks to the following lemma:

Lemma 1.5. *[25] If $F : (0,\infty) \mapsto \mathbb{R}$ is an increasing mapping and (t_n) a sequence of positive numbers, then*

(a) If $\lim_{n\to\infty} F(t_n) = -\infty$, then $\lim_{n\to\infty} t_n = 0$;

(b) If $\inf(F) = -\infty$ and $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} F(t_n) = -\infty$.

The classes $\mathcal{F}^*$ and $\mathcal{F}$ are not the same because (F_3) and (F_3^*) are not in the direct relation (see [24] for adequate examples). In a same manner as in the case for F-contraction, it is possible to define a F^*-contraction depending on $F \in \mathcal{F}^*$.

Definition 1.6. *Let (X,d) be a metric space and $T : X \to X$ a mapping. If there exist $F \in \mathcal{F}^*$ and $\tau > 0$ such that*

$$(\forall x, y \in X)\, d(Tx,Ty) > 0 \implies \tau + F(d(Tx,Ty)) \le F(d(x,y)),$$

then a mapping T is called a F^*-contraction.

Same as in the case of F-contraction, F^*-contraction also has a unique fixed point on a complete metric space but the proof techniques are different than in the case of F-contraction.

Theorem 1.7. *[24] Let (X, d) be a complete metric space and $T : X \mapsto X$ a F^*-contraction. Then T has a unique fixed point $x^* \in X$ and, for every $x \in X$, a sequence $(T^n x)$ converges to x^*.*

The concept of Boyd-Wong contraction, also known as φ-contraction or ψ-contraction, was firstly presented in 1969. in the article by Boyd and Wong [10]. In 1976. this idea was modified by Matkowski [21] where different prerequisites were imposed to the function φ. Further along this idea was generalized and modified in numerous manuscripts.

Theorem 1.8. *[10] Let (X, d) be a complete metric space and $T : X \mapsto X$ a mapping. If there exists an upper semicontinuous from the right function $\varphi : [0, \infty) \mapsto [0, \infty)$ such that $\varphi(t) < t$ for $t \in (0, \infty)$ and*

$$(\forall\, x, y \in X)\, d(Tx, Ty) \leq \varphi\left(d(x, y)\right), \tag{2}$$

then the mapping T has a unique fixed point x^* and the iterative sequence $(T^n x)$ converges to x^* for any $x \in X$.

Matkowski contraction is a modification of Boyd-Wong contraction where the assumption of upper semicontinuity from the right is omitted.

Definition 1.9. *A function $\varphi : [0, \infty) \mapsto [0, \infty)$ is called a comparison function if it is an increasing function and $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \in [0, \infty)$.*

Theorem 1.10. *[21] Let (X, d) be a complete metric space and $T : X \mapsto X$ a mapping. If there exists a comparison function φ such that*

$$(\forall\, x, y \in X)\, d(Tx, Ty) \leq \varphi\left(d(x, y)\right), \tag{3}$$

then the mapping T has a unique fixed point x^* and the iterative sequence $(T^n x)$ converges to x^* for any $x \in X$.

Note that we will make difference between increasing and, previously mentioned, strictly increasing functions, where in the case of increasing function we have $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$ (but we cannot claim strict inequality in this case).
Recall also the class of contractive mapping that was introduced by Meir and Keeler in 1969. (see [22]).

Definition 1.11. *[22] Let (X, d) be a metric space and $T : X \mapsto X$ a mapping. If for every $\varepsilon > 0$, there exists some $\delta > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon, \tag{4}$$

then a mapping T is Meir-Keeler contraction.

Theorem 1.12. *[22] If (X, d) is a complete metric space and $T : X \mapsto X$ a Meir-Keeler contraction, then it possesses a unique fixed point $x^* \in X$. Moreover, for any $x \in X$,*

$$\lim_{n\to\infty} T^n x = x^*.$$

Regarding the relation of Boyd-Wong contraction and Matkowski contraction, it is known that any Boyd-Wong contraction on a metric space (X, d) is Matkowski contraction, but the converse does not hold. Makowski contraction is also not necessarily a Meir-Keeler mapping.

2. Main results

First of all, we will discuss on the relation between F^*-contraction on one side and Boyd-Wong contracton, Meir-Keeler contraction mapping and Matkowski contraction, respectfully, on the other.

