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New parameterized inequalities for twice differentiable functions

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Abstract. The present paper first establishes that an identity involving generalized fractional integrals is proved for twice differentiable functions by using a parameter. By using this equality, we obtain some parameterized inequalities for the functions whose second derivatives in absolute value are convex. Finally, we show that our main results reduce to trapezoid, midpoint Simpson and Bullen-type inequalities which are proved in earlier published papers.

1. Introduction

In the literature, the theory of inequalities has an important place in mathematics. There are many studies on the well-known Hermite–Hadamard inequality. Many researchers have studied the Hermite–Hadamard inequality and related inequalities such as trapezoid, midpoint, Simpson's inequality, and Bullen's inequality and have contributed to science.

Over the years, numerous articles have focused on obtaining trapezoid and midpoint type inequalities that give bounds for the right-hand side and left-hand side of the Hermite–Hadamard inequality, respectively. For example, Dragomir and Agarwal first established trapezoid inequalities for convex functions in [10], whereas Kirmacı first, obtained midpoint inequalities for convex functions in [18]. Moreover, in [22], Qaisar and Hussain presented several generalized midpoint type inequalities. Sarikaya et al. and Iqbal et al. proved some fractional trapezoid and midpoint type inequalities for convex functions in [29] and [16], respectively. In [6] and [7], researchers established some generalized midpoint type inequalities for Riemann-Liouville fractional integrals.

Many mathematicians have researched the results of Simpson-type for convex functions. More precisely, some inequalities of Simpson's type for s-convex functions are proved by using differentiable functions [1]. In the papers [27, 28], it is investigated the new variants of Simpson's type inequalities based on the differentiable convex mapping. For more information about Simpson type inequalities for various convex classes, we refer the reader to Refs. [11, 14, 17, 19, 21, 23, 24] and the references therein.

In [8], Bullen established the well-known Bullen-type inequalities in the literature in 1978. In [30], Sarikaya et al. proved generalized Bullen inequality for generalized convex function. Erden and Sarikaya established the generalized Bullen-type inequalities involving local fractional integrals on fractal sets in [13]. Du et al. used the generalized fractional integrals to obtain Bullen-type inequalities in [12].

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Researches on the differentiable functions of these inequalities also have an important place in the literature. Many researchers have focused on twice differentiable functions to obtain many important inequalities. For example, Barani et al. established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality in [3, 4]. In [20], some new generalized fractional integral inequalities of midpoint and trapezoid type for twice differentiable convex functions are obtained. In [25], authors obtained some new inequalities of the Simpson and the Hermite–Hadamard type for functions whose absolute values of derivatives are convex. In [5] and [15], several fractional Simpson's inequality for twice differentiable functions were obtained. In [9], some generalizations of integral inequalities of Bullen–type for twice differentiable functions involving Riemann-Liouville fractional integrals were obtained.

Here, we give some definitions and notations which are used frequently in main section.

The well-known gamma and beta are defined as follows: For $0 < \varkappa, \gamma < \infty$, and $\varkappa, \gamma \in \mathbb{R}$,

$$\Gamma(\varkappa) := \int_0^\infty \tau^{\varkappa-1} e^{-\tau} d\tau$$

and

$$\beta(\varkappa, \gamma) := \int_0^1 \tau^{\varkappa-1} (1-\tau)^{\gamma-1} d\tau = 2\int_0^{\frac{\pi}{2}} \sin(\tau)^{2\varkappa-1} \cos(\tau)^{2\gamma-1} d\tau = \frac{\Gamma(\varkappa)\Gamma(\gamma)}{\Gamma(\varkappa+\gamma)}.$$

The generalized fractional integrals were introduced by Sarikaya and Ertuğral as follows:

Definition 1.1. *[26]Let us note that a function $\varphi : [0,\infty) \to [0,\infty)$ satisfy the following condition:*

$$\int_0^1 \frac{\varphi(\tau)}{\tau} d\tau < \infty.$$

We consider the following left-sided and right-sided generalized fractional integral operators

$$_{(\sigma)+}I_\varphi F(\varkappa) = \int_\sigma^\varkappa \frac{\varphi(\varkappa-\tau)}{\varkappa-\tau} F(\tau) d\tau, \quad \varkappa > \sigma \tag{1}$$

and

$$_{(\rho)-}I_\varphi F(\varkappa) = \int_\varkappa^\rho \frac{\varphi(\tau-\varkappa)}{\tau-\varkappa} F(\tau) d\tau, \quad \varkappa < \rho, \tag{2}$$

respectively.

The most significant feature of generalized fractional integrals is that they generalize some important types of fractional integrals such as Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Hadamard fractional integrals, Katugampola fractional integrals, conformable fractional integral, etc. These important special cases of the integral operators (1) and (2) are mentioned as follows:

1. Let us consider $\varphi(\tau) = \tau$. Then, the operators (1) and (2) reduce to the Riemann integral.
2. If we choose $\varphi(\tau) = \frac{\tau^\alpha}{\Gamma(\alpha)}$ and $\alpha > 0$, then the operators (1) and (2) reduce to the Riemann-Liouville fractional integrals $J_{\sigma+}^\alpha F(\varkappa)$ and $J_{\rho-}^\alpha F(\varkappa)$, respectively. Here, Γ is Gamma function.
3. For $\varphi(\tau) = \frac{1}{k\Gamma_k(\alpha)} \tau^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators (1) and (2) reduce to the k-Riemann-Liouville fractional integrals $J_{\sigma+,k}^\alpha F(\varkappa)$ and $J_{\rho-,k}^\alpha F(\varkappa)$, respectively. Here, Γ_k is k-Gamma function defined by

$$\Gamma_k(\alpha) = \int_0^\infty \tau^{\alpha-1} e^{-\frac{\tau^k}{k}} d\tau, \quad \mathcal{R}(\alpha) > 0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1} \Gamma\left(\frac{\alpha}{k}\right), \quad \mathcal{R}(\alpha) > 0; k > 0.$$

2. A new identity for twice differentiable functions

In this section, we prove equality involving a reel parameter for twice differentiable functions by the view of generalized fractional integrals.

