Filomat 37:12 (2023), 4029–4045
https://doi.org/10.2298/FIL2312029L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation of functions belonging to Hölder's class and solution of Lane-Emden differential equation using Gegenbauer wavelets

Shyam Lal[a], **Harish Chandra Yadav**[a]

[a]*Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India*

Abstract. In this paper, a very new technique based on the Gegenbauer wavelet series is introduced to solve the Lane-Emden differential equation. The Gegenbauer wavelets are derived by dilation and translation of an orthogonal Gegenbauer polynomial. The orthonormality of Gegenbauer wavelets is verified by the orthogonality of classical Gegenbauer polynomials. The convergence analysis of Gegenbauer wavelet series is studied in Hölder's class. Hölder's class $H^{\alpha}[0,1)$ and $H^{\phi}[0,1)$ of functions are considered, $H^{\phi}[0,1)$ class consides with classical Hölder's class $H^{\alpha}[0,1)$ if $\phi(t) = t^{\alpha}$, $0 < \alpha \leq 1$. The Gegenbauer wavelet approximations of solution functions of the Lane-Emden differential equation in these classes are determined by partial sums of their wavelet series. In briefly, four approximations $E^{(1)}_{2^{k-1},0}$, $E^{(1)}_{2^{k-1},M}$, $E^{(2)}_{2^{k-1},0}$, $E^{(2)}_{2^{k-1},M}$ of solution functions of classes $H^{\alpha}[0,1)$, $H^{\phi}[0,1)$ by $(2^{k-1},0)^{th}$ and $(2^{k-1},M)^{th}$ partial sums of their Gegenbauer wavelet expansions have been estimated. The solution of the Lane-Emden differential equation obtained by the Gegenbauer wavelets is compared to its solution derived by using Legendre wavelets and Chebyshev wavelets. It is observed that the solutions obtained by Gegenbauer wavelets are better than those obtained by using Legendre wavelets and Chebyshev wavelets, and they coincide almost exactly with their exact solutions. This is an accomplishment of this research paper in wavelet analysis.

1. Introduction

The idea of a wavelet is used in engineering, biotechnology, viscoelastic materials, biosciences, statistical mechanics, the detection of submarines and aircraft, and other models of the real-life problem. Wavelet theory is based on a new and emerging area of numerical research. The wavelet analysis is the disintegration of a function into repositioned and scaled designs of a basic wavelet. The approximations of function belonging to the Hölder's class of order $\alpha, \alpha \in (0,1]$, by the trigonometric polynomial are at the commonplace of Fourier analysis (Zygmund [19]). To the best of our knowledge, there is no work related to the approximation of function belonging to Hölder's class $H^{\alpha}[0,1)$ and $H^{\phi}[0,1)$ by Gegenbauer wavelet expansion. Working on the solution of differential equations, Izadi et al. ([7],[8]) have developed numerical techniques to obtain the solution of linear as well as non-linear Lane-Emden differential equations by generalized Bessel quasilinearization technique and approximation method whereas Singh et al. ([13]) provide a reliable algorithm for this type of differential equation. Srivastava et al. ([17],[14]) have proposed a novel and efficient collocation method based on Fibonacci wavelets for the numerical solution of the

2020 *Mathematics Subject Classification*. 42C40, 65T60, 65L05.
Keywords. Gegenbauer wavelet, $H^{\alpha}[0,1)$ and $H^{\phi}[0,1)$ class, Approximation of function and Lane-Emden differential equation.
Received: 13 June 2022; Revised: 12 October 2022; Accepted: 08 December 2022
Communicated by Miodrag Spalević
Email addresses: shyam_lal@rediffmail.com (Shyam Lal), harishchandrayadav20395@gmail.com (Harish Chandra Yadav)

non-linear Hunter-Saxton equation and dual-phase-lag heat transfer model in multi-layer skin tissue during hyperthermia treatment. Chouhan et al. ([1]) studied the numerical problem of anomolous infiltration and diffusion modelling expressed in non-linear fractional differential equations by Bernoulli wavelets. Srivastava et al. ([16],[15]) introduced and developed a generalized wavelet method together with the quasilinearization technique to solve the Volterras population growth model of fractional order and the numerical solution of the fractional Bagley-Torvik equation by the Gegenbauer wavelet operational matrix of integration. To make an advanced analysis, in the direction of Srivastava et al., Gegenbauer wavelets have been considered in this paper. In Hölder's class $H^{\alpha}[0,1)$ and $H^{\phi}[0,1)$, the convergence analysis of solution function f by Gegenbauer wavelet series has been investigated. A method has been proposed to find the solution of the Lane-Emden differential equation by Gegenbauer wavelets. The importance of the algorithm is discribed in section 7. This Gegenbauer wavelets contains the Chebyshev and Legendre wavelets as particular cases.
The objective of this research paper are as follows.
(i). To define Hölder's class and generalized Hölder's class in the interval $[0,1)$.
(ii). To define the Gegenbauer polynomial and Gegenbauer wavelet.
(iii). To derive the approximation of function f belonging to classes $H^{\alpha}[0,1)$ & $H^{\phi}[0,1)$.
(iv). To introduce the process for computing numerical solution of differential equation and to illustrate the effectiveness of this process by an example.
(v). To compare the solution of Lane-Emden differential equation obtained by Gegenbauer wavelets with solution derived by Chebyshev as well as Legendre wavelet methods.
The remaining part of this paper is classified as follows: in Section 2, some fundamental definitions and properties of the Hölder's class, the Gegenbauer wavelet, and the orthonormality of the Gegenbauer wavelet are discussed. In Section 3, the convergence analysis of the solution function f by Gegenbauer wavelet series has been investigated. In Section 4, definition of Gegenbauer wavelet approximation and its properties are studied which are required for our subsequent investigation. Two estimators for Gegenbauer wavelet approximations have been developed. In Section 5, the proof of the theorems have been estimated. In Section 6, the algorithm for the solution of the differential equation has been developed in the interval [0,1), and it is used to obtain the solution of the Lane-Emden differential equation. In Section 7, the solution of the Lane-Emden differential equation by Gegenbauer, Chebyshev, and Legendre wavelet methods and their absolute error have been obtained. Section 8 is designated for the conclusions of this research paper.

2. Definitions and Preliminaries

2.1. *Function of Hölder's class H^{α}[0,1)*

A function $f \in H^{\alpha}[0,1)$, $\alpha \in (0,1]$ if f is continuous and satisfies the condition

$$f(x) - f(y) = O(|x-y|^{\alpha}), \forall\ x, y \in [0,1) \text{ (Titchmarsh[18]).}$$

Proposition 2.1 $H^{\alpha}[0,1) \subsetneq H^{\beta}[0,1)$, for $0 < \beta < \alpha$, for all $\alpha, \beta \in (0,1]$.
Proof: Let $f \in H^{\alpha}[0,1)$,

$$\begin{aligned}
\text{Then } \lim_{t\to 0+} \frac{|f(x+t)-f(x)|}{|t|^{\alpha}} &= C_f > 0. \\
\text{Now, } \lim_{t\to 0+} \frac{|f(x+t)-f(x)|}{|t|^{\beta}} &= \lim_{t\to 0+} \frac{|f(x+t)-f(x)|}{|t|^{\alpha}|t|^{\beta-\alpha}} \\
&\leq \lim_{t\to 0+} \frac{|f(x+t)-f(x)|}{|t|^{\alpha}} \max(|t|^{\alpha-\beta}) \\
&\leq \lim_{t\to 0} \frac{|f(x+t)-f(x)|}{|t|^{\alpha}} = C_f,\ \max(|t|^{\alpha-\beta}) \leq 1 \\
\lim_{t\to 0+} \frac{|f(x+t)-f(x)|}{|t|^{\beta}} &\leq C_f,\ \text{Thus } f \in H^{\beta}[0,1).
\end{aligned}$$

$$\text{Let } f(x) = |x|^{\beta},\ |f(x+t) - f(x)| = O(|t|^{\beta})$$
$$\lim_{t\to 0+} \frac{|f(x+t) - f(x)|}{|t|^{\alpha}} = \lim_{t\to 0+} \frac{|t|^{\beta}}{|t|^{\alpha}} = \lim_{t\to 0+} \frac{1}{|t|^{\alpha-\beta}} = +\infty.$$

Therefore, $f \notin H^{\alpha}[0,1)$ but $f \in H^{\beta}[0,1)$.