Theorem 2.1. *F^*-contraction on a complete metric space is Boyd-Wong contraction.*

Proof. Assume that (X, d) is a complete metric space and $T : X \mapsto X$ is a F^*-contraction for some $F \in \mathcal{F}^*$ and $\tau > 0$. Since F is strictly increasing continuous function, it has continuous strictly increasing inverse F^{-1}, so we can easily transform (1) in order to obtain Boyd-Wong contraction in the following way:

$$Tx \neq Ty \implies d(Tx, Ty) \leq F^{-1}\left(F(d(x, y)) - \tau\right).$$

Let $\varphi(t) = F^{-1}(F(t) - \tau)$ for $t > 0$ and $\varphi(0) = 0$. Since F and F^{-1} are both continuous, we conclude that φ is continuous on $(0, \infty)$. Regarding upper semicontinuity from the right at zero, we have $t_n \searrow 0 \Leftrightarrow F(t_n) \searrow -\infty \Leftrightarrow F^{-1}(F(t_n) - \tau) \searrow 0$, meaning that $\lim_{n\to\infty} t_n = 0+$ implies $\lim_{n\to\infty} \varphi(t_n) = 0$. Function φ is upper semicontinuous from the right (and more than that) on $[0, \infty)$. Also, if $t > 0$, then

$$F(t) - \tau < F(t) \Rightarrow F^{-1}(F(t) - \tau) < t,$$

i.e., $\varphi(t) < t$ for any $t \in (0, \infty)$.
As a consequence, we may conclude that T is Boyd-Wong contraction for the defined φ. □

Corollary 2.2. *F^*-contraction is a Meir-Keeler contraction mapping.*

Proof. F^*-contraction is a Boyd-Wong contraction and since each Boyd-Wong contraction is Meir-Keeler contraction mapping, the adequate conclusion easily follows. □

Note that this proof can be alternatively given in a direct form.

Theorem 2.3. *F^*-contraction on a complete metric space is Matkowski contraction.*

Proof. Assume that (X, d) is a complete metric space and $T : X \mapsto X$ is a F^*-contraction for some $F \in \mathcal{F}^*$ and $\tau > 0$. Define φ same as in the previous proof. Same comment regarding $\varphi(t) < t$ for $t \in (0, \infty)$ is applicable in this case. We will prove, by the principle of mathematical induction, that, for any $n \in \mathbb{N}$,

$$\varphi^n(t) = F^{-1}(F(t) - n\tau). \tag{5}$$

Indeed, it holds for $n = 1$ and assume that it is true for some $n > 1$. Then,

$$\begin{aligned}
\varphi^{n+1}(t) &= \varphi(\varphi^n(t)) \\
&= \varphi(F^{-1}(F(t) - n\tau)) \\
&= F^{-1}\left(F\left(F^{-1}(F(t) - n\tau)\right) - \tau\right) \\
&= F^{-1}((F(t) - n\tau) - \tau) \\
&= F^{-1}(F(t) - (n+1)\tau),
\end{aligned}$$

leads to the conclusion that (5) holds for any $n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} \varphi^n(t) = \lim_{n\to\infty} F^{-1}(F(t) - n\tau) = 0$.
Obviously, φ is an (strictly) increasing function due to

$$\begin{aligned}
u < v \quad &\Leftrightarrow \quad F(u) < F(v) \Leftrightarrow F(u) - \tau < F(v) - \tau \\
&\Leftrightarrow \quad F^{-1}(F(u) - \tau) < F^{-1}(F(v) - \tau) \Leftrightarrow \varphi(u) < \varphi(v).
\end{aligned}$$

In conclusion, T is a Matkowski contraction.
□

Discussion regarding the relation between F-contraction and Boyd-Wong, Matkowski or Meir-Keeler contraction is different than in the case of F^*-contraction since the main idea for the case of F^*-contraction was existence of the continuous strictly increasing inverse F^{-1}, which is not the case for F-contraction because of the lack of continuity. In that instance, since F is not necesserally continuous, it does not need to be onto mapping, so it may have countably many jump discontinuities, as well-known.