Lemma 2.1. *Let $F:[\sigma,\rho]\to\mathbb{R}$ be an absolutely continuous mapping (σ,ρ) such that $F''\in L_1([\sigma,\rho])$. Then, the following equality holds:*

$$(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)^{+}}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)^{-}}I_{\varphi}F(\sigma)\right] \tag{3}$$

$$=\frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\int_0^1\omega(\tau)F''(\tau\sigma+(1-\tau)\rho)\,d\tau.$$

Here,

$$\omega(\tau)=\begin{cases}\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau, & \tau\in\left[0,\frac{1}{2}\right),\\ \Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau), & \tau\in\left[\frac{1}{2},1\right]\end{cases}$$

where $\Delta(\tau)=\int_0^\tau\Lambda(s)\,ds$ and $\Lambda(s)=\int_0^s\frac{\varphi((\rho-\sigma)u)}{u}du$.

Proof. By using integration by parts, we obtain

$$\begin{aligned}I_1&=\int_0^{\frac{1}{2}}\left(\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right)F''(\tau\sigma+(1-\tau)\rho)\,d\tau\\ &=-\frac{\left(\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right)F'(\tau\sigma+(1-\tau)\rho)}{\rho-\sigma}\Bigg|_0^{\frac{1}{2}}+\frac{1}{\rho-\sigma}\int_0^{\frac{1}{2}}\left(\Lambda(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\right)F'(\tau\sigma+(1-\tau)\rho)\,d\tau\\ &=-\frac{\Delta\left(\frac{1}{2}\right)-\frac{\lambda}{2}\Lambda\left(\frac{1}{2}\right)}{\rho-\sigma}F'\left(\frac{\sigma+\rho}{2}\right)+\frac{1}{\rho-\sigma}\left[-\frac{\Lambda(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)}{\rho-\sigma}F(\tau\sigma+(1-\tau)\rho)\Bigg|_0^{\frac{1}{2}}\right.\\ &\quad\left.+\frac{1}{\rho-\sigma}\int_0^{\frac{1}{2}}\frac{\varphi((\rho-\sigma)\tau)}{\tau}F(\tau\sigma+(1-\tau)\rho)\,d\tau\right].\end{aligned}\tag{4}$$

With help of the equality (4) and using the change of the variable $\varkappa=\tau\sigma+(1-\tau)\rho$ for $\tau\in\left[0,\frac{1}{2}\right)$, it can be rewritten as follows

$$\begin{aligned}I_1&=-\frac{\Delta\left(\frac{1}{2}\right)-\frac{\lambda}{2}\Lambda\left(\frac{1}{2}\right)}{\rho-\sigma}F'\left(\frac{\sigma+\rho}{2}\right)\\ &\quad-\frac{\Lambda\left(\frac{1}{2}\right)-\lambda\Lambda\left(\frac{1}{2}\right)}{(\rho-\sigma)^2}F\left(\frac{\sigma+\rho}{2}\right)-\frac{\lambda\Lambda\left(\frac{1}{2}\right)}{(\rho-\sigma)^2}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)^{+}}I_{\varphi}F(\rho).\end{aligned}\tag{5}$$

Similarly, we get

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^{1}\left(\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right)F''(\tau\sigma+(1-\tau)\rho)\,d\tau \\ &= -\left.\frac{\left(\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right)F'(\tau\sigma+(1-\tau)\rho)}{\rho-\sigma}\right|_{\frac{1}{2}}^{1} \\ &\quad+\frac{1}{\rho-\sigma}\int_{\frac{1}{2}}^{1}\left(-\Lambda(1-\tau)+\lambda\Lambda\left(\frac{1}{2}\right)\right)F'(\tau\sigma+(1-\tau)\rho)\,d\tau \\ &= \frac{\Delta\left(\frac{1}{2}\right)-\frac{\lambda}{2}\Lambda\left(\frac{1}{2}\right)}{\rho-\sigma}F'\left(\frac{\sigma+\rho}{2}\right) \\ &\quad+\frac{1}{\rho-\sigma}\left[-\left.\frac{-\Lambda(1-\tau)+\lambda\Lambda\left(\frac{1}{2}\right)}{\rho-\sigma}F(\tau\sigma+(1-\tau)\rho)\right|_{\frac{1}{2}}^{1}\right. \\ &\quad\left.+\frac{1}{\rho-\sigma}\int_{\frac{1}{2}}^{1}\frac{\varphi((\rho-\sigma)(1-\tau))}{(1-\tau)}F(\tau\sigma+(1-\tau)\rho)\,d\tau\right] \\ &= \frac{\Delta\left(\frac{1}{2}\right)-\frac{\lambda}{2}\Lambda\left(\frac{1}{2}\right)}{\rho-\sigma}F'\left(\frac{\sigma+\rho}{2}\right)+\frac{-\Lambda\left(\frac{1}{2}\right)+\lambda\Lambda\left(\frac{1}{2}\right)}{(\rho-\sigma)^2}F\left(\frac{\sigma+\rho}{2}\right) \\ &\quad-\frac{\lambda\Lambda\left(\frac{1}{2}\right)}{(\rho-\sigma)^2}F(\sigma)+{}_{\left(\frac{\sigma+\rho}{2}\right)^-}I_{\varphi}F(\sigma). \end{aligned} \tag{6}$$