Hence, $H^{\alpha}[0,1) \subsetneq H^{\beta}[0,1)$.

Example: Let $f(x) = |x|^{\frac{1}{3}},\ |f(x+t) - f(x)| = O(|t|^{\frac{1}{3}})$
$$\lim_{t\to 0+} \frac{|f(x+t) - f(x)|}{|t|^{\frac{1}{2}}} = \lim_{t\to 0+} \frac{|t|^{\frac{1}{3}}}{|t|^{\frac{1}{2}}} = \lim_{t\to 0+} \frac{1}{|t|^{\frac{1}{6}}} = +\infty.$$

Therefore, $f \notin H^{\frac{1}{2}}[0,1)$ but $f \in H^{\frac{1}{3}}[0,1)$.

Hence, $H^{\frac{1}{2}}[0,1) \subsetneq H^{\frac{1}{3}}[0,1)$.

2.2. Function of Hölder's class H^{ϕ}[0,1)

Let $\phi(t)$ be positive monotonic increasing function of t such that $\phi(|t|) \to 0$ as $t \to 0$. A function $f \in H^{\phi}[0,1)$, if f is continuous and satisfies the condition

$$f(x+t) - f(x) = O(\phi(|t|)), \forall\ x, t, x+t \in [0,1),$$

If $\phi(t) = t^{\alpha}$ then H^{ϕ}[0,1) consides with classical Hölder's class $H^{\alpha}[0,1)$ of functions.

2.3. Gegenbauer Polynomial and Gegenbauer Wavelet

The Gegenbauer polynomials denoted by $C_m^{(\lambda)}(t)$, for $\lambda > -\frac{1}{2}$, it is also known as ultraspherical harmonics polynomials of order m and satisfy the following singular Sturm-Liouville equation in the interval [-1,1] as:

$$\frac{d}{dt}[(1-t^2)\omega(t)\frac{d}{dt}C_m^{(\lambda)}(t)] + m(2\lambda + m)\omega(t)C_m^{(\lambda)}(t) = 0,\ \lambda > -\frac{1}{2},\ m \in \{1,2,3,...\}.$$

where $\omega(t) = (1-t^2)^{\lambda-\frac{1}{2}}$ is the weight function of the Gegenbauer wavelet.
The recurrence formula of the Gegenbauer polynomials is given by:

$$C_0^{(\lambda)}(t) = 1,\ C_1^{(\lambda)}(t) = 2\lambda t;$$

$$C_{m+1}^{(\lambda)}(t) = \frac{1}{m+1}\left(2t(m+\lambda)C_m^{(\lambda)}(t) - (m+2\lambda-1)C_{m-1}^{(\lambda)}(t)\right), m \in \{1,2,3,...\}.$$

The Gegenbauer polynomials are orthogonal with respect to weight function $\omega(t) = (1-t^2)^{\lambda-\frac{1}{2}}$, for $\lambda > -\frac{1}{2}$, on the interval [-1,1] as:

$$\int_{-1}^{1} C_m^{(\lambda)}(t)C_n^{(\lambda)}(t)\omega(t)dt = L_m^{(\lambda)}\delta_{mn}, \lambda > -\frac{1}{2}.$$

where $L_m^{(\lambda)} = \frac{\pi 2^{1-2\lambda\Gamma(m+2\lambda)}}{m!(m+\lambda)(\Gamma\lambda)^2}$ is the normalizing factor, and δ is the Kronecker delta function.

Particular cases:
(i) For $\lambda = \frac{1}{2}$, Gegenbauer polynomials reduces to Legendre polynomials.
(ii) For $\lambda = 0$ and $\lambda = 1$, it reduces to Chebyshev polynomials of first and second kind respectively.
The Gegenbauer polynomials holds the inequality,

$$|C_m^{(\lambda)}(\cos\theta)|\sin^{\lambda}\theta < \frac{\Gamma(m+\frac{3\lambda}{2})2^{1-\lambda}}{\Gamma\lambda\Gamma(1+m+\frac{\lambda}{2})},\ 0\le\theta\le\pi. \tag{1}$$

From the Rodrigues formula ;

$$\int C_m^{(\lambda)}(t)\omega(t)dt = -\frac{2\lambda(1-t^2)^{\lambda+\frac{1}{2}}}{m(m+2\lambda)}C_{m-1}^{(\lambda+1)}(t),\ m\ge 1. \tag{2}$$

The Gegenbauer wavelets denoted by $\psi_{n,m}^{(\lambda)}$, are defined on [0,1) by

$$\psi_{n,m}^{(\lambda)}(t) = \begin{cases} \frac{2^{\frac{k}{2}}}{\sqrt{L_m^{(\lambda)}}}C_m^{(\lambda)}(2^k t-2n+1), & if\ t\in[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}), \\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

where $n = 1,2,3,...,2^{k-1}$, $m = 0,1,2,...,M-1(M>0)$ is the order of the Gegenbauer polynomials and $k = 1,2,3,...,$ is the level of resolution (Guo Ben-Yu[6], Elgindy and Smith[5]).

2.4. Orthonormality of Gegenbauer wavelets

Proposition 2.4 : $\{\psi_{n,m}^{(\lambda)}(t)\}$ forms an orthonormal set for $n = 1,2,...,2^{k-1}, m = 0,1,2,3...,$ with respect to weight function $\omega_{n,k}(t) = (1-(2^k t-2n+1)^2)^{\lambda-\frac{1}{2}}$. i.e.

$$<\psi_{n,m}^{(\lambda)},\psi_{n',m'}^{(\lambda)}>\omega_{n,k}(t) = \begin{cases} 1, & if\ \ n=n', m=m', \\ 0, & \text{otherwise.}\end{cases}$$

Proof :(i) For $m = 0$ & $n = n'$;

$$\begin{aligned} <\psi_{n,0}^{(\lambda)},\psi_{n,0}^{(\lambda)}>\omega_{n,k}(t) &= \int_0^1 \psi_{n,0}^{(\lambda)}(t)\overline{\psi_{n,0}^{(\lambda)}(t)}\omega_{n,k}(t)dt \\ &= \int_0^1 (\psi_{n,0}^{(\lambda)}(t))^2\omega_{n,k}(t)dt \\ &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \frac{2^k}{L_0^{(\lambda)}}C^{2}{}_0^{(\lambda)}(2^k t-2n+1)\omega_{n,k}(t)dt \\ &= \frac{2^k}{L_0^{(\lambda)}}\int_{-1}^1 C^{2}{}_0^{(\lambda)}(v)\omega(v)\frac{dv}{2^k},\ 2^k t-2n+1 = v \\ &= \frac{\lambda(\Gamma\lambda)^2}{\pi 2^{1-2\lambda}\Gamma(2\lambda)}\frac{\Gamma(\frac{2\lambda+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\lambda+1)} = 1. \end{aligned}$$

(ii). For $m \neq 0$, $n = n'$ & $m = m'$;

$$\begin{aligned} <\psi_{n,m}^{(\lambda)},\psi_{n,m}^{(\lambda)}>\omega_{n,k}(t) &= \int_0^1 \psi_{n,m}^{(\lambda)}(t)\overline{\psi_{n,m}^{(\lambda)}(t)}\omega_{n,k}(t)dt \\ &= \int_0^1 (\psi_{n,m}^{(\lambda)}(t))^2\omega_{n,k}(t)dt \\ &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \frac{2^k}{L_m^{(\lambda)}}C^{2}{}_m^{(\lambda)}(2^k t-2n+1)\omega_{n,k}(t)dt \\ &= \frac{2^k}{L_m^{(\lambda)}}\int_{-1}^1 C^{2}{}_m^{(\lambda)}(v)\omega(v)\frac{dv}{2^k},\ 2^k t-2n+1 = v \\ &= \frac{1}{L_m^{(\lambda)}}\int_{-1}^1 C^{2}{}_m^{(\lambda)}(v)\omega(v)dv = \frac{1}{L_m^{(\lambda)}}L_m^{(\lambda)} = 1. \end{aligned}$$

(iii). For $n \neq n'$,

$$< \psi^{(\lambda)}_{n,m}, \psi^{(\lambda)}_{n',m} > \omega_{n,k}(t) \quad = \quad \int_0^1 \psi^{(\lambda)}_{n,m}(t)\overline{\psi^{(\lambda)}_{n',m}(t)}\omega_{n,k}(t)dt$$

$\psi^{(\lambda)}_{n,m}(t)$ is defined in $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}) \subset [0,1)$ and $\psi^{(\lambda)}_{n',m}(t)$ is defined in $[\frac{n'-1}{2^{k-1}}, \frac{n'}{2^{k-1}}) \subset [0,1)$. If $n \neq n'$, then the intervals $[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}})$ and $[\frac{n'-1}{2^{k-1}}, \frac{n'}{2^{k-1}})$ are disjoint i.e.

$$\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \cap \left[\frac{n'-1}{2^{k-1}}, \frac{n'}{2^{k-1}}\right) = \phi.$$

Therefore $< \psi^{(\lambda)}_{n,m}, \psi^{(\lambda)}_{n',m} > \omega_{n,k}(t) = 0$ if $n \neq n' \ \forall \ m, m'$.