Lemma 2.4. *Monotone real-valued function defined on an interval $I \subseteq \mathbb{R}$ has most countably many discontinuities. All the discontinuities are of the first kind (jump discontinuities).*

This result is known as Froda's theorem or Darboux-Froda's theorem dedicated to the results on this topic published by the Romanian mathematician A. Froda [15] in 1929. and the prior work on discontinuities by the well-known French mathematician J. G. Darboux [14] in his memoir in 1875.

It is important to emphasize that I can be any type of interval on the real line (not related to its closedness or boundedness) since the proof is based on the fact that any interval of real numbers may be represented as a (mostly) countable union of closed and bounded intervals. Also, note that the standard classification of discontinuities is on three classes: jump, removable and essential discontinuities. Let F_l be the set of all left discontinuities and F_r the set of all right discontinuites of F, i.e.,

$$\begin{aligned} F_l &= \{t \in (0,\infty) \mid \limsup_{s\nearrow t} F(s) \neq F(t)\} \\ F_r &= \{t \in (0,\infty) \mid \liminf_{s\searrow t} F(s) \neq F(t)\}. \end{aligned}$$

Evidently, all the discontinuities of F are in the union of those two sets.

Theorem 2.5. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a F-contraction for some $F \in \mathcal{F}$ and maximal $\tau > 0$. If $F_l = \emptyset$ or for each $t \in F_l$, $F(t) - \limsup_{s\nearrow t} F(s) < \tau$, then F-contraction is a Matkowski contraction.*

Proof. If $T : X \mapsto X$ is a F-contraction for some $F \in \mathcal{F}$ and $\tau > 0$, then (1) holds. Assume that τ is maximal, meaning

$$\tau = \inf\{F(d(x, y)) - F(d(Tx, Ty)) \mid x, y \in X, Tx \neq Ty\}.$$

Define $\varphi : [0,\infty) \mapsto [0,\infty)$ such that $\varphi(0) = 0$ and for each $t > 0$,

$$\varphi(t) = \sup\{s \in (0,\infty) \mid F(s) \leq F(t) - \tau\}.$$

From the definition of function φ, we clearly have $\varphi(t) \leq t$ for $t > 0$. If $\varphi(t) = t$ for some $t > 0$, then for any $s < t$, $F(s) \leq F(t) - \tau$ and $\limsup_{s\nearrow t} F(s) \leq F(t) - \tau$ which contradicts our presumed criteria about F_l. Hence, $\varphi(t) < t$ for any $t > 0$.

In order to prove that $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t \in (0,\infty)$, we will prove that $F(\varphi^n(t)) \leq F(t) - (n-1)\tau$ for any $t > 0$ and any $n \in \mathbb{N}$. For this purpose, let $t > 0$ be arbitrary. Indeed, for $n = 1$, the statement is valid since $\varphi(t) < t$ leads to $F(\varphi(t)) < F(t)$. In the case of $n = 2$ we have

$$\varphi(\varphi(t)) = \sup\{s \in (0,\infty) \mid F(s) \leq F(\varphi(t)) - \tau\},$$

hence $F(\varphi(\varphi(t)))$ must be less than $F(t) - \tau$ because otherwise we would have $\varphi(\varphi(t)) = \varphi(t)$ since $\varphi(\varphi(t))$ would preferably be equal to $\sup\{s \in (0,\infty) \mid F(s) \leq F(\varphi(t)) - \tau\}$ than $\varphi(t)$. It can be also understood that if $F(t) - \tau \notin ran(F)$, then there can't be any point in $ran(F)$ between $F(\varphi(t))$ and $F(t) - \tau$. It is also true even in the case that $F(t) - \tau \in ran(F)$, since then $\varphi(t) = F^{-1}(F(t) - \tau)$ and $F(\varphi(t)) = F(t) - \tau$.