From (5) and (6), we have

$$I_1+I_2=\frac{2\Lambda\left(\frac{1}{2}\right)(\lambda-1)}{(\rho-\sigma)^2}F\left(\frac{\sigma+\rho}{2}\right)-\frac{\lambda\Lambda\left(\frac{1}{2}\right)}{(\rho-\sigma)^2}(F(\sigma)+F(\rho)) \tag{7}$$
$$+\frac{1}{(\rho-\sigma)^2}\left({}_{\left(\frac{\sigma+\rho}{2}\right)^+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)^-}I_{\varphi}F(\sigma)\right).$$

Multiplying the both sides of (7) by $\frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}$, we obtain desired identity (3). This ends the proof of Lemma 2.1. □

3. Some parameterized inequalities for generalized fractional integrals

By utilizing generalized fractional integrals, we prove some parameterized inequalities for functions whose various power of absolute value of second derivatives are convex function.

Theorem 3.1. *Let us consider that the assumptions of Lemma 2.1 are valid. Let us also consider that the mapping $|F''|$ is convex on $[\sigma,\rho]$. Then, we get the following inequality for generalized fractional integrals*

$$\left|(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)^+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)^-}I_{\varphi}F(\sigma)\right]\right| \tag{8}$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\Upsilon_{\varphi}(\lambda)\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right],$$

where $\Upsilon_{\varphi}(\lambda)$ *is defined by*

$$\Upsilon_{\varphi}(\lambda)=\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|d\tau.$$

Proof. By taking modulus in Lemma 2.1, we have

$$\left|(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right| \tag{9}$$

$$=\frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left|\int_0^1\omega(\tau)F''(\tau\sigma+(1-\tau)\rho)\,d\tau\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left[\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|\left|F''(\tau\sigma+(1-\tau)\rho)\right|d\tau\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|\left|F''(\tau\sigma+(1-\tau)\rho)\right|d\tau\right].$$

By using convexity of $|F''|$, we obtain

$$\left|(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left[\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|\left[\tau\left|F''(\sigma)\right|+(1-\tau)\left|F''(\rho)\right|\right]d\tau\right.$$

$$\left.+\int_{\frac{1}{2}}^1\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|\left[\tau\left|F''(\sigma)\right|+(1-\tau)\left|F''(\rho)\right|\right]d\tau\right]$$

$$=\frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left[\left(\int_0^{\frac{1}{2}}\tau\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|d\tau+\int_{\frac{1}{2}}^1\tau\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|d\tau\right)\left|F''(\sigma)\right|\right.$$

$$\left.+\left(\int_0^{\frac{1}{2}}(1-\tau)\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|d\tau+\int_{\frac{1}{2}}^1(1-\tau)\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|d\tau\right)\left|F''(\rho)\right|\right]$$

$$=\frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right]\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Delta\left(\frac{1}{2}\right)\tau\right|d\tau$$

$$= \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\Upsilon_{\varphi}(\lambda)\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right].$$

This finishes the proof of Theorem 3.1. □

Theorem 3.2. *Let us note that the assumptions of Lemma 2.1 hold. If the mapping $|F''|^q$, $q>1$ is convex on $[\sigma,\rho]$, then we have the following inequality for generalized fractional integrals*

$$\begin{aligned}
&\left|(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right| \\
&\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_1^{\varphi}(\lambda,p)\right)^{\frac{1}{p}}\left[\left(\frac{\left|F''(\sigma)\right|^q+3\left|F''(\rho)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|F''(\sigma)\right|^q+\left|F''(\rho)\right|^q}{8}\right)^{\frac{1}{q}}\right] \\
&\leq \frac{(\rho-\sigma)^2}{2^{\frac{2}{q}}\Lambda\left(\frac{1}{2}\right)}\left(2\Psi_1^{\varphi}(\lambda,p)\right)^{\frac{1}{p}}\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right].
\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and $\Psi_1^{\varphi}(\lambda,p)$ is defined by

$$\Psi_1^{\varphi}(\lambda,p)=\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|^p d\tau.$$

Proof. By using the Hölder inequality in (9), we obtain

$$\begin{aligned}
&\left|(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right| \\
&\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left[\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|F''(\tau\sigma+(1-\tau)\rho)\right|^q d\tau\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left|F''(\tau\sigma+(1-\tau)\rho)\right|^q d\tau\right)^{\frac{1}{q}}\right].
\end{aligned}$$