(iv). For $m \neq m'$,

$$< \psi^{(\lambda)}_{n,m}, \psi^{(\lambda)}_{n,m'} > \omega_{n,k}(t) \quad = \quad \int_0^1 \psi^{(\lambda)}_{n,m}(t)\overline{\psi^{(\lambda)}_{n,m'}(t)}\omega_{n,k}(t)dt$$

If $m+m'$ is an even number, then the integrand $\psi^{(\lambda)}_{n,m}(v)\overline{\psi^{(\lambda)}_{n,m'}(v)}\omega(v)$ is an odd function on [-1,1], $2^k t - 2n + 1 = v$.
Therefore $< \psi^{(\lambda)}_{n,m}(t), \psi^{(\lambda)}_{n,m'}(t) > \omega_{n,k}(t) = 0, \ \forall \ n, n'$.
If $m + m'$ is an odd number, then the integral under weight function $\omega(v)$ on [-1,1] is zero.
Therefore $< \psi^{(\lambda)}_{n,m}(t), \psi^{(\lambda)}_{n,m'}(t) > \omega_{n,k}(t) = 0$ if $m \neq m' \ \forall \ n, n'$.

3. Convergence of Gegenbauer Wavelet Series

In this section, the convergence analysis of the solution function $f \in L^2[0,1)$ of Lane-Emden differential equation by Gegenbauer wavelet expansion has been discussed.

Theorem 3.1. If $f(t)$ is the exact solution of the Lane-Emden differential equation then its Gegenbauer wavelet series $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi^{(\lambda)}_{n,m}(t)$ converges uniformly to $f(t)$.

$$\begin{aligned}
\textbf{Proof}: \text{ Let } f(t) &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi^{(\lambda)}_{n,m}(t). \\
\text{Then } < f, f > &= \left\langle \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi^{(\lambda)}_{n,m}(t), \sum_{n'=1}^{2^{k-1}} \sum_{m'=0}^{\infty} c_{n',m'}\psi^{(\lambda)}_{n',m'}(t) \right\rangle \\
&= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} \sum_{n'=1}^{2^{k-1}} \sum_{m'=0}^{\infty} c_{n,m}\overline{c_{n',m'}} < \psi^{(\lambda)}_{n,m}, \psi^{(\lambda)}_{n',m'} > \\
&= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} |c_{n,m}|^2 \|\psi^{(\lambda)}_{n,m}\|_2^2 \\
&= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} |c_{n,m}|^2, \\
& \quad \{\psi^{(\lambda)}_{n,m}\} \text{ is an orthonormal basis of } L^2[0,1) \ \& \ \|\psi^{(\lambda)}_{n,m}\|_2 = 1. \\
\text{Thus, } \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} |c_{n,m}|^2 &= \ < f, f > = \int_0^1 |f(t)|^2 dt < \infty, \ f \in L^2[0,1).
\end{aligned}$$

Therefore the wavelet series $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{\infty} c_{n,m}\psi^{(\lambda)}_{n,m}(t)$ is convergent and by Bessel's inequality, $\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} |c_{n,m}|^2 \leq \|f\|_2^2 < \infty, \ \forall \ M \geq 0$.

$$\begin{aligned}
\text{Let } (S_{2^{k-1},M}f)(t) &= \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(\lambda)}(t).\\
\text{For } N>M, \quad & \|(S_{2^{k-1},N}f)-(S_{2^{k-1},M}f)\|_2^2\\
&= \left\|\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{N-1} c_{n,m}\psi_{n,m}^{(\lambda)}(t)-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(\lambda)}(t)\right\|_2^2\\
&= \left\|\sum_{n=1}^{2^{k-1}}\sum_{m=M}^{N-1} c_{n,m}\psi_{n,m}^{(\lambda)}(t)\right\|_2^2\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{N-1} |c_{n,m}|^2 \to 0 \text{ as } M\to\infty,\ N\to\infty.\\
\text{Therefore,} \quad & \|(S_{2^{k-1},N}f)-(S_{2^{k-1},M}f)\|_2^2 \to 0 \text{ as } M\to\infty,\ N\to\infty.
\end{aligned}$$

Hence $(S_{2^{k-1},N}f)_{N=0}^{\infty}$ is a Cauchy sequence in $L^2[0,1)$. Since $L^2[0,1)$ is a Banach space, therefore the Cauchy sequence $(S_{2^{k-1},N}f)_{N=0}^{\infty}$ converges to a function $g(t)$, say.
Here, $g(t)=\lim_{N\to\infty}(S_{2^{k-1},N}f)=\lim_{N\to\infty}\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{N-1}c_{n,m}\psi_{n,m}^{(\lambda)}(t)$.
Now we need to show that $g(t)=f(t)$, for this

$$\begin{aligned}
< g(t)-f(t),\psi_{n,m}^{(\lambda)}(t) > &= < g(t),\psi_{n,m}^{(\lambda)}(t) > - < f(t),\psi_{n,m}^{(\lambda)}(t) >\\
&= \lim_{N\to\infty} < (S_{2^{k-1},N}f),\psi_{n,m}^{(\lambda)}(t) > -c_{n,m}\\
&= c_{n,m}-c_{n,m}=0.
\end{aligned}$$

Therefore $g-f=0$ i.e. $g=f$.

Hence the wavelet series $\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{N-1}c_{n,m}\psi_{n,m}^{(\lambda)}(t)$ converges uniformly to $f(t)$ as $N\to\infty$.

4. Gegenbauer wavelet approximation and theorems

In this section, approximation and theorems based on Gegenbauer wavelet have been establish.

4.1. Gegenbauer wavelet approximation

Since $\{\psi_{n,m}^{(\lambda)}(t)\}$ forms an orthonormal basis for $L^2[0,1)$, therefore a function $f\in L^2[0,1)$ can be expressed into Gegenbauer wavelet series as:

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{n,m}\psi_{n,m}^{(\lambda)}(t) \tag{4}$$

$$\text{where } c_{n,m} =< f,\psi_{n,m}^{(\lambda)} > \omega_{n,k}(t),\ \omega_{n,k}(t)=(1-(2^kt-2n+1)^2)^{\lambda-\frac{1}{2}} \tag{5}$$

The $(2^{k-1},M)^{\text{th}}$ partial sum $(S_{2^{k-1},M}f)(t)$ of Gegenbauer wavelet series equation (4) is given by

$$(S_{2^{k-1},M}f)(t)=\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}c_{n,m}\psi_{n,m}^{(\lambda)}(t)=C^T\psi^{(\lambda)}(t). \tag{6}$$

where C and $\psi^{(\lambda)}(t)$ are given by; $C=[c_{1,0},c_{1,1},...,c_{1,M-1},c_{2,0},...,c_{2,M-1},...,c_{2^{k-1},0},...,c_{2^{k-1},M-1}]^T$ and
$\psi^{(\lambda)}(t)=[\psi_{1,0}^{(\lambda)}(t),\psi_{1,1}^{(\lambda)}(t),...,\psi_{1,M-1}^{(\lambda)}(t),\psi_{2,0}^{(\lambda)}(t),...,\psi_{2,M-1}^{(\lambda)}(t),...,\psi_{2^{k-1},0}^{(\lambda)}(t),...,\psi_{2^{k-1},M-1}^{(\lambda)}(t)]^T$.