Assume that $F(\varphi^n(t)) \leq F(t) - (n-1)\tau$ for some $n > 1$ and observe $\varphi^{n+1}(t)$. Thus,

$$\varphi^{n+1}(t) = \varphi(\varphi^n(t)) < \varphi^n(t)$$

and $F(t) - n\tau < F(\varphi^{n+1}(t)) < F(\varphi^n(t))$ would again lead to the contradiction. Hence, for any $t > 0$ and $n \in \mathbb{N}$, $\varphi^n(t) \leq F(t) - (n-1)\tau$ by the principle of mathematical induction. Moreover, for $n \in \mathbb{N}$,

$$F(\varphi^n(t)) \leq F(t) - (n-1)\tau,$$

implies that $\lim_{n\to\infty} F(\varphi^n(t)) = -\infty$ and so $\lim_{n\to\infty} \varphi^n(t) = 0$. Perceive that we have assumed that $\varphi^n(t) \neq 0$ for each $n \in \mathbb{N}$ in previous estimations. But if we get that $\varphi^{n_0}(t) = 0$ for some $n_0 \in \mathbb{N}$, then $\varphi^n(t) = 0$ for any $n \geq n_0$, and once more $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t > 0$.

Remaining is the question of increasingness of function φ. Let $t < u$, then $F(t) < F(u)$ and

$$\{s \in (0,\infty) \mid F(s) \leq F(t) - \tau\} \subseteq \{s \in (0,\infty) \mid F(s) \leq F(u) - \tau\}.$$

Hence, $\varphi(t) \leq \varphi(u)$, so φ is increasing. It will be proven that $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x, y \in X$. Assume that $Tx \neq Ty$, then the inequality

$$F(d(Tx, Ty)) \leq F(d(x, y)) - \tau$$

implies $d(Tx, Ty) \leq \varphi(d(x, y))$ whenever $Tx \neq Ty$. Hence, $d(Tx, Ty) \leq \varphi(d(x, y))$ for any $x, y \in X$. From all of the above, T is Matkowski contraction. $\square$

This result generalizes results of Turinici presented in [27].

Remark 2.6. *Observe that not even once in the previous proof we have used* (F_3)*, therefore we may state the following result.*

Theorem 2.7. *Let* (X, d) *be complete metric space and* $T : X \mapsto X$ *such that (1) holds for some* $\tau > 0$ *and* $F : (0, \infty) \mapsto \mathbb{R}$ *satisfying* (F_1) *and* (F_2) *and* $F_l = \emptyset$ *or for each* $t \in F_l$*,* $F(t) - \limsup_{s\nearrow t} F(s) < \tau$*,. Then* T *is a Matkowski contraction on* (X, d)*.*

Remark 2.8. *Note that F-contraction such that F fulfills only* (F_1) *and* (F_2) *is, in some articles, called weak-F-contraction and it possesses a unique fixed point on a complete metric space. Even more, for any* $x \in X$*, the sequence of successive approximations converges to the fixed point.*

Remark 2.9. *Recently, in [12], the concept of simple F-contraction is introduced where a function F is only increasing and if (1) holds, then a mapping T on a complete metric space* (X, d) *has a unique fixed point. Therefore, similar result may be stated for the simple F-contraction even in the case that the assumption of strict increasingness is replaced by F being increasing (nondecreasing).*

Theorem 2.10. *Let* (X, d) *be complete metric space and* $T : X \mapsto X$ *such that (1) holds for some* $\tau > 0$ *and* $F : (0, \infty) \mapsto \mathbb{R}$ *such that F is increasing and* $F_l = \emptyset$ *or for each* $t \in F_l$*,* $F(t) - \limsup_{s\nearrow t} F(s) < \tau$*,. Then* T *is a Matkowski contraction on* (X, d)*.*

Due to well-known relation between Matkowski contraction and Boyd-Wong contraction, in remains to analyze can we, under the same assumptions as in Theorem 2.5, assert that F is Boyd-Wong contraction? The answer is partially positive since we need to include additional constraint.

Theorem 2.11. *Let* (X, d) *be a complete metric space and* $T : X \mapsto X$ *a F-contraction for some* $F \in \mathcal{F}$ *and maximal* $\tau > 0$*. If* $F_l = \emptyset$ *or for each* $t \in F_l$*,* $F(t) - \limsup_{s\nearrow t} F(s) < \tau$ *and if for each* $t > 0$ *such that* $t \in F_r$ *and* $F(t) - \tau \notin ran(F)$*,*

$$\inf\{F(s) \mid F(s) \geq F(t) - \tau\} \geq \liminf_{s\searrow t} F(s) - \tau,$$

then F-contraction is a Boyd-Wong contraction.