With the help of the convexity of $|F''|^q$, we get

$$\begin{aligned}
&\left|(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right| \\
&\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left[\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[\tau\left|F''(\sigma)\right|^q+(1-\tau)\left|F''(\rho)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1}\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[\tau\left|F''(\sigma)\right|^q+(1-\tau)\left|F''(\rho)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$= \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|^p d\tau\right)^{\frac{1}{p}}$$

$$\times\left[\left(\frac{|F''(\sigma)|^q+3|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\sigma)|^q+|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}\right]$$

$$= \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_1^{\varphi}(\lambda,p)\right)^{\frac{1}{p}}\left[\left(\frac{|F''(\sigma)|^q+3|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\sigma)|^q+|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}\right].$$

For the proof of second inequality, let $\sigma_1 = |F''(\sigma)|^q$, $\rho_1 = 3|F''(\rho)|^q$, $\sigma_2 = 3|F''(\sigma)|^q$ and $\rho_2 = |F''(\rho)|^q$. Using the facts that,

$$\sum_{k=1}^{n}(\sigma_k+\rho_k)^s \leq \sum_{k=1}^{n}\sigma_k^s+\sum_{k=1}^{n}\rho_k^s,\ 0\leq s<1 \tag{10}$$

and $1+3^{\frac{1}{q}} \leq 4$. the desired result can be obtained straightforwardly. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let us note that the assumptions of Lemma 2.1 hold. If the mapping $|F''|^q$, $q \geq 1$ is convex on $[\sigma,\rho]$, then we have the following inequality*

$$\left|(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_2^{\varphi}(\lambda)\right)^{1-\frac{1}{q}}\left[\left(\Omega_1^{\varphi}(\lambda)|F''(\sigma)|^q+\Omega_2^{\varphi}(\lambda)|F''(\rho)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\Omega_2^{\varphi}(\lambda)|F''(\sigma)|^q+\Omega_1^{\varphi}(\lambda)|F''(\rho)|^q\right)^{\frac{1}{q}}\right].$$

Here,

$$\begin{cases}\Psi_2^{\varphi}(\lambda)=\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|d\tau,\\ \Omega_1^{\varphi}(\lambda)=\int_0^{\frac{1}{2}}\tau\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|d\tau,\\ \Omega_2^{\varphi}(\lambda)=\int_0^{\frac{1}{2}}(1-\tau)\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|d\tau.\end{cases}.$$

Proof. By applying power-mean inequality in (9), we obtain

$$\left|(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left[\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|d\tau\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|\left|F''(\tau\sigma+(1-\tau)\rho)\right|^q d\tau\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^1\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|d\tau\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_{\frac{1}{2}}^1\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|\left|F''(\tau\sigma+(1-\tau)\rho)\right|^q d\tau\right)^{\frac{1}{q}}\right].$$

Since $|F''|^q$ is convex, we have

$$\begin{aligned}
&\left|(\lambda-1)F\left(\frac{\sigma+\rho}{2}\right)-\lambda\left(\frac{F(\sigma)+F(\rho)}{2}\right)+\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_\varphi F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_\varphi F(\sigma)\right]\right|\\
\leq\ & \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left[\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|d\tau\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|\left[\tau\left|F''(\sigma)\right|^q+(1-\tau)\left|F''(\rho)\right|^q\right]d\tau\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|d\tau\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_{\frac{1}{2}}^1\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|\left[\tau\left|F''(\sigma)\right|^q+(1-\tau)\left|F''(\rho)\right|^q\right]d\tau\right)^{\frac{1}{q}}\right]\\
=\ & \frac{(\rho-\sigma)^2}{\Lambda\left(\frac{1}{2}\right)}\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\lambda\Lambda\left(\frac{1}{2}\right)\tau\right|d\tau\right)^{1-\frac{1}{q}}\\
&\times\left[\int_0^{\frac{1}{2}}\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|\left[\tau\left|F''(\sigma)\right|^q+(1-\tau)\left|F''(\rho)\right|^q\right]d\tau\right]^{\frac{1}{q}}\\
=\ & \frac{(\rho-\sigma)^2}{\Lambda\left(\frac{1}{2}\right)}\left(\Psi_2^\varphi(\lambda)\right)^{1-\frac{1}{q}}\\
&\times\left[\left(\left|F''(\sigma)\right|^q\int_0^{\frac{1}{2}}\tau\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right|d\tau\right.\right.
\end{aligned}$$

$$+\left|F''(\rho)\right|^q \int_0^{\frac{1}{2}} (1-\tau)\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right| d\tau\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\left|F''(\sigma)\right|^q \int_0^{\frac{1}{2}} (1-\tau)\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right| d\tau$$

$$+\left|F''(\rho)\right|^q \int_0^{\frac{1}{2}} \tau\left|\Delta(1-\tau)-\lambda\Lambda\left(\frac{1}{2}\right)(1-\tau)\right| d\tau\Bigg)^{\frac{1}{q}}\Bigg]$$

$$= \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_2^{\varphi}(\lambda)\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\Omega_1^{\varphi}(\lambda)\left|F''(\sigma)\right|^q+\Omega_2^{\varphi}(\lambda)\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}+\left(\Omega_2^{\varphi}(\lambda)\left|F''(\sigma)\right|^q+\Omega_1^{\varphi}(\lambda)\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}\right].$$

Then, we obtain the desired result of Theorem 3.3. □

4. Special cases of main results

In this section, we present some special cases of our results. We first show that our results reduce several inequalities given earlier published papers. We also give several new inequalities trapezoid, midpoint Simpson and Bullen type inequalities by special choice of real parameter λ.

