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Existence of solution for some φ-Caputo fractional differential inclusions via Wardowski-Mizoguchi-Takahashi multi-valued contractions

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Abstract. In this study, we examine the existence of solution for some φ-Caputo fractional differential inclusions with arbitrary coefficients with boundary values using Wardowski-Mizoguchi-Takahashi multi-valued contractions. Our results utilize some existence results regarding φ-Caputo fractional differential inclusions, in particular the results of Belmor et al. (2021). Our key findings are illustrated with an example.

1. Introduction

Recently, many researchers have been studying the mathematical modelings of some physical phenomenon which appear in some technological fields, for instance, physics, mechanics and chemistry based on fractional integro-differential operators (see, for example [5–8]). The Riemann-Liouville (R-L) and Caputo integro-differential operators are the most famous fractional operators which have been used. For having a great range of investigations of the mathematical models, a new fractional integro-differential operator, namely φ-Caputo fractional derivative was introduced in [4] and used in [9] which means that fractional order derivative with respect to an another strictly increasing differentiable function φ. For any $a, b \in \mathbb{R}$, denote by $E = C([a,b],\mathbb{R})$, the space of all continuous functions $\jmath$ from $[a,b]$ into $\mathbb{R}$ endowed the supremum norm $\|\jmath\| = sup_{y\in[a,b]}|\jmath(y)|$. $L^1([a,b],\mathbb{R})$ be the Banach space of measurable functions $\jmath : [a,b] \to \mathbb{R}$ with the norm $\|\jmath\|_1 = \int_a^b |\jmath(\xi)|d\xi$. $AC([a,b],\mathbb{R})$ stands for the set of absolutely continuous functions from $[a,b]$ into $\mathbb{R}$. We define $AC_g^n([a,b],\mathbb{R})$ by

$$AC_g^n([a,b],\mathbb{R}) = \left\{\jmath : [a,b] \to \mathbb{R}; (\delta_g^{n-1}\jmath)(y) \in AC([a,b],\mathbb{R}), \delta_g = \frac{1}{g'(y)}\frac{d}{dy}\right\}$$

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which is endowed with the norm given by

$$\|\jmath\|_{C_g^n} = \Sigma_{k=0}^{n-1}\|\delta_g^k \jmath(y)\|,$$

where $g \in C^n([a,b],\mathbb{R})$, with $g'(y) > 0$ on $[a,b]$, and $\delta_g^k = \underbrace{\delta_g\delta_g...\delta_g}_{k-times}.$. Recently, Belmor et al. [9] investigated the following fractional differential inclusion (FDI) with respect to an assumed strictly increasing differentiable function g:

$$^cD^{\eta}_{0^+;g}\jmath(\ell) \in \mathfrak{R}(\ell, \jmath(\ell)), \quad \ell \in [0,l], 1 < \eta \leq 2,$$

equipped with the following boudary value conditions:

$$\jmath(0) - \delta_g\jmath(0) = \frac{a}{\Gamma(\theta)}\int_0^p g'(\hbar)(g(p) - g(\hbar))^{\theta-1}\kappa(\hbar, \jmath(\hbar))d\hbar = a\mathcal{I}^{\theta}_{0^+;g}\kappa(p, \jmath(p)),$$

$$\jmath(l) + \delta_g\jmath(l) = \frac{b}{\Gamma(\mu)}\int_0^q g'(\hbar)(g(q) - g(\hbar))^{\mu-1}\chi(\hbar, \jmath(\hbar))d\hbar = b\mathcal{I}^{\mu}_{0^+;g}\chi(q, \jmath(q)),$$