The Gegenbauer wavelet approximation $E_{2^{k-1},M}(f)$ of f by $(2^{k-1},M)^{\text{th}}$ partial sum $(S_{2^{k-1},M}f)$ of Gegenbauer wavelet series equation (4) is defined by

$$E_{2^{k-1},M}(f) = \min_{(S_{2^{k-1},M}f)} \|f - (S_{2^{k-1},M}f)\|_2. \tag{7}$$

$E_{2^{k-1},M}(f)$ is said to be best approximation of the function f by $(S_{2^{k-1},M}f)$ if $E_{2^{k-1},M}(f) \to 0$ as $k \to \infty$, $M \to \infty$ (Zygmund [19]).

4.2. Theorems

In this paper, the following theorems have been developed:

Theorem 4.2.1. If the solution function f of Lane-Emden differential equation belongs to $H^{\alpha}[0,1)$ class and its Gegenbauer wavelet expansion be
$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(\lambda)}(t)$ having $(2^{k-1},M)^{\text{th}}$ partial sums

$$(S_{2^{k-1},M}f)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(\lambda)}(t)$$

then the Gegenbauer wavelet approximation of f by $(S_{2^{k-1},M}f)$ under $\|\cdot\|_2$ is given by;

(i) For $m = 0$, $E_{2^{k-1},0}^{(1)}(f) = \min\|f - \sum_{n=1}^{2^{k-1}} c_{n,0}\psi_{n,0}^{(\lambda)}(t)\|_2 = O\left(\frac{1}{2^{k\alpha}}\right)$, $k \geq 1$.

(ii) For $m \geq 1$, $E_{2^{k-1},M}^{(1)}(f) = \min\|f - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(\lambda)}(t)\|_2 = O\left(\frac{1}{2^{(k-1)\alpha}\sqrt{M}}\right)$, $M \geq 1$.

Theorem 4.2.2. If a function $f \in H^{\phi}[0,1)$ class such that $\phi(|t|) \to 0$ as $t \to 0$ then the Gegenbauer wavelet approximation of f by $(S_{2^{k-1},M}f)$ satisfies;

(i) For $m = 0$, $E_{2^{k-1},0}^{(2)}(f) = \min\|f - \sum_{n=1}^{2^{k-1}} c'_{n,0}\psi_{n,0}^{(\lambda)}(t)\|_2 = O\left(\phi\left(\frac{1}{2^k}\right)\right)$, $k \geq 1$.

(ii) For $m \geq 1$, $E_{2^{k-1},M}^{(2)}(f) = \min\|f - \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c'_{n,m}\psi_{n,m}^{(\lambda)}(t)\|_2 = O\left(\frac{\phi\left(\frac{1}{2^{k-1}}\right)}{\sqrt{M}}\right)$, $M \geq 1$.

5. Proof of theorems:

In this section, the proofs of theorems have been developed.

5.1. Proof of theorem 4.2.1

For $m = 0$, error between Gegenbauer wavelet expansion and $f(t)$ in the interval $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right)$ is given by:

$$\begin{aligned}
e_n^{(\lambda)}(f)(t) &= c_{n,0}\psi_{n,0}^{(\lambda)}(t) - f(t)\chi_{\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right)}. \\
c_{n,0} &= < f, \psi_{n,0}^{(\lambda)} > \omega_{n,k}(t) \\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)\psi_{n,0}^{(\lambda)}(t)\omega_{n,k}(t)dt \\
&\leq f(t_1)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \psi_{n,0}^{(\lambda)}(t)\omega_{n,k}(t)dt, \text{ by mean value theorem, } t_1 \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right) \\
&= f(t_1)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \frac{2^{\frac{k}{2}}}{\sqrt{L_0^{(\lambda)}}} C_0^{(\lambda)}(2^k t - 2n + 1)(1 - (2^k t - 2n + 1)^2)^{\lambda - \frac{1}{2}} dt
\end{aligned}$$

$$
\begin{aligned}
&= f(t_1)\int_0^{\pi}\frac{2^{\frac{k}{2}}}{\sqrt{L_0^{(\lambda)}}}\sin^{2\lambda}\theta\frac{d\theta}{2^k},\ 2^k t-2n+1=\cos\theta\\
&= \frac{2f(t_1)}{2^{\frac{k}{2}}}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{L_0^{(\lambda)}}}\sin^{2\lambda}\theta d\theta\\
&= \frac{2f(t_1)}{2^{\frac{k}{2}}\sqrt{L_0^{(\lambda)}}}\frac{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}{2\Gamma(\lambda+1)}.\\
|e_n^{(\lambda)}(f)| &\le \left|\frac{2f(t_1)}{2^{\frac{k}{2}}\sqrt{L_0^{(\lambda)}}}\frac{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}{2\Gamma(\lambda+1)}\frac{2^{\frac{k}{2}}}{\sqrt{L_0^{(\lambda)}}}-f(t)\right|\\
&= |f(t_1)-f(t)|\\
&\le N|t_1-t|^{\alpha},\ f\in H^{\alpha}[0,1),\ \text{N is suitable positive real number,}\\
&\le \frac{N}{2^{\alpha k}}.\\
\text{Then }\ \|e_n^{(\lambda)}(f)\|_2^2 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|e_n^{(\lambda)}(f)|^2\omega_{n,k}(x)dx\\
&\le \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left(\frac{N}{2^{\alpha k}}\right)^2\omega_{n,k}(x)dx\\
&= \left(\frac{N}{2^{\alpha k}}\right)^2\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\omega_{n,k}(x)dx\\
&= \frac{N^2}{2^{2\alpha k}}\frac{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}{2^k\Gamma(\lambda+1)}. \qquad (8)\\
(E^{(1)}_{2^{k-1},0}(f))^2 &= \int_0^1\left(\sum_{n=1}^{2^{k-1}}e_n^{(\lambda)}(f)\right)^2\omega_{n,k}(t)dt\\
&= \int_0^1\sum_{n=1}^{2^{k-1}}(e_n^{(\lambda)}(f))^2\omega_{n,k}(t)dt+\sum\sum_{1\le n\ne n'\le 2^{k-1}}\int_0^1 e_n(f)e_{n'}(f)\omega_{n,k}(t)dt\\
&= \sum_{n=1}^{2^{k-1}}\int_0^1(e_n^{(\lambda)}(f))^2\omega_{n,k}(t)dt+0,\\
&\quad \text{since support of } e_n^{(\lambda)}(f) \text{ and } e_{n'}^{(\lambda)}(f) \text{ are disjoint.}\\
&= \sum_{n=1}^{2^{k-1}}\|e_n^{(\lambda)}(f)\|_2^2\\
&\le \sum_{n=1}^{2^{k-1}}\frac{N^2}{2^{2\alpha k}}\frac{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}{2^k\Gamma(\lambda+1)}\\
&= \frac{N^2}{2^{2\alpha k}}\frac{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}{2\Gamma(\lambda+1)}.\\
\text{Therefore, } E^{(1)}_{2^{k-1},0}(f) &= O\left(\frac{1}{2^{k\alpha}}\right), k\ge 1.
\end{aligned}
$$