Remark 4.1. *If we assign $\lambda = 1$ in Theorem 3.1, then we have the following trapezoid type inequality*

$$\left|\frac{F(\sigma)+F(\rho)}{2}-\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\Delta\left(\frac{1}{2}\right)\tau\right| d\tau\right)\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right].$$

It can be easily seen that this result is the same as [20, Theorem 3.1].

Remark 4.2. *If we assign $\lambda = 0$ in Theorem 3.1, then we have the following midpoint type inequality*

$$\left|\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]-F\left(\frac{\sigma+\rho}{2}\right)\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)\right| d\tau\right)\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right].$$

It can be easily seen that this result is the same as [20, Theorem 2.1].

Remark 4.3. *If we assign $\lambda = \frac{1}{3}$ in Theorem 3.1, then we have thr following Simpson type inequality*

$$\left|\frac{1}{6}\left[F(\sigma)+4F\left(\frac{\sigma+\rho}{2}\right)+F(\rho)\right]-\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\int_0^{\frac{1}{2}}\left|\Delta(\tau)-\frac{\tau}{3}\Lambda\left(\frac{1}{2}\right)\right|d\tau\right)\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right].$$

which proved by Ali et al. in [2].

Corollary 4.4. *If we assign $\lambda=\frac{1}{2}$ in Theorem 3.1, then the following inequality Bullen-type inequality*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[\,_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+\,_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$
$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\Upsilon_{\varphi}\left(\frac{1}{2}\right)\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right]$$

is valid.

Corollary 4.5. *Let us consider $\varphi(\tau)=\tau$ in Corollary 4.4. Then, we have*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}F(\tau)\,d\tau\right|$$
$$\leq \frac{(\rho-\sigma)^2}{96}\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right]$$

which is given by Sarikaya and Aktan in [25, Proposition 4].

Corollary 4.6. *If we take $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$, $\alpha>0$ in Corollary 4.4, then we obtain the following inequality Bullen-type inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)+}F(\rho)+J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)-}F(\sigma)\right]\right|$$
$$\leq \frac{(\rho-\sigma)^2}{8}\left(\frac{1}{4}-\frac{1}{(\alpha+1)(\alpha+2)}\right)\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right].$$

Remark 4.7. *If we assign $\lambda=1$ in Theorem 3.2, then we have the following trapezoid type inequality for generalized fractional integrals*

$$\left|\frac{F(\sigma)+F(\rho)}{2}-\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[\,_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+\,_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$
$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_1^{\varphi}(1,p)\right)^{\frac{1}{p}}\left[\left(\frac{\left|F''(\sigma)\right|^q+3\left|F''(\rho)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|F''(\sigma)\right|^q+\left|F''(\rho)\right|^q}{8}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(\rho-\sigma)^2}{2^{\frac{2}{q}}\Lambda\left(\frac{1}{2}\right)}\left(2\Psi_1^{\varphi}(1,p)\right)^{\frac{1}{p}}\left[\left|F''(\sigma)\right|+\left|F''(\rho)\right|\right].$$

It can be easily seen that this result is the same as [20, Theorem 3.2].

Remark 4.8. *If we assign $\lambda = 0$ in Theorem 3.2, then we have the folllowing midpoint type inequalities for generalized fractional integrals*

$$\left|\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]-F\left(\frac{\sigma+\rho}{2}\right)\right|$$
$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\int_0^{\frac{1}{2}}|\Delta(\tau)|^p\,d\tau\right)^{\frac{1}{p}}\left[\left(\frac{|F''(\sigma)|^q+3|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\sigma)|^q+|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(\rho-\sigma)^2}{2^{\frac{2}{q}}\Lambda\left(\frac{1}{2}\right)}\left(2\int_0^{\frac{1}{2}}|\Delta(\tau)|^p\,d\tau\right)^{\frac{1}{p}}\left[|F''(\sigma)|+|F''(\rho)|\right].$$

It can be easily seen that this result is the same as [20, Theorem 2.2].

Remark 4.9. *If we assign $\lambda = \frac{1}{3}$ in Theorem 3.2, then we have the folllowing Simpson type inequalities for generalized fractional integrals*

$$\left|\frac{1}{6}\left[F(\sigma)+4F\left(\frac{\sigma+\rho}{2}\right)+F(\rho)\right]-\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$
$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_1^{\varphi}\left(\frac{1}{3},p\right)\right)^{\frac{1}{p}}\left[\left(\frac{|F''(\sigma)|^q+3|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\sigma)|^q+|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(\rho-\sigma)^2}{2^{\frac{2}{q}}\Lambda\left(\frac{1}{2}\right)}\left(2\Psi_1^{\varphi}\left(\frac{1}{3},p\right)\right)^{\frac{1}{p}}\left[|F''(\sigma)|+|F''(\rho)|\right].$$

which is given by Ali et al. in [2].

Corollary 4.10. *If we assign $\lambda = \frac{1}{2}$ in Theorem 3.2, then the following Bullen-type inequality for generalized fractional integrals*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$
$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_1^{\varphi}\left(\frac{1}{2},p\right)\right)^{\frac{1}{p}}\left[\left(\frac{|F''(\sigma)|^q+3|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\sigma)|^q+|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(\rho-\sigma)^2}{2^{\frac{2}{q}}\Lambda\left(\frac{1}{2}\right)}\left(2\Psi_1^{\varphi}\left(\frac{1}{2},p\right)\right)^{\frac{1}{p}}\left[|F''(\sigma)|+|F''(\rho)|\right]$$

is valid.