where $^cD^{\eta}_{0^+;g}$ is the g-Caputo fractional derivative presented by Jarad et al. [4], $\mathfrak{R} : [0,l] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map, $\mathcal{P}(\mathbb{R})$ is the collection of nonempty subsets of $\mathbb{R}$, $\mathcal{I}^{z}_{0^+;g}$ stands for g-R-L fractional integral (g-RLFI) of fractional order z on $[0,l]$, $0 < p, q < l$, $\kappa, \chi : [0,l] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\delta_g = \frac{1}{g'(\ell)}\frac{d}{d\ell}$ and a, b are two suitable chosen constants. The authors investigated the solvability of the above mentioned problem by using the endpoint result via φ-weak contractions given by Moradi and Khojasteh [3]. Moreover, in 2005, Echenique [1] began to combine two theories of fixed-point and graph. Consider a directed graph K such that $V(K) = \Lambda$ and the set of its edges $E(K)$ is such that $E(K) \supseteq \Delta$, where $\Delta = \{(\varsigma, \varsigma) : \varsigma \in \Lambda\}$. Also suppose that K possesses no parallel edges. The pair $(V(K), E(K))$ can be used to identify K. The graph K is called a (C)-graph, if for any sequence $\{\varsigma_n\}$ in Λ, that $\varsigma_n \to \varsigma$ and $(\varsigma_n, \varsigma_{n+1}) \in E(K)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{\varsigma_{n_k}\}$ such that $(\varsigma_{n_k}, \varsigma) \in E(K)$ for all $k \in \mathbb{N}$. In this paper, first, we demonstrate the existence of a fixed point for a weakly generalized Wardowski-Mizoguchi-Takahashi multi-valued contraction on graphs. Then, we study the solvability of the following φ-Caputo FDI with arbitrary coefficients and with boundary value conditions via weakly generalized Wardowski-Mizoguchi-Takahashi multi-valued contractions:

$$^cD^{r}_{a^+;g}\jmath(\ell) \in \mathfrak{R}(\ell, \jmath(\ell)), \quad \ell \in [a,b],\ 1 < r \leq 2, \tag{1}$$

$$c_1\jmath(a) + c_2\delta_g\jmath(a) = \mathcal{I}^{\theta}_{a^+;g}\mathcal{K}(p, \jmath(p)), \tag{2}$$

$$c_3\jmath(b) + c_4\delta_g\jmath(b) = \mathcal{I}^{\mu}_{a^+;g}\chi(q, \jmath(q)), \tag{3}$$

where $a < p, q < b$, $0 < \theta, \mu \leq 1$ and $c_i, i = 1, ..., 4$ are some coefficients, $\mathfrak{R} : [a,b]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map and $\mathcal{K}, \chi : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

2. Preliminaries and auxiliary notions

We collect in $\mathfrak{B}$ all functions $\aleph : [0,\infty) \longrightarrow [0,1]$ such that

$$\limsup_{\jmath \longrightarrow t^+} \aleph(\jmath) < 1,$$

for all $t > 0$.

Let (Λ, σ) be a metric space. Following [15], let $\mathbb{CB}(\Lambda)$ be the collection of all nonempty closed bounded subsets of Λ. Let $\mathcal{H}$ be the Hausdorff-Pompieu metric on $\mathbb{CB}(\Lambda)$ generated by the metric σ which is defined by

$$\mathcal{H}(\Re_1, \Re_2) = \max\left\{\sup_{\varpi_1\in\Re_1} \sigma(\varpi_1, \Re_2), \sup_{\varpi_2\in\Re_2} \sigma(\varpi_2, \Re_1)\right\},$$

for every $\Re_1, \Re_2 \in \mathbb{CB}(\Lambda)$.

$\theta \in \Lambda$ is a fixed point of multi-valued mapping $\Re : \Lambda \to \mathcal{P}(\Lambda)$ provided that $\theta \in \Re\theta$.

The following theorem has been proved by Mizoguchi and Takahashi [14]:

Theorem 2.1. *[14] Let (Λ, σ) be a complete metric space (c.m.s.) and let $\Re : \Lambda \to \mathbb{CB}(\Lambda)$ be such that*

$$\mathcal{H}(\Re\jmath, \Re\jmath') \leq \aleph(\sigma(\jmath, \jmath'))\sigma(\jmath, \jmath'),$$

for all $\jmath, \jmath' \in \Lambda$, where $\aleph \in \mathfrak{B}$. Then $\Re$ possesses a fixed point.