$$
\begin{aligned}
\textbf{(ii)}\text{ For m} \geq 1,\ f(t) &= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(\lambda)}(t).\\
c_{n,m} &= < f, \psi_{n,m}^{(\lambda)} > \omega_{n,k}(t)\\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} f(t)\psi_{n,m}^{(\lambda)}(t)\omega_{n,k}(t)dt\\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(f(t) - f\left(\frac{n-1}{2^{k-1}}\right)\right)\psi_{n,m}^{(\lambda)}(t)\omega_{n,k}(t)dt\\
&+ f\left(\frac{n-1}{2^{k-1}}\right)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \psi_{n,m}^{(\lambda)}(t)\omega_{n,k}(t)dt\\
&= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left(f(t) - f\left(\frac{n-1}{2^{k-1}}\right)\right)\frac{2^{\frac{k}{2}}}{\sqrt{L_0^{(\lambda)}}} C_m^{(\lambda)}(2^k t - 2n + 1)\omega_{n,k}(t)dt + 0\\
|c_{n,m}| &\leq \frac{N2^{\frac{k}{2}}}{\sqrt{L_0^{(\lambda)}}}\left(\frac{1}{2^{k-1}}\right)^{\alpha}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left|C_m^{(\lambda)}(2^k t - 2n+1)\omega_{n,k}(t)\right| dt,\ f \in H^{\alpha}[0,1)\\
&= \frac{N2^{\frac{k}{2}}}{\sqrt{L_0^{(\lambda)}}2^{(k-1)\alpha}}\int_{-1}^{1}\left|C_m^{(\lambda)}(v)\omega(v)\right|\frac{dv}{2^k},\ 2\mu^k t - 2n + 1 = v\\
&= \frac{N}{\sqrt{L_0^{(\lambda)}}2^{(k-1)\alpha}2^{\frac{k}{2}}}\int_{-1}^{1}\left|C_m^{(\lambda)}(v)(1-v^2)^{\lambda-\frac{1}{2}}\right| dv\\
&\leq \frac{N}{\sqrt{L_0^{(\lambda)}}2^{(k-1)\alpha}2^{\frac{k}{2}}}\left|-\frac{2\lambda(1-v^2)^{\lambda+\frac{1}{2}}}{m(m+2\lambda)}C_{m-1}^{(\lambda+1)}(v)\right|_{-1}^{1},\ \text{by integral (2)},\\
&\leq \frac{N}{\sqrt{L_0^{(\lambda)}}2^{(k-1)\alpha}2^{\frac{k}{2}}}\frac{2\lambda}{m(m+2\lambda)}\max_{0\leq\theta\leq\pi}\left[C_{m-1}^{(\lambda+1)}(\cos\theta)\sin^{2\lambda+1}\theta\right],\ v = \cos\theta\\
&\leq \frac{2N\lambda}{\sqrt{L_0^{(\lambda)}}2^{(k-1)\alpha}2^{\frac{k}{2}}m(m+2\lambda)}\frac{2^{-\lambda}\Gamma(m+\frac{3\lambda+1}{2})}{\Gamma(\lambda+1)\Gamma(m+\frac{\lambda+1}{2})},\ \text{by inequality (1)},\\
&= \frac{2N}{\sqrt{2\pi}2^{(k-1)\alpha}2^{\frac{k}{2}}m(m+2\lambda)}\sqrt{\frac{m!(m+\lambda)}{\Gamma(m+2\lambda)}}\frac{\Gamma(m+\frac{3\lambda+1}{2})}{\Gamma(m+\frac{\lambda+1}{2})}\\
&\leq \sqrt{\frac{2}{\pi}}\frac{N}{2^{(k-1)\alpha}2^{\frac{k}{2}}m}. \qquad (9)
\end{aligned}
$$

$$
\begin{aligned}
\text{Now, } f(t) - (S_{2^{k-1},M}f)(t) &= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty} c_{n,m}\psi_{n,m}^{(\lambda)}(t).\\
\text{Then, } (E_{2^{k-1},M}^{(1)}(f))^2 &= \|f(t) - (S_{2^{k-1},M}f)(t)\|_2^2\\
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|c_{n,m}|^2\|\psi_{n,m}^{(\lambda)}(t)\|_2^2\\
&+ \sum_{n=1}^{2^{k-1}}\sum\sum_{M\leq m\neq m'\leq\infty}|c_{n,m}||c_{n,m'}||\psi_{n,m}^{(\lambda)}(t)||\psi_{n,m'}^{(\lambda)}(t)|\\
&+ \sum\sum_{1\leq n\neq n'\leq 2^{k-1}}\sum_{m=M}^{\infty}|c_{n,m}||c_{n',m}||\psi_{n,m}^{(\lambda)}(t)||\psi_{n',m}^{(\lambda)}(t)|
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|c_{n,m}|^2+0\\
&\le \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\left[\sqrt{\frac{2}{\pi}}\frac{N}{2^{(k-1)\alpha}2^{\frac{k}{2}}m}\right]^2\\
&= \frac{N^2}{\pi 2^{2(k-1)\alpha}}\sum_{m=M}^{\infty}\frac{1}{m^2}\\
&\le \frac{N^2}{\pi 2^{2(k-1)\alpha}}\left[\frac{1}{M^2}+\int_M^{\infty}\frac{dm}{m^2}\right], \text{by Cauchy's integral test,}\\
&\le \frac{N^2}{\pi 2^{2(k-1)\alpha}}\left[\frac{1}{M^2}+\frac{1}{M}\right]\le\frac{2N^2}{\pi 2^{2(k-1)\alpha}M}.
\end{aligned}
$$

$$\text{Therefore, } E^{(1)}_{2^{k-1},M}(f) = O\left(\frac{1}{2^{(k-1)\alpha}\sqrt{M}}\right),\ M\ge 1.$$

This completes the proof of the Theorem 4.2.1.

5.2. Proof of theorem 4.2.2

(i) For $m=0$, following the proof of first part of Theorem 4.2.1 and for $f\in H^{\phi}[0,1)$ class,

$$
\begin{aligned}
|e_n^{'(\lambda)}(f)| &\le |f(t_1)-f(t)|\\
&\le N\phi(|t_1-t|),\ f\in H^{\phi}[0,1)\\
&\le N\phi\left(\frac{1}{2^k}\right).\\
\|e_n^{'(\lambda)}(f)\|_2^2 &= \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}|e_n^{'(\lambda)}(f)|^2\omega_{n,k}(x)dx\\
&\le \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left(N\phi\left(\frac{1}{2^k}\right)\right)^2\omega_{n,k}(x)dx\\
&= N^2\phi^2\left(\frac{1}{2^k}\right)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\omega_{n,k}(x)dx\\
&= N^2\phi^2\left(\frac{1}{2^k}\right)\frac{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}{2^k\Gamma(\lambda+1)}.\\
(E^{(2)}_{2^{k-1},0}(f))^2 &= \sum_{n=1}^{2^{k-1}}\|e_n^{'(\lambda)}(f)\|_2^2\\
&\le \sum_{n=1}^{2^{k-1}}N^2\phi^2\left(\frac{1}{2^k}\right)\frac{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}{2^k\Gamma(\lambda+1)}\\
&= N^2\phi^2\left(\frac{1}{2^k}\right)\frac{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}{2\Gamma(\lambda+1)}.\\
\text{Therefore, } E^{(2)}_{2^{k-1},0}(f) &= O\left(\phi\left(\frac{1}{2^k}\right)\right), k\ge 1.
\end{aligned}
$$

(ii) For $m\ge 1$, following the proof of second part of Theorem 4.2.1 and for $f\in H^{\phi}[0,1)$ class,

$$
\begin{aligned}
|c'_{n,m}| &\leq \frac{N2^{\frac{k}{2}}}{\sqrt{L_0^{(\lambda)}}}\phi\left(\frac{1}{2^{k-1}}\right)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left|C_m^{(\lambda)}(2^k t - 2n + 1)\omega_{n,k}(t)\right| dt,\ f \in H^{\phi}[0,1) \\
&\leq \sqrt{\frac{2}{\pi}}\phi\left(\frac{1}{2^{k-1}}\right)\frac{N}{2^{\frac{k}{2}}m}. \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
\text{Then, } (E^{(2)}_{2^{k-1},M}(f))^2 &= \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}|c'_{n,m}|^2 \\
&\leq \sum_{n=1}^{2^{k-1}}\sum_{m=M}^{\infty}\left[\sqrt{\frac{2}{\pi}}\phi\left(\frac{1}{2^{k-1}}\right)\frac{N}{2^{\frac{k}{2}}m}\right]^2 \\
&= \frac{N^2}{\pi}\phi^2\left(\frac{1}{2^{k-1}}\right)\sum_{m=M}^{\infty}\frac{1}{m^2} \\
&\leq \frac{N^2}{\pi}\phi^2\left(\frac{1}{2^{k-1}}\right)\left[\frac{1}{M^2} + \int_M^{\infty}\frac{dm}{m^2}\right] \\
&\leq \frac{N^2}{\pi}\phi^2\left(\frac{1}{2^{k-1}}\right)\left[\frac{1}{M^2} + \frac{1}{M}\right] \leq \frac{2N^2}{\pi M}\phi^2\left(\frac{1}{2^{k-1}}\right).
\end{aligned}
$$

$$
\text{Therefore, } E^{(2)}_{2^{k-1},M}(f) = O\left(\frac{\phi\left(\frac{1}{2^{k-1}}\right)}{\sqrt{M}}\right),\ M \geq 1.
$$

This completes the proof of the Theorem 4.2.2.