Proof. If $T : X \mapsto X$ is a F-contraction for some $F \in \mathcal{F}$ and $\tau > 0$, then (1) holds. Assume that τ is maximal, meaning

$$\tau = \inf\{F(d(x, y)) - F(d(Tx, Ty)) \mid x, y \in X, Tx \neq Ty\}.$$

Define $\varphi : [0, \infty) \mapsto [0, \infty)$ such that $\varphi(0) = 0$ and for each $t > 0$,

$$\varphi(t) = \sup\{s \in (0, \infty) \mid F(s) \leq F(t) - \tau\}.$$

Same as in the proof of Theorem 2.5, we have $\varphi(t) < t$ for any $t > 0$. In order to prove that φ is upper semicontinuous from the right, we will discuss on several different cases. Let us take some $t > 0$ and the sequence $(t_n) \subseteq (t, \infty)$ such that $\lim_{t_n\searrow t} t_n = t$.

(1) If $t \notin F_r$, then $\lim_{n\to\infty} F(t_n) - \tau = F(t) - \tau$. We will split this case in several subcases regarding the fact are $F(t) - \tau$ and $F(t_n) - \tau$ or its subsequence in the range of F or not and also regarding the question of

semicontinuity from the right.
(1.1) Observe $F(t)-\tau \in ran(F)$ and if $F(t)-\tau = F(u)$ for some u, then assume that there is some subsequence $(t_{n_k}) \subseteq (t_n)$ such that $F(t_{n_k})-\tau = F(u_{n_k})$ must be in the range of F for any $k \in \mathbb{N}$. If $F(t)-\tau = F(u)$ and $F(t_{n_k})-\tau = F(u_{n_k})$ for any $k \in \mathbb{N}$, we have that $\limsup_{k\to\infty} u_{n_k} = u$ and $\varphi(t_{n_k}) = u_{n_k}$, $k \in \mathbb{N}$ along with $\varphi(t) = u$. For the rest of the sequence, we may claim that starting from some $n_0 \in \mathbb{N}$, $\varphi(t_n) \leq u_{n_k}$ for some $k \in \mathbb{N}$. Therefore,

$$\varphi(t) = \limsup_{k\to\infty} \varphi(t_{n_k}) = \limsup_{n\to\infty} \varphi(t_n).$$

(1.2) If $F(t)-\tau \in ran(F)$ and $F(t)-\tau \in F_r$, that means $F(t_n)-\tau$ is not in $ran(F)$ starting from some $n_0 \in \mathbb{N}$. If $F(t_n)-\tau \notin ran(F)$ starting from some $n_0 \in \mathbb{N}$, and

$$\varphi(t) = \varphi(t_n),\ n \geq n_1 \geq n_0, \Rightarrow \limsup_{n\to\infty} \varphi(t_n) = \varphi(t).$$

Otherwise, there is a sequence of discontinuities $(v_n) \subseteq F_l \cup F_r$ converging to u from the above and the values of $\varphi(t_n)$ are among the elements of the sequence $(v(n))$, meaning that $\varphi(t_n) = v_{m(n)}$ for some $m(n) \in \mathbb{N}$ such that $\lim_{n\to\infty} m(n) = \infty$. Hence,

$$\limsup_{n\to\infty} \varphi(t_n) = \limsup_{n\to\infty} v_{m(n)} = u$$

Observe that even if we would have the existence of a subsequence such that $F(t_{n_k})-\tau \in ran(F)$, we could use similar analysis as in (1.1).
(1.3) If $F(t)-\tau \notin ran(F)$, then $F(t_n)-\tau$ may or may not be in $ran(F)$.
If $F(t_{n_k})-\tau \in ran(F)$ such that $n_k \geq k$, $k \in \mathbb{N}$, then $F(t_{n_k})-\tau = F(u_{n_k})$ and $\varphi(t_{n_k}) = u_{n_k}$ for any $k \in \mathbb{N}$. Again, starting from some $n_0 \in \mathbb{N}$, $\varphi(t_n) \leq u_{n_k}$ for some $k \in \mathbb{N}$, so instead of $\limsup_{n\to\infty} \varphi(t_n)$ we may observe $\limsup_{k\to\infty} \varphi(t_{n_k})$. However, let $\limsup_{k\to\infty} u_{n_k} = v$ and assume contrary $v > u$ where $u = \varphi(t)$. Obviously, $v > 0$, since $F(t)-\tau \leq F(u_{n_k})$, $k \in \mathbb{N}$. Also, $F(v) \leq F(t_{n_k})-\tau$ for each $k \in \mathbb{N}$ and since $\lim_{n\to\infty} F(t_n)-\tau = F(t)-\tau$, we have