Let Q represents the collection of all nondecreasing lower semi-continuous maps $q : [0, \infty) \longrightarrow [0, \infty)$ so that $q(s) = 0$ if and only if $s = 0$ and $\limsup_{\kappa\longrightarrow 0^+} \frac{\kappa}{q(\kappa)} < \infty$.

We say that Λ enjoys the property *(H)* provided that for any increasing sequence $\{\varpi_n\} \subseteq \Lambda$, $\varpi_n \to \varsigma$ as $n \to \infty$ yields that $\varpi_n \preceq \varsigma$ for each $n \geq 0$. Also it is called that $\Re : \Lambda \to \mathcal{P}(\Lambda)$ admits comparable approximative valued property whenever for every $\varsigma \in \Lambda$ there exists $\ell \in \Re\varsigma$ such that $(\varsigma, \ell) \in E(K)$ and $d(\varsigma, \Re\varsigma) = d(\varsigma, \ell)$. Another Theorem 2.1 for single-valued mappings has been studied by Gordji and Ramezani [12].

Theorem 2.2 ([12]). *In a complete ordered metric space $(\Lambda, d, \preceq)$, and for an increasing mapping $\Re : \Lambda \longrightarrow \mathcal{P}(\Lambda)$, let $\varpi_0 \preceq \Re(\varpi_0)$ for some $\varpi_0 \in \Lambda$ and*

$$q(d(\Re\jmath, \Re\jmath')) \leq \aleph(q(d(\jmath, \jmath')))q(d(\jmath, \jmath'))$$

for all comparable elements $\jmath, \jmath' \in \Lambda$ and for some $q \in Q$, where $\aleph \in \mathfrak{B}$. If either $\Re$ is continuous, or, Λ enjoys the property (H), then there is a fixed point of $\Re$.

First, recall some counterproductive definitions of fractional differential equations. For a continuous function $\Re : [0, \infty) \to \mathbb{R}$, the Reimann-Liouville integral (R-L integral) of fractional order r is defined by

$$\mathcal{I}^r_a\Re(t) = \frac{1}{\Gamma(r)}\int_a^t (t - \tau)^{r-1}\Re(\tau)d\tau. \tag{4}$$

The Caputo-derivative of fractional order α is:

$$^cD^\alpha\Re(t) = \frac{1}{\Gamma(n - \alpha)}\int_0^t (t - \tau)^{n-\alpha-1}\Re^{(n)}(\tau)d\tau \quad (n - 1 < \alpha < n, n = [\alpha] + 1), \tag{5}$$

and the R-L derivative of fractional order α is:

$$D^\alpha\Re(t) = \frac{1}{\Gamma(n - \alpha)}(\frac{d}{dt})^n\int_0^t (t - \tau)^{n-\alpha-1}\Re(\tau)d\tau \quad (n - 1 < \alpha < n, n = [\alpha] + 1). \tag{6}$$

Definition 2.3. *For an increasing map g with $g'(s) > 0$ for any $s \in [a, b]$, the g-R-L integral of order r of an integrable function $\Re : [a, b] \to \mathbb{R}$ with respect to g is defined as*

$$\mathcal{I}^r_{a^+;g}\Re(t) = \frac{1}{\Gamma(r)}\int_a^t g'(\hbar)(g(t) - g(\hbar))^{r-1}\Re(\hbar)d\hbar, \tag{7}$$

when the right side of the above equality is evaluated to a limited extent.

If $g(t) = t$, then the g-R-L integral 7 is the standard R-L integral 4.

Definition 2.4. *([4]) Let $n = [r] + 1$. For a real mapping $\mathfrak{R} \in C([a, b], \mathbb{R})$, the g-R-L derivative of fractional order r is formulated as*

$$D^r_{a^+;g}\mathfrak{R}(t) = \frac{1}{\Gamma(n-r)}\Big(\frac{1}{g'(t)}\frac{d}{dt}\Big)^n \int_a^t g'(\hbar)(g(t) - g(\hbar))^{n-r-1}\mathfrak{R}(\hbar)d\hbar, \tag{8}$$

provided that the right side of the above equality is evaluated to a limited extent.