Corollary 4.11. *Let us consider $\varphi(\tau) = \tau$ in Corollary 4.10. Then, we have*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}F(\tau)\,d\tau\right|$$

$$\leq \frac{(\rho-\sigma)^2}{16}\left[\frac{(\Gamma(p+1))^2}{\Gamma(2p+2)}\right]^{\frac{1}{p}}\left[\left(\frac{|F''(\sigma)|^q+3|F''(\rho)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\sigma)|^q+|F''(\rho)|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\rho-\sigma)^2}{16}\left[\frac{4(\Gamma(p+1))^2}{\Gamma(2p+2)}\right]^{\frac{1}{p}}\left[|F''(\sigma)|+|F''(\rho)|\right].$$

Proof. For $\varphi(\tau)=\tau$, we have

$$\begin{aligned}\Psi_1^{\varphi}(1,p) &= \int_0^{\frac{1}{2}}\left|\frac{\rho-\sigma}{2}\tau^2-\frac{\rho-\sigma}{4}\tau\right|^p d\tau \\ &= \left(\frac{\rho-\sigma}{2}\right)^p\int_0^{\frac{1}{2}}\tau^p\left(\frac{1}{2}-\tau\right)^p d\tau \\ &= \frac{1}{2^{2p+1}}\left(\frac{\rho-\sigma}{2}\right)^p\int_0^1 u^p(1-u)^p du \\ &= \frac{(\rho-\sigma)^p}{2^{3p+1}}\beta(p+1,p+1) \\ &= \frac{(\rho-\sigma)^p}{2^{3p+1}}\frac{(\Gamma(p+1))^2}{\Gamma(2p+2)}.\end{aligned}$$

This gives the required results. □

Corollary 4.12. *If we take $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$, $\alpha>0$ in Corollary 4.10, then we obtain the following Bullen-type inequalities for Riemann-Liouville fractional integrals*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)+}F(\rho)+J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)-}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2^{1-\alpha}}(\delta(\alpha))^{\frac{1}{p}}\left[\left(\frac{|F''(\sigma)|^q+3|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\sigma)|^q+|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\rho-\sigma)^2}{2^{\frac{2}{q}-\alpha}}(2\delta(\alpha))^{\frac{1}{p}}\left[|F''(\sigma)|+|F''(\rho)|\right].$$

Here,

$$\delta(\alpha)=\int_0^{\frac{1}{2}}\tau^p\left|\frac{\tau^{\alpha}}{\alpha+1}-\frac{1}{2^{\alpha+1}}\right|^p d\tau.$$

Corollary 4.13. *In Corollary 4.10, if we use $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\sigma)}$, $\alpha,k>0$, then we obtain the following Bullen-type inequalities for k-Riemann-Liouville fractional integrals*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)+,k}F(\rho)+J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)-,k}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2^{1-\alpha}}(\delta(\alpha,k))^{\frac{1}{p}}\left[\left(\frac{|F''(\sigma)|^q+3|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{3|F''(\sigma)|^q+|F''(\rho)|^q}{8}\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(\rho-\sigma)^2}{2^{\frac{2}{q}-\frac{\alpha}{k}}}(2\delta(\alpha,k))^{\frac{1}{p}}\left[|F''(\sigma)|+|F''(\rho)|\right].$$

Here,

$$\delta(\alpha,k)=\int_0^{\frac{1}{2}}\tau^p\left|\frac{k\tau^{\frac{\alpha}{k}}}{\alpha+k}-\frac{1}{2^{\frac{\alpha+k}{k}}}\right|^p d\tau.$$

Corollary 4.14. *In Corollary 4.4, if we use $\varphi(\tau)=\frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\sigma)}$, $\alpha,k>0$, then we obtain the following inequality Bullen-type inequality for k-Riemann-Liouville fractional integrals*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)+,k}F(\rho)+J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)-,k}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{8}\left(\frac{1}{4}-\frac{k^2}{(\alpha+k)(\alpha+2k)}\right)\left[|F''(\sigma)|+|F''(\rho)|\right].$$

Corollary 4.15. *If we assign $\lambda=1$ in Theorem 3.3, then the following inequality holds:*

$$\left|\frac{F(\sigma)+F(\rho)}{2}-\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_2^{\varphi}(1)\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\Omega_1^{\varphi}(1)|F''(\sigma)|^q+\Omega_2^{\varphi}(1)|F''(\rho)|^q\right)^{\frac{1}{q}}+\left(\Omega_2^{\varphi}(1)|F''(\sigma)|^q+\Omega_1^{\varphi}(1)|F''(\rho)|^q\right)^{\frac{1}{q}}\right].$$

Remark 4.16. *If we choose $\varphi(\tau)=\tau$ in Corollary 4.15, then Corollary 4.15 reduces to [25, Proposition 6].*

Corollary 4.17. *If we select $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$, $\alpha>0$, in Corollary 4.15, then we obtain*

$$\left|\frac{F(\sigma)+F(\rho)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)+}F(\rho)+J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)-}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{16}\left(1-\frac{2}{(\alpha+1)(\alpha+2)}\right)^{1-\frac{1}{q}}\left[\left(\Theta_1(\alpha)|F''(\sigma)|^q+\Theta_2(\alpha)|F''(\rho)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\Theta_2(\alpha)|F''(\sigma)|^q+\Theta_1(\alpha)|F''(\rho)|^q\right)^{\frac{1}{q}}\right],$$

where

$$\begin{cases}\Theta_1(\alpha)=\frac{1}{3}-\frac{1}{(\alpha+1)(\alpha+3)},\\[1ex]\Theta_2(\alpha)=\frac{2}{3}-\frac{\alpha+4}{(\alpha+1)(\alpha+2)(\alpha+3)}.\end{cases}$$