$$F(v) \leq F(t) - \tau.$$

From the definition of u, we get $\varphi(t) = u \geq v$ which is a contradiction. Therefore,

$$\limsup_{n\to\infty} \varphi(t_n) = \varphi(t).$$

If that kind of a sequence does not exist, then $F(t_n)-\tau \notin ran(F)$ for $n \geq n_0$, and since $\lim_{n\to\infty} F(t_n)-\tau = F(t)-\tau$, in a similar way of reasoning (by observing the discontinuities in between), we get the same conclusion that $\limsup_{n\to\infty} \varphi(t_n) = \varphi(t)$.

(2) Otherwise, if $t \in F_r$, then we certainly know that there is a gap between $F(t)$ and $\limsup_{n\to\infty} F(t_n)$, so

$$F(t) - \tau < \limsup_{n\to\infty} F(t_n) - \tau.$$

(2.1) If both $F(t)-\tau$ and $F(t_n)-\tau$ would be in the range of F, starting from some $n \geq n_0$ and for some $u, u_n \in (0,\infty)$, $n \geq n_0$ such that $F(t)-\tau = F(u)$ and $F(t_n) = F(u_n)-\tau$, $n \geq n_0$, then assume that $\limsup_{n\to\infty} u_n = v > u$. But, $\inf\{F(s) \mid F(s) \geq F(t)-\tau\} \geq \liminf_{s\searrow t} F(s) - \tau$ would lead to the obvious observation that $u \geq v$. Hence, it follows that our assumption was false.
(2.2) Suppose $F(t)-\tau \in ran(F)$, where $F(t)-\tau = F(u)$ and $F(t_{n_k})-\tau \notin ran(F)$ for some subsequence $(t_{n_k}) \subseteq (t_n)$ where $n_k \geq k$, $k \in \mathbb{N}$. Having in mind the definition of φ, evidently

$$\varphi(t_{n_k}) \leq \varphi(t_{n(k)}),$$

for some $n(k) \in \mathbb{N}$ such that $F(t_{n(k)})-\tau \in ran(F)$ and previous estimations are again applicable, claiming that the limit superior of $(\varphi(t_n))$ is $\varphi(t)$.
(2.3) If $F(t)-\tau \notin ran(F)$ and $F(t_n)-\tau \in ran(F)$ for $n \geq n_0$, then $F(t_n)-\tau = F(u_n)$ and $\varphi(t_n) = u_n$ for $n \geq n_0$.

Again, we will prove that $\limsup_{n\to\infty} u_n = u$ for some $u > 0$ due to the lower boundary $F(t) - \tau$. However, let $\limsup_{n\to\infty} u_n = v \geq u$, we must have

$$\inf\{F(s) \mid F(s) \geq F(t) - \tau\} = \liminf_{n\to\infty} F(t_n) - \tau,$$

because, if not, we would have that $F(t_n) - \tau \notin ran(F)$ starting from some $n_0 \in \mathbb{N}$. From the definition of v, it follows that $\varphi(t) = v$ or, equivalently, $u = v$. Thus,

$$\limsup_n \varphi(t_n) = \varphi(t).$$

(2.4) Assume $F(t) - \tau \notin ran(F)$ and $F(t_{n_k}) - \tau \notin ran(F)$ for some subsequence $(t_{n_k}) \subseteq (t_n)$ where $n_k \geq k, k \in \mathbb{N}$. Then the presumption,

$$\inf\{F(s) \mid F(s) \geq F(t) - \tau\} \geq \liminf_{n\to\infty} F(t_n) - \tau$$

gives us the same conclusion regarding upper semicontinuity from the right.