The g-R-L derivative of fractional order r 8 will be the standard R-L derivative 6 if $g(t) = t$. Based on these operators, a new g version of the Caputo derivative has been introduced by Almeida as follows.

Definition 2.5. *([2]) Let $n = [r] + 1$ and $\mathfrak{R} \in A^n_c([a, b], \mathbb{R})$ be an increasing map with $g'(s) > 0$ for any $s \in [a, b]$. The g-Caputo derivative of fractional order r of $\mathfrak{R}$ with respect to g is*

$$^cD^r_{a^+;g}\mathfrak{R}(t) = \frac{1}{\Gamma(n-r)}\Big(\frac{1}{g'(\hbar)}\frac{d}{d\hbar}\Big)^n \int_a^t g'(\hbar)(g(t) - g(\hbar))^{n-r-1}\mathfrak{R}(\hbar)d\hbar, \tag{9}$$

provided the right hand side of equality possesses values finitely.

If $g(s) = s$, then the g-Caputo derivative of fractional order r 9 will be the standard Caputo derivative of fractional order r 29. In the following, some useful specs from the g-Caputo and g-R-L integro-derivative operators are visible.

Let $A_c([0, l], \mathbb{R})$ be the family of absolutely continuous functions from $[0, l]$ into $\mathbb{R}$. Define $A^n_{c_g}([0, l], \mathbb{R})$ by

$$A^n_{c_g}([0, l], \mathbb{R}) = \Big\{w : [0, l] \to \mathbb{R} | \delta^{n-1}_g w \in A_c([0, l], \mathbb{R}), \delta_g = \frac{1}{g'(\ell)}\frac{d}{d\ell}\Big\}.$$

Lemma 2.6. *([4]) Let $n = [r] + 1$. For a real mapping $\mathfrak{R} \in A^n_c([a, b], \mathbb{R})$,*

$$\mathcal{I}^r_{a^+;g}\,{}^cD^r_{a^+;g}\mathfrak{R}(t) = \mathfrak{R}(t) - \Sigma^{n-1}_{k=1}\frac{(\delta^k_g\mathfrak{R})(a)}{k!}(\mathfrak{R}(t) - \mathfrak{R}(a))^k, \tag{10}$$

where $\delta^k_g = \delta_g\delta_g \cdots \delta_g$.

Proposition 2.7. *([2], [4]) Let $n = [r] + 1$. For a real mapping $\mathfrak{R} \in A^n_c([a, b], \mathbb{R})$,*

(i) $^cD^\alpha_{a^+;g}(\mathfrak{R}(t) - \mathfrak{R}(a))^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(\mathfrak{R}(t) - \mathfrak{R}(a))^{\beta-\alpha}, \alpha > 0, \beta > -1,$

(ii) $\mathcal{I}^\alpha_{a^+;g}(\mathfrak{R}(t) - \mathfrak{R}(a))^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(\mathfrak{R}(t) - \mathfrak{R}(a))^{\beta+\alpha}, \alpha > 0, \beta > -1,$

(iii) $^cD^\beta_{a^+;g}(\mathcal{I}^\alpha_{a,g}\mathfrak{R})(t) = \mathcal{I}^{\alpha-\beta}_{a^+;g}\mathfrak{R}(t), 0 < \beta \leq \alpha.$

3. Main results

Let Ξ be the set of all strictly increasing continuous functions $\wp : [0, \infty) \to [-\infty, \infty]$ so that $\wp(s) = 0$ if and only if $s = 1$.