6. Algorithm of Lane-Emden differential equation

In this section, the algorithm for the solution of Lane-Emden differential equation is to be discussed. For this, let us derive the eight basis functions of the Gegenbauer wavelet for $\lambda = 2, k = 2, M = 4$ as follows;

$$
\left.
\begin{aligned}
\psi^{(\lambda)}_{1,0}(t) &= 4\sqrt{\frac{2}{3\pi}} \\
\psi^{(\lambda)}_{1,1}(t) &= \frac{8}{\sqrt{\pi}}(4t-1) \\
\psi^{(\lambda)}_{1,2}(t) &= 4\sqrt{\frac{2}{15\pi}}(12(4t-1)^2-2) \\
\psi^{(\lambda)}_{1,3}(t) &= \frac{2}{\sqrt{3\pi}}(32(4t-1)^3-12(4t-1))
\end{aligned}
\right\},\ t \in \left[0, \frac{1}{2}\right), \qquad (11)
$$

$$
\left.
\begin{aligned}
\psi^{(\lambda)}_{2,0}(t) &= 4\sqrt{\frac{2}{3\pi}} \\
\psi^{(\lambda)}_{2,1}(t) &= \frac{8}{\sqrt{\pi}}(4t-3) \\
\psi^{(\lambda)}_{2,2}(t) &= 4\sqrt{\frac{2}{15\pi}}(12(4t-3)^2-2) \\
\psi^{(\lambda)}_{2,3}(t) &= \frac{2}{\sqrt{3\pi}}(32(4t-3)^3-12(4t-3))
\end{aligned}
\right\},\ t \in \left[\frac{1}{2}, 1\right). \qquad (12)
$$

Let $y(t)$ be the solution of Lane -Emden differential equation :

$$
y'' + \frac{\beta}{t}y' + f(t,y) = g(t),\ t \in (0,1],\ \beta \geq 0,\ y(0) = a,\ y'(0) = b, \text{(Narayan and Rajesh[12]).} \qquad (13)
$$

$$\text{Then, } y(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}^{(\lambda)}(t) \tag{14}$$

and $(2^{k-1}, M)^{th}$ partial sum of series (14) as:

$$y(t) = (S_{2^{k-1},M}f)(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(\lambda)}(t) = C^T\psi^{(\lambda)}(t) \tag{15}$$

By initial conditions of Eqn. (13), the Eqn. (15) reduces to

$$y(0) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(\lambda)}(0) = a, \;\; y'(0) = \frac{d}{dt}\left(\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}^{(\lambda)}(t)\right)_{t=0} = b.$$

In Eqn. (15), C^T contains $2^{k-1}M$ unknown coefficients. Hence, excluding initial conditions, $2^{k-1}M-2$ extra conditions are needed for the solution of the differential equation. For determining the values of $2^{k-1}M$ unknown coefficients $c_{n,m}$, collocation points $t_i = \frac{i-1}{2^{k-1}M}, i = 1,2,...,2^{k-1}M$, are substituted in Eqn. (15) to obtain $2^{k-1}M-2$ equations. Hence the values of unknown coefficients $c_{n,m}$ are obtained by these $2^{k-1}M$ equations. This algorithm is also applicable to higher-order differential equations.

7. Results and Discussion

In this section, the validity of the proposed method for the numerical solution of the Lane-Emden differential equation and its error analysis have been discussed. This example has an exact solution; compare it with the proposed method and the Chebyshev and Legendre wavelet methods.

7.1. Example

Consider the Lane-Emden differential equation

$$y'' + \frac{2}{t}y' + y = t^2(t+1) + 6(2t+1), \; t \in (0,1], \; y(0) = 0, \; y'(0) = 0. \tag{16}$$

The exact solution of the Eqn. (16) is $y(t) = t^2 + t^3$.

$$\begin{aligned} y(x+t) - y(x) &= ((x+t)^2 + (x+t)^3) - (x^2 + x^3) \\ &= 2xt + t^2 + 3x(x+t) + t^3 \\ &\leq 2t + t + 6t + t = 10t \\ &\leq 10t^\alpha = O(|t|^\alpha), \; \alpha \in [0,1). \end{aligned}$$

Hence, $y(t) \in H^\alpha[0,1)$, by proposition (2.1).
By using the algorithm of the Gegenbauer wavelet approach described in Section 6, the differential equation has now been solved. For the approximate solution of the Eqn. (16), take $\lambda = 2$, $M = 4$ and $k = 2$. Then the approximate solution $y(t)$ will be

$$\begin{aligned} y(t) &= \sum_{m=0}^{3} c_{1,m}\psi_{1,m}(t) = c_{1,0}\psi_{1,0}(t) + c_{1,1}\psi_{1,1}(t) + c_{1,2}\psi_{1,2}(t) + c_{1,3}\psi_{1,3}(t) \\ &= 4\sqrt{\frac{2}{3\pi}}c_{1,0} + \frac{8}{\sqrt{\pi}}(4t-1)c_{1,1} + 4\sqrt{\frac{2}{15\pi}}(12(4t-1)^2 - 2)c_{1,2} \\ &+ \frac{2}{\sqrt{3\pi}}(32(4t-1)^3 - 12(4t-1))c_{1,3}, \; t \in \left[0, \frac{1}{2}\right). \end{aligned} \tag{17}$$

By Eqns. (16) and (17),

$$4\sqrt{\frac{2}{3\pi}}c_{1,0}+\left(\frac{8}{\sqrt{\pi}}(4t-1)+\frac{64}{t\sqrt{\pi}}\right)c_{1,1}+\left(4\sqrt{\frac{2}{15\pi}}(12(4t-1)^2-2)+\frac{768}{t}\sqrt{\frac{2}{15\pi}}(4t-1)+1536\sqrt{\frac{2}{15\pi}}\right)c_{1,2}$$
$$+\quad\frac{2}{\sqrt{3\pi}}\left(32(4t-1)^3-12(4t-1)+\frac{4}{t\sqrt{3\pi}}(384(4t-1)^2-48)+\frac{6144}{\sqrt{3\pi}}(4t-1)\right)c_{1,3}=t^2(t+1)+6(2t+1). \quad (18)$$

For values of unknowns $c_{1,0}$, $c_{1,1}$, $c_{1,2}$ and $c_{1,3}$, we collocate Eqn. (18) at $t = 0.125$ and $t = 0.375$ and using the initial condition in Eqn. (17), four systems of linear equations are obtained. Solving these systems of equations, the values of the unknowns are as follows:

$$c_{1,0} = 0.052291496911437,\ c_{1,1} = 0.039378247175879,$$
$$c_{1,2} = 0.011060694488843,\ c_{1,3} = 0.000749506866172. \quad (19)$$

Putting the values of $c_{1,0}$, $c_{1,1}$, $c_{1,2}$ and $c_{1,3}$ from Eqn. (19) into Eqn. (17),

$$\begin{aligned} y(t) &= 0.052291496911437\left(4\sqrt{\frac{2}{3\pi}}\right)+0.039378247175879\left(\frac{8}{\sqrt{\pi}}(4t-1)\right) \\ &+ 0.011060694488843\left(4\sqrt{\frac{2}{15\pi}}(12(4t-1)^2-2)\right) \\ &+ 0.000749506866172\left(\frac{2}{\sqrt{3\pi}}(32(4t-1)^3-12(4t-1))\right),\ t\in\left[0,\frac{1}{2}\right). \end{aligned}$$

Following above process for approximate solution of $y(t)$ in the interval $\left[\frac{1}{2},1\right)$,

$$\begin{aligned} y(t) &= 4\sqrt{\frac{2}{3\pi}}c_{2,0}+\frac{8}{\sqrt{\pi}}(4t-3)c_{2,1}+4\sqrt{\frac{2}{15\pi}}(12(4t-3)^2-2)c_{2,2} \\ &+ \frac{2}{\sqrt{3\pi}}(32(4t-3)^3-12(4t-3))c_{2,3},\ t\in\left[\frac{1}{2},1\right). \end{aligned} \quad (20)$$

For values of unknowns $c_{2,0}$, $c_{2,1}$, $c_{2,2}$ and $c_{2,3}$, we collocate Eqn. (20) at $t = 0.5$, $t = 0.625$, $t = 0.75$, and $t = 0.875$, four systems of linear equations obtained. Solving these systems of equations, the values of the unknowns are as follows:

$$c_{20} = 0.552593926822261,\ c_{21} = 0.177851204277836,$$
$$c_{22} = 0.020541289764991,\ c_{23} = 0.000749506866172. \quad (21)$$