Summarizing all of the above, we conclude that T is a Boyd-Wong contraction on a complete metric space (X, d). $\square$

Example 2.12. *Define the function $F : (0, \infty) \mapsto \mathbb{R}$ such that for any $t > 0$*

$$F(t) = \begin{cases} \lceil \log t \rceil, & \text{if } \lceil \log t \rceil = \log t \\ t \cdot e^{\frac{2}{3(-1+e)}} + \frac{4-5e}{3(-1+e)}, & \text{if } e^{-2} < t < e^{-1} \\ t \cdot \frac{e^{-\lfloor \log t \rfloor}}{2(e-1)} + \lceil \log t \rceil - \frac{3}{4} - \frac{1}{2(e-1)}, & \text{otherwise} \end{cases} .$$

In order to achieve $\limsup_{t_n \nearrow t} F(t_n) - F(t) < \tau$, we must take $\tau > \frac{1}{3}$. Assume contrary of what this example is intended to show that there exists upper semicontinuous from the right function $\varphi : [0, \infty) \mapsto [0, \infty)$ such that $\varphi(t) < t$ and there exist $x, y, x_n, y_n \in X$ and $n \in \mathbb{N}$ such that

$$d(x_n, y_n) \searrow d(x, y) = e^{-1} \text{ if } n \to \infty.$$

Moreover, assume that $F(d(Tx, Ty)) = F(d(x, y)) - \frac{1}{2}$ and $F(d(Tx_n, Ty_n)) = F(d(x_n, y_n)) - \frac{1}{2}$ starting from some $n \geq n_0$. Then, $F(d(Tx, Ty)) = -\frac{3}{2}$ gives us $\varphi(d(x, y)) = 0.251607$ since function is continuous at this point and some open neighbourhood of it. But

$$\limsup_n F(d(x_n, y_n)) = -\frac{3}{4},$$

and since there does not exist $u \in (0, \infty)$ such that $F(u) = -\frac{3}{4}$, we have that $\limsup_n \varphi(d(x_n, y_n)) = e^{-1}$. Therefore, φ is not upper semi-continuous from the right at e^{-1}.

Let us mention that both Boyd-Wong contraction and Matkowski contraction can be reformulated (as presented in [21]) by restricting the function φ.
In a metric space (X, d) let $P = \{d(x, y) \mid x, y \in X\}$, and denote with $\overline{P}$ closure of the set P with respect to the topology on X induced by the metric d.

Theorem 2.13. *[21] Let (X, d) be a complete metric space and $T : X \mapsto X$ a mapping. If there exists an increasing function $\varphi : \overline{P} \mapsto [0, \infty)$ such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for $t \in [0, \infty)$ and*

$$(\forall\, x, y \in X)\, d(Tx, Ty) \leq \varphi\left(d(x, y)\right),$$

then the mapping T has a unique fixed point x^* and the iterative sequence $(T^n x)$ converges to x^* for any $x \in X$.

Having in mind this wider class of φ which will be denoted with Φ, we may state the following extension of Theorem 2.5 based on what can be easily observed, that if those discontinuities with saltus not less than τ are not in $\overline{P}$, then all the conclusions of Theorem 2.5 hold.

Theorem 2.14. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a F-contraction for some $F \in \mathcal{F}$ and maximal $\tau > 0$. If $F_l = \emptyset$ or for each $t \in F_l \cap \overline{P}$, $F(t) - \limsup_{s \nearrow t} F(s) < \tau$, then F-contraction is a Matkowski contraction.*

We can do the same restriction for Boyd-Wong contraction since in original paper of Boyd and Wong [10] it is already stated for a domain $\overline{P}$.

Theorem 2.15. *[10] Let (X, d) be a complete metric space and $T : X \mapsto X$ a mapping. If there exists an upper semicontinuous from the right function $\varphi : \overline{P} \mapsto [0, \infty)$ such that $\varphi(t) < t$ for $t \in (0, \infty)$ and*

$$(\forall\, x, y \in X)\, d(Tx, Ty) \leq \varphi\left(d(x, y)\right),$$

then the mapping T has a unique fixed point x^* and the iterative sequence $(T^n x)$ converges to x^* for any $x \in X$.