Corollary 4.18. *In Corollary 4.15, if we use* $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\sigma)}$, $\alpha, k > 0$, *then we obtain the following inequality,*

$$\left|\frac{F(\sigma)+F(\rho)}{2} - \frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)+,k}F(\rho) + J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)-,k}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{16}\left(1-\frac{2k^2}{(\alpha+k)(\alpha+2k)}\right)^{1-\frac{1}{q}}\left[\left(\Theta_1(\alpha,k)\left|F''(\sigma)\right|^q + \Theta_2(\alpha,k)\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\Theta_2(\alpha,k)\left|F''(\sigma)\right|^q + \Theta_1(\alpha,k)\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}\right],$$

where

$$\begin{cases} \Theta_1(\alpha,k) = \frac{1}{3} - \frac{k^2}{(\alpha+k)(\alpha+3k)}, \\ \Theta_2(\alpha,k) = \frac{2}{3} - \frac{(\alpha+4)k^3}{(\alpha+k)(\alpha+2k)(\alpha+3k)}. \end{cases}$$

Corollary 4.19. *Let us consider* $\lambda = 0$ *in Theorem 3.3, then the following inequality*

$$\left|F\left(\frac{\sigma+\rho}{2}\right) - \frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho) + {}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_2^{\varphi}(0)\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\Omega_1^{\varphi}(0)\left|F''(\sigma)\right|^q + \Omega_2^{\varphi}(0)\left|F''(\rho)\right|^q\right)^{\frac{1}{q}} + \left(\Omega_2^{\varphi}(0)\left|F''(\sigma)\right|^q + \Omega_1^{\varphi}(0)\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}\right]$$

is valid.

Remark 4.20. *If we choose* $\varphi(\tau) = \tau$ *in Corollary 4.19, then Corollary 4.19 reduces to [25, Proposition 5].*

Corollary 4.21. *Consider* $\varphi(\tau) = \frac{\tau^{\alpha}}{\Gamma(\alpha)}$, $\alpha > 0$, *in Corollary 4.19. Then, we obtain the following midpoint type inequalities for Riemann-Liouville fractional integrals*

$$\left|F\left(\frac{\sigma+\rho}{2}\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\left[J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)+}F(\rho) + J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)-}F(\sigma)\right]\right|$$

$$\leq \frac{(\rho-\sigma)^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2(\alpha+3)}\left|F''(\sigma)\right|^q + \frac{(\alpha+4)}{2(\alpha+2)(\alpha+3)}\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\alpha+4)}{2(\alpha+2)(\alpha+3)}\left|F''(\sigma)\right|^q + \frac{1}{2(\alpha+3)}\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}\right].$$

which is proved by Tomar et al. in [31].

Corollary 4.22. *In Corollary 4.19, if we use* $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\sigma)}$, $\alpha, k > 0$, *then we obtain the following midpoint type inequalities for k-Riemann-Liouville fractional integrals*

$$\left|F\left(\frac{\sigma+\rho}{2}\right) - \frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)+,k}F(\rho) + J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)-,k}F(\sigma)\right]\right|$$

$$\leq \frac{k(\rho-\sigma)^2}{8(\alpha+k)}\left(\frac{k}{\alpha+2k}\right)^{1-\frac{1}{q}}\left[\left(\frac{k}{2(\alpha+3k)}\left|F''(\sigma)\right|^q + \frac{k(\alpha+4k)}{2(\alpha+2k)(\alpha+3k)}\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{k(\alpha+4k)}{2(\alpha+2k)(\alpha+3k)}\left|F''(\sigma)\right|^{q}+\frac{k}{2(\alpha+3k)}\left|F''(\rho)\right|^{q}\right)^{\frac{1}{q}}\Bigg].$$

Remark 4.23. *If we assign $\lambda=\frac{1}{3}$ in Theorem 3.3, then the following inequality holds:*

$$\left|\frac{1}{6}\left[F(\sigma)+4F\left(\frac{\sigma+\rho}{2}\right)+F(\rho)\right]-\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$

$$\leq\frac{(\rho-\sigma)^{2}}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_{2}^{\varphi}\left(\frac{1}{3}\right)\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\Omega_{1}^{\varphi}\left(\frac{1}{3}\right)\left|F''(\sigma)\right|^{q}+\Omega_{2}^{\varphi}\left(\frac{1}{3}\right)\left|F''(\rho)\right|^{q}\right)^{\frac{1}{q}}+\left(\Omega_{2}^{\varphi}\left(\frac{1}{3}\right)\left|F''(\sigma)\right|^{q}+\Omega_{1}^{\varphi}\left(\frac{1}{3}\right)\left|F''(\rho)\right|^{q}\right)^{\frac{1}{q}}\right].$$

which is given by Ali et al. in [2].