Putting the values of $c_{1,0}$, $c_{1,1}$, $c_{1,2}$ and $c_{1,3}$ from Eqn. (21) into Eqn. (20),

$$\begin{aligned} y(t) &= 0.552593926822261\left(4\sqrt{\frac{2}{3\pi}}\right)+0.177851204277836\left(\frac{8}{\sqrt{\pi}}(4t-3)\right) \\ &+ 0.020541289764991\left(4\sqrt{\frac{2}{15\pi}}(12(4t-3)^2-2)\right) \\ &+ 0.000749506866172\left(\frac{2}{\sqrt{3\pi}}(32(4t-3)^3-12(4t-3))\right),\ t\in\left[\frac{1}{2},1\right). \end{aligned}$$

The exact solution (ES) and approximate solution of the Lane-Emden differential equation obtained by Gegenbauer wavelet method (GWM) for different values of t in the interval [0,1) have been obtained. Also compare this solution by Legendre wavelet method (LWM), first kind Chebyshev wavelet method (FKCWM) and second kind Chebyshev wavelet method (SKCWM) are given in the Table 1 (Lal and Patel[10]) .

t	ES	LWM	FKCWM	SKCWM	GWM
0.1	0.011	0.1105200243	0.0109999998	0.0110000002	0.011000000000001
0.2	0.048	0.0483697994	0.0479999999	0.0480000001	0.048000000000000
0.3	0.117	0.1181960706	0.1169999999	0.1170000000	0.116999999999998
0.4	0.224	0.2267734978	0.2239999999	0.2240000001	0.223999999999999
0.5	0.375	0.3803447612	0.3750000001	0.3750000001	0.375000000002715
0.6	0.576	0.5851525426	0.5760000001	0.5760000001	0.576000000002668
0.7	0.833	0.8474395224	0.8330000003	0.8330000002	0.833000000002610
0.8	1.152	1.1734483820	1.1520000000	1.1520000000	1.152000000002543
0.9	1.539	1.5694218030	1.5390000010	1.5390000000	1.539000000002469

Table 1

The graphs of the exact solution and approximate solution of the Lane-Emden differential equation by Gegenbauer wavelet method, Legendre wavelet method and Chebyshev wavelet method are shown in the Figure 1.

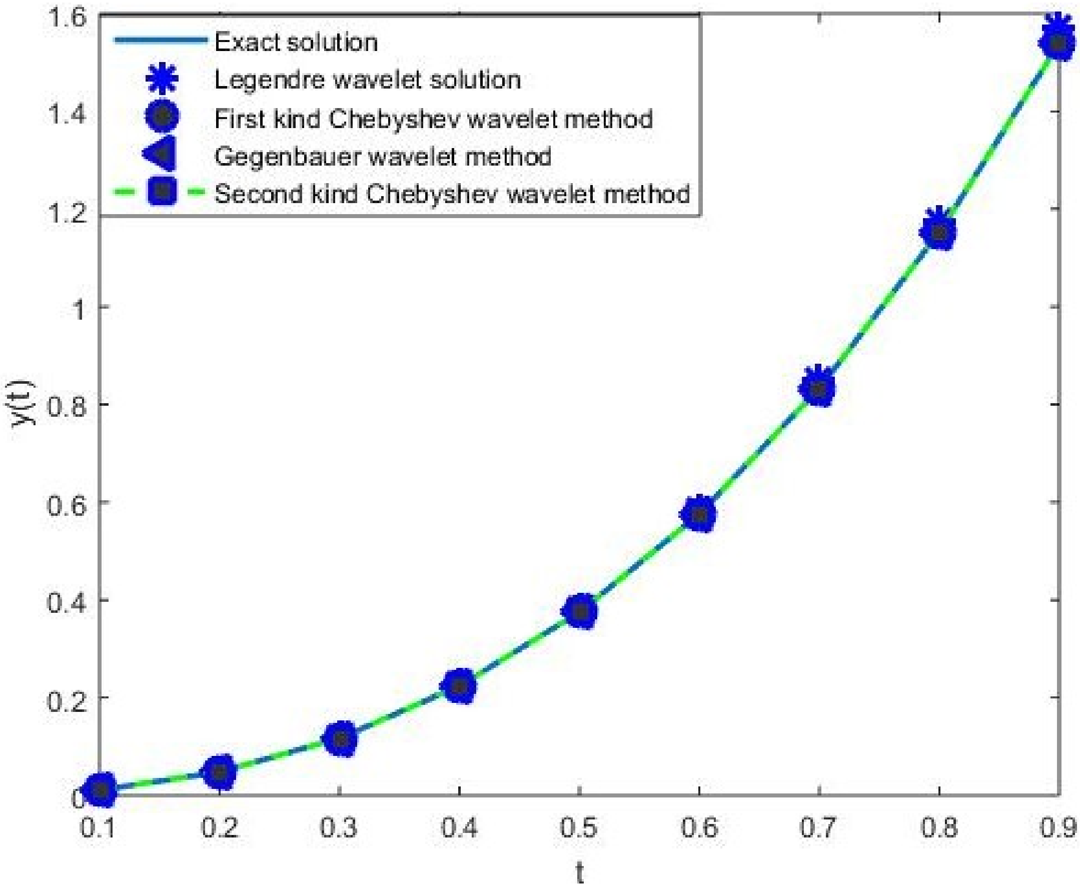

Figure 1

By Table 1 and Figure 1, it is evident that the exact and Gegenbauer wavelet solutions of the Lane-Emden differential equation coincide almost everywhere.

7.2. Absolute Error

The absolute error between the exact solution and approximate solution of the Lane-Emden differential equation by the Gegenbauer wavelet method, the Legendre wavelet method, and the Chebyshev wavelet method is given in Table 2. The absolute error is negligible by the Gegenbauer wavelet method, as compared to the Legendre wavelet method, and the Chebyshev wavelet method are shown in Table 2.

t	LWM	FKCWM($\times 10^{-8}$)	SKCWM($\times 10^{-9}$)	GWM($\times 10^{-11}$)
0.1	0.00005200243	0.019999999920084	0.200000000935563	0.000100093544564
0.2	0.00036979940	0.010000000133514	0.100000001335143	0
0.3	0.00119607060	0.010000000827404	0	0.000201227923213
0.4	0.00277349780	0.010000000827404	0.100000008274037	0.000099920072216
0.5	0.00534476120	0.010000000827404	0.100000008274037	0.271499489556959
0.6	0.00915254260	0.010000000827404	0.100000008274037	0.266808797277918
0.7	0.01443952240	0.030000002482211	0.200000016548074	0.261002330859128
0.8	0.02144838200	0	0	0.254307686020638
0.9	0.03042180300	0.100000008274037	0	0.246913600676635

Table 2

The graphs of the absolute error in the solution of the Lane-Emden differential equation by the Gegenbauer wavelet method and the Chebyshev wavelet method are shown in Figure 2.