Consequently, we can modify the statement of Theorem 2.11 as follows:

Theorem 2.16. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a F-contraction for some $F \in \mathcal{F}$ and maximal $\tau > 0$. If $F_l = \emptyset$ or for each $t \in F_l \cap \overline{P}$, $F(t) - \limsup_{s \nearrow t} F(s) < \tau$ and if for each $t > 0$ such that $t \in F_r$ and $F(t) - \tau \notin ran(F)$,*

$$\inf\{F(s) \mid F(s) \geq F(t) - \tau\} - F(t) + \tau \geq \liminf_{s \searrow t} F(s) - F(t),$$

then F-contraction is a Boyd-Wong contraction on (X, d).

Clearly, we can exclude abundant condition (F_3) and state both results Theorem 2.14 and Theorem 2.16 for weak F-contraction meaning that it fulfills only (F_1) and (F_2).

Theorem 2.17. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a weak F-contraction for some $F \in \mathcal{F}$ and maximal $\tau > 0$. If $F_l = \emptyset$ or for each $t \in F_l \cap \overline{P}$, $F(t) - \limsup_{s \nearrow t} F(s) < \tau$, then F-contraction is a Matkowski contraction.*

Theorem 2.18. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a weak F-contraction for some $F \in \mathcal{F}$ and maximal $\tau > 0$. If $F_l = \emptyset$ or for each $t \in F_l \cap \overline{P}$, $F(t) - \limsup_{s \nearrow t} F(s) < \tau$ and if for each $t > 0$ such that $t \in F_r \cap \overline{P}$ and $F(t) - \tau \notin ran(F)$,*

$$\inf\{F(s) \mid F(s) \geq F(t) - \tau\} - F(t) + \tau \geq \liminf_{s \searrow t} F(s) - F(t),$$

then F-contraction is a Boyd-Wong contraction on (X, d).

What is common for both types of contraction Boyd-Wong and Matkowski is that $\varphi(t) < t$ for any $t \in (0, \infty)$. Question that should be answered is existence of this kind of a function? Evidently, if

$$F(t) - \limsup_{s \nearrow t} F(s) \geq \tau,$$

we cannot use previously defined φ since in those cases $\varphi(t) = t$. One step in that direction could be one easy but significant observation, stated in the next lemma.

Lemma 2.19. *Let (X, d) be a complete metric space and $T : X \mapsto X$ a F-contraction for some $F \in \mathcal{F}$ and maximal $\tau > 0$. If for some $x, y \in X$ such that $Tx \neq Ty$, $d(x, y) \in F_l$ and*

$$F(d(Tx, Ty)) = F(d(x, y)) - \tau,$$

then

$$F(d(x, y)) - \limsup_{s \nearrow d(x,y)} F(d(x, y)) < \tau.$$

Proof. If $T : X \mapsto X$ is a F-contraction for some $F \in \mathcal{F}$ and maximal $\tau > 0$, we will prove that there exists increasing function $\varphi : [0, \infty) \mapsto [0, \infty)$ such that

$$\lim_{n\to\infty} \varphi^n(t) = 0 \text{ for } t \in \overline{P}$$

and

$$d(Tx, Ty) \le \varphi(d(x, y)),\ x, y \in X.$$

Assume that $x_0, y_0 \in X$ are such that

$$F(d(Tx_0, Ty_0)) = F(d(x_0, y_0)) - \tau,$$

and recall that $d(Tx, Ty) < d(x, y)$ whenever $x \neq y$, then

$$F(d(x_0, y_0)) - \limsup_{s\nearrow d(x,y)} F(d(x, y)) < F(d(x_0, y_0)) - F(d(Tx_0, Ty_0)) = \tau.$$

□

Regarding the relation between F-contraction and Meir-Keeler contraction, we will state the result from [11].

Theorem 2.20. *[11] If (X, d) is a complete metric space and $T : X \mapsto X$ a F-contraction for some $F \in \mathcal{F}$ and $\tau > 0$, then it is a Meir-Keeler mapping.*

However, we cannot claim that converse holds.

In conclusion, F-contraction is, with some presumptions, Boyd-Wong and Matkowski contraction, but it is always Meir-Keeler contraction. What remains open is to answer is the F-contraction on complete metric space (X, d), for F satisfying (F_1) and (F_2) and maximal $\tau > 0$, Matkowski or Boyd-Wong contraction on (X, d) even in the case that F has left discontinuities on $\overline{P}$ with saltus greater or equal to τ?

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