Corollary 4.24. *If we assign $\lambda=\frac{1}{2}$ in Theorem 3.3, then the following Bullen-type inequality for generalized fractional integrals*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{1}{2\Lambda\left(\frac{1}{2}\right)}\left[{}_{\left(\frac{\sigma+\rho}{2}\right)+}I_{\varphi}F(\rho)+{}_{\left(\frac{\sigma+\rho}{2}\right)-}I_{\varphi}F(\sigma)\right]\right|$$

$$\leq\frac{(\rho-\sigma)^{2}}{2\Lambda\left(\frac{1}{2}\right)}\left(\Psi_{2}^{\varphi}\left(\frac{1}{2}\right)\right)^{1-\frac{1}{q}}\Bigg[\left(\Omega_{1}^{\varphi}\left(\frac{1}{2}\right)\left|F''(\sigma)\right|^{q}+\Omega_{2}^{\varphi}\left(\frac{1}{2}\right)\left|F''(\rho)\right|^{q}\right)^{\frac{1}{q}}$$

$$+\left(\Omega_{2}^{\varphi}\left(\frac{1}{2}\right)\left|F''(\sigma)\right|^{q}+\Omega_{1}^{\varphi}\left(\frac{1}{2}\right)\left|F''(\rho)\right|^{q}\right)^{\frac{1}{q}}\Bigg]$$

is valid.

Corollary 4.25. *Let us consider $\varphi(\tau)=\tau$ in Corollary 4.24. Then, we have*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}F(\tau)\,d\tau\right|$$

$$\leq\frac{(\rho-\sigma)^{2}}{96}\left[\left(\frac{\left|F''(\sigma)\right|^{q}+3\left|F''(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|F''(\sigma)\right|^{q}+\left|F''(\rho)\right|^{q}}{4}\right)^{\frac{1}{q}}\right].$$

Corollary 4.26. *If we take $\varphi(\tau)=\frac{\tau^{\alpha}}{\Gamma(\alpha)}$, $\alpha>0$ in Corollary 4.24, then we obtain the following Bulen type inequality for Riemann-Liouville fractional integrals*

$$\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\rho-\sigma)^{\alpha}}\left[J_{\left(\frac{\sigma+\rho}{2}\right)+}^{\alpha}F(\rho)+J_{\left(\frac{\sigma+\rho}{2}\right)-}^{\alpha}F(\sigma)\right]\right|$$

$$\leq\frac{(\rho-\sigma)^{2}}{8}\left(\frac{1}{4}-\frac{1}{(\alpha+1)(\alpha+2)}\right)^{1-\frac{1}{q}}\Bigg[\left(\vartheta_{1}(\alpha)\left|F''(\sigma)\right|^{q}+\vartheta_{2}(\alpha)\left|F''(\rho)\right|^{q}\right)^{\frac{1}{q}}$$

$$+\left(\vartheta_{2}(\alpha)\left|F''(\sigma)\right|^{q}+\vartheta_{1}(\alpha)\left|F''(\rho)\right|^{q}\right)^{\frac{1}{q}}\Bigg].$$

Here,

$$\begin{cases}\vartheta_{1}(\alpha)=\frac{1}{12}-\frac{1}{2(\alpha+1)(\alpha+3)},\\[2mm]\vartheta_{2}(\alpha)=\frac{1}{6}-\frac{\alpha+4}{2(\alpha+1)(\alpha+2)(\alpha+3)}.\end{cases}$$

Corollary 4.27. *In Corollary 4.24, if we use* $\varphi(\tau) = \frac{\tau^{\frac{\alpha}{k}}}{k\Gamma_k(\sigma)}$, $\alpha, k > 0$, *then we obtain the following Bulen type inequality for k-Riemann-Liouville fractional integrals*

$$\begin{aligned}&\left|\frac{1}{2}\left[F\left(\frac{\sigma+\rho}{2}\right)+\frac{F(\sigma)+F(\rho)}{2}\right]-\frac{2^{\frac{\alpha-k}{k}}\Gamma_k(\alpha+k)}{(\rho-\sigma)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)+,k}F(\rho)+J^{\alpha}_{\left(\frac{\sigma+\rho}{2}\right)-,k}F(\sigma)\right]\right| \\ \leq\ & \frac{(\rho-\sigma)^2}{8}\left(\frac{1}{4}-\frac{k^2}{(\alpha+k)(\alpha+2k)}\right)^{1-\frac{1}{q}}\left[\left(\vartheta_1(\alpha,k)\left|F''(\sigma)\right|^q+\vartheta_2(\alpha,k)\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\vartheta_2(\alpha,k)\left|F''(\sigma)\right|^q+\vartheta_1(\alpha,k)\left|F''(\rho)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Here,

$$\begin{cases}\vartheta_1(\alpha,k)=\frac{1}{12}-\frac{k^2}{2(\alpha+k)(\alpha+3k)}, \\ \vartheta_2(\alpha,k)=\frac{1}{6}-\frac{(\alpha+4)k^3}{2(\alpha+k)(\alpha+2k)(\alpha+3k)}.\end{cases}$$

5. Conclusion

In this work, midpoint, trapezoid, Simpson, and Bullen-type inequality for twice differentiable functions using generalized fractional integrals are obtained. Also, we prove that our results generalize the inequalities obtained by Mohammed and Sarikaya [20], Sarikaya and Aktan [25] and Hezenci et al. [15]. Some new inequalities for *k*-Riemann-Liouville are obtained by special choices of main findings. In the future works, authors can try to generalize our results by utilizing some other kinds of convex function classes.

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