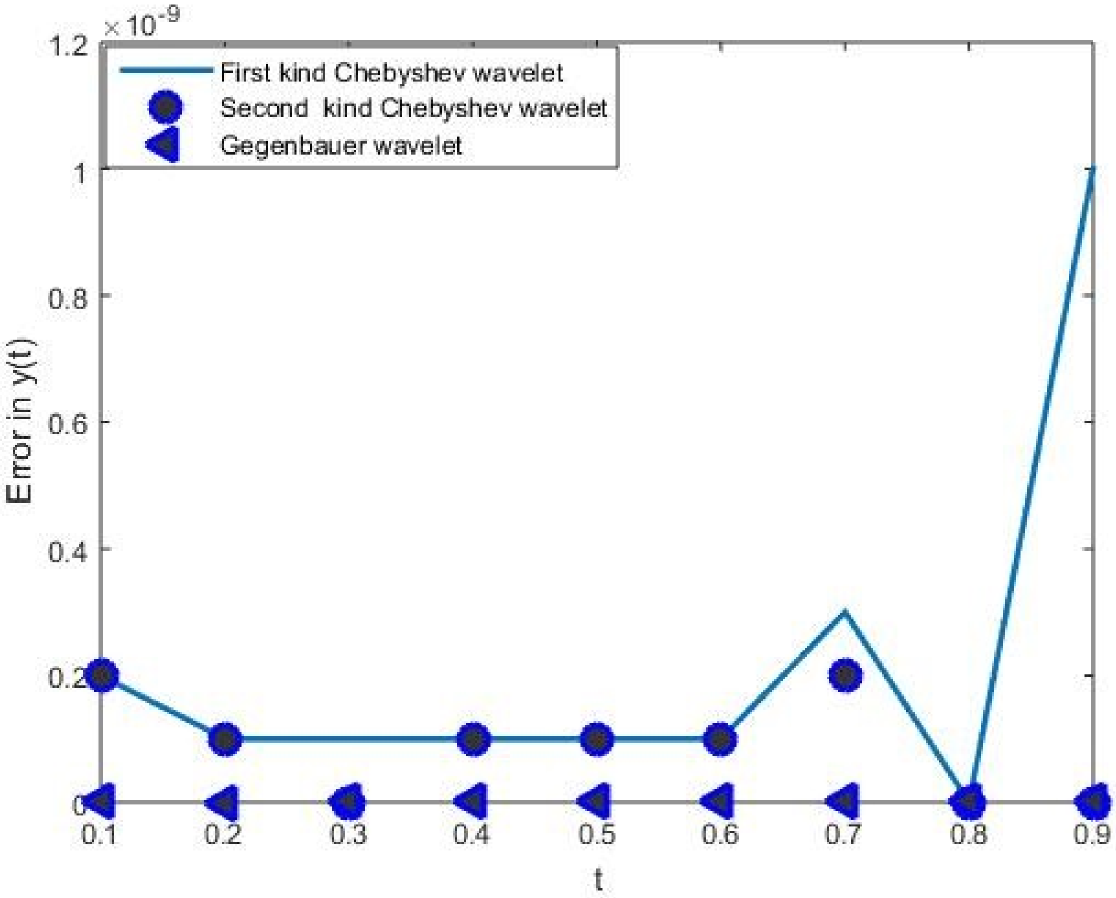

Figure 2

8. Conclusions

1. Gegenbauer wavelet approximation of Theorems 4.2.1 and 4.2.2 are given by
$E^{(1)}_{2^{k-1},0}(f) = O\left(\frac{1}{2^{k\alpha}}\right)$, $k \geq 1$. $(E^{(1)}_{2^{k-1},0}(f)) \to 0$ as $k \to \infty$;
$E^{(1)}_{2^{k-1},M}(f) = O\left(\frac{1}{2^{(k-1)\alpha}\sqrt{M}}\right)$, $M \geq 1$. $(E^{(1)}_{2^{k-1},M}(f)) \to 0$ as $k \to \infty$ and $M \to \infty$;
$E^{(2)}_{2^{k-1},0}(f) = O\left(\phi\left(\frac{1}{2^k}\right)\right)$, $k \geq 1$. $(E^{(2)}_{2^{k-1},0}(f)) \to 0$ as $k \to \infty$;
$E^{(2)}_{2^{k-1},M}(f) = O\left(\phi\left(\frac{1}{2^{k-1}}\right)\frac{1}{\sqrt{M}}\right)$, $M \geq 1$. $(E^{(2)}_{2^{k-1},M}(f)) \to 0$ as $k \to \infty$ and $M \to \infty$.
Therefore, approximation $E^{(1)}_{2^{k-1},0}$, $E^{(1)}_{2^{k-1},M}$, $E^{(2)}_{2^{k-1},0}$ and $E^{(2)}_{2^{k-1},M}$ of function f belonging to $H^{\alpha}[0,1)$ and $H^{\phi}[0,1)$ are the best possible in wavelet analysis.
2. By Table 1 and Figure 1, it is apparent that the exact and Gegenbauer wavelet solutions of the Lane-Emden differential equation coincide almost everywhere in the interval [0,1).
3. By Table 2 and Figure 2, it is comprehensible that the absolute error in the Gegenbauer wavelet method is negligible as compared to the Legendre wavelet method and the Chebyshev wavelet method.
4. The illustrated example shows the validity and accuracy of the proposed method of Section 6 to solve the Lane-Emden differential equation.
5. The proposed method in Section 6 is also applicable to solve the natural problems in higher-order differential equations.

Acknowledgments
Shyam Lal, one of the authors, is thankful to DST-CIMS for encouragement to this work.

Harish Chandra Yadav, one of the authors, is grateful to U.G.C. (India) for providing financial assistance in the form of a Junior Research Fellowship vide UGC-Ref.No.:1071/(CSIR-UGC NET DEC-2018) Dated:24-07-2019 for his research work.

The authors are grateful to the anonymous reviewer for taking the time to give his valuable comments and suggestions, which improve the quality and presentation of this research paper.

References

[1] D. Chouhan, V. Mishra, & H. M. Srivastava, Bernoulli wavelet method for numerical solution of anomalous infiltration and diffusion modeling by nonlinear fractional differential equations of variable order, Results in Applied Mathematics 10 (2021) 100146.
[2] C.K. Chui, An introduction to Wavelets (Wavelet Analysis and its Applications), vol.1 Academic Press Cambridge, 1992.
[3] C.-Y. Lee, H.M. Srivastava, W.-C. Yueh Explicit of some linear ordinary and partial fractional differintegral equations, Appl. Math. Comput., vol.144 (2003) 11-25 .
[4] L. Debnath, Wavelet Transforms and Their Applications, Birkh$\ddot{a}$user, Boston, 2002.
[5] KT Elgindy and KA Smith-Miles, Solving boundary value problems, integral, and integro-differential equations using Gegenbauer integration matrices, J Comput Appl Math, 237(1) (2013) 307–325.
[6] Guo Ben-Yu, Gegenbauer approximation and its applications to differential equations on the whole line, J. Math. Anal. Appl., 226 (1998), pp. 180-206.
[7] M. Izadi and H.M. Srivastava, Generalized Bessel Quasilinearization Technique Applied to Bratu and Lane–Emden-Type Equations of Arbitrary Order, Fractal Fract, 5 (2021), 179.
[8] Mohammad Izadi & H.M. Srivastava, An efficient approximation technique applied to a non-linear Lane–Emden pantograph delay differential model, Applied Mathematics and Computation, Elsevier, vol. 401(C) (2021).
[9] Lal and Kumar, CAS wavelet approximation of functions of H$\ddot{o}$lder's class $H^{\alpha}[0,1)$ and Solution of Fredholm Integral Equations, Ratio Mathematics, vol.39, (2020) 187-212.
[10] Lal & Patel, Chebyshev wavelet approximation of functions having first derivative of Hölder's class, São Paulo Journal of Mathematical Sciences, (2021).
[11] Y. Meyer and S. Roques, Wavelets their post and their future, Progress in Wavelet Analysis and Applications (Toulouse,1992),pp. 9-18. Frontiers, Gif-sur-Yvette (1993).
[12] Narayan Kumar, Rajesh K. Pandey, Carlo Cattani, Solution of the Lane-Emden Equation Using the Bernstein Operational Matrix of Integration, International Scholarly Research Notices, vol. 2011, Article ID 351747, 7 pages, 2011.
[13] H. Singh, H. M. Srivastava, & D. Kumar, A reliable algorithm for the approximate solution of the nonlinear Lane-Emden type equations arising in astrophysics, Numerical Methods for Partial Differential Equations, 34(5) (2017), 1524–1555.
[14] H.M. Srivastava, M. Irfan, F.A. Shah, A Fibonacci Wavelet Method for Solving Dual-Phase-Lag Heat Transfer Model in Multi-Layer Skin Tissue during Hyperthermia Treatment, Energies 14 (2021), 2254.

[15] H.M. Srivastava, F.A. Shah, & R. Abass, An Application of the Gegenbauer Wavelet Method for the Numerical Solution of the Fractional Bagley-Torvik Equation, Russ. J. Math. Phys. 26 (2019), 77–93.
[16] H. M. Srivastava, F. A. Shah, & M. Irfan, Generalized wavelet quasilinearization method for solving population growth model of fractional order, Mathematical Methods in the Applied Sciences,(2020).
[17] H.M. Srivastava, F.A. Shah, N.A. Nayied, Fibonacci Wavelet Method for the Solution of the Non-Linear Hunter–Saxton Equation, Appl. Sci. 12 (2022), 7738.
[18] E.C. Titchmarsh, The Theory of functions, (2nd edn.), Oxford University Press, Oxford, 1939.
[19] A. Zygmund, Trigonometric Series, vol.I. Cambridge University Press, Cambridge, 1959.