As examples of elements of Ξ:

(i) $\wp_1(\omega) = \begin{cases} \ln(\omega), & \omega \in (0, \infty), \\ -\infty, & \omega = 0, \\ 1, & \omega = \infty, \end{cases}$

(ii) $\wp_2(\omega) = \begin{cases} \ln(\omega) + \omega, & \omega \in (0, \infty), \\ -\infty, & \omega = 0, \\ 1, & \omega = \infty, \end{cases}$

(iii) $\wp_3(\omega) = \begin{cases} -\frac{1}{\sqrt{\omega}} + 1, & \omega \in (0, \infty), \\ -\infty, & \omega = 0, \\ 1, & \omega = \infty, \end{cases}$

(iv) $\wp_4(\omega) = \begin{cases} -\frac{1}{\omega} + 1, & \omega \in (0, \infty), \\ -\infty, & \omega = 0, \\ 1, & \omega = \infty. \end{cases}$

Definition 3.1. *In a metric space (Λ, d), assume that K is a directed graph on Λ and $\mathfrak{R} : \Lambda \to \mathcal{P}(\Lambda)$ be a multivalued mapping. $\mathfrak{R}$ is called a weakly generalized Wardowski-Mizoguchi-Takahashi multi-valued contraction if there exist $\wp \in \Xi$ and $\aleph \in \mathfrak{B}$ such that*

$$\wp(H(\mathfrak{R}\varsigma, \mathfrak{R}\ell))) \leq \wp(\aleph(d(\varsigma, \ell))) + \wp(M(\varsigma, \ell)) \tag{11}$$

for all $\varsigma, \ell \in \Lambda$ with $(\varsigma, \ell) \in E(K)$ and

$$M(\varsigma, \ell) = \max\{d(\varsigma, \ell), d(\varsigma, \mathfrak{R}\varsigma), d(\ell, \mathfrak{R}\ell), \frac{1}{2}[d(\varsigma, \mathfrak{R}\ell) + d(\ell, \mathfrak{R}\varsigma)]\}.$$

Theorem 3.2. *In a c.m.s. (Λ, d) and for a directed graph K on Λ, assume that $\mathfrak{R} : \Lambda \to \mathcal{P}(\Lambda)$ be a weakly generalized Wardowski-Mizoguchi-Takahashi multi-valued contraction satisfying comparable approximate valued property. If K be a (C)-graph, then $\mathfrak{R}$ possesses a fixed point.*

Proof. Choose a fixed element $\varpi_0 \in \Lambda$. If $\varpi_0 \in \mathfrak{R}\varpi_0$, then we have nothing to prove. Suppose that $\varpi_0 \notin \mathfrak{R}\varpi_0$. Since $\mathfrak{R}$ admits comparable approximative valued property, there exists $\varpi_1 \in \mathfrak{R}\varpi_0$ such that $(\varpi_0, \varpi_1) \in E(K)$ and $d(\varpi_0, \mathfrak{R}\varpi_0) = d(\varpi_0, \varpi_1)$. It is clear that $\varpi_1 \neq \varpi_0$. If $\varpi_1 \in \mathfrak{R}\varpi_1$, then ϖ_1 is a fixed point of $\mathfrak{R}$. Suppose that $\varpi_1 \notin \mathfrak{R}\varpi_1$. Then, there exists $\varpi_2 \in \mathfrak{R}\varpi_1$ such that $(\varpi_1, \varpi_2) \in E(K)$ and $d(\varpi_1, \mathfrak{R}\varpi_1) = d(\varpi_1, \varpi_2)$. It is clear that $\varpi_2 \neq \varpi_1$. According to this process, we will have a sequence $\{\varpi_n\}$ in Λ such that $\varpi_n \in \mathfrak{R}\varpi_{n-1}$, $(\varpi_{n-1}, \varpi_n) \in E(K)$, $\varpi_n \neq \varpi_{n-1}$ and $d(\varpi_{n-1}, \varpi_n) = d(\varpi_{n-1}, \mathfrak{R}\varpi_{n-1})$ for all $n \in \mathbb{N}$.

In view of (29), we obtain that

$$\begin{aligned} \wp(d(\varpi_{n+1}, \varpi_{n+2})) = \wp(d(\varpi_{n+1}, \mathfrak{R}\varpi_{n+1})) &= \wp(H(\mathfrak{R}\varpi_n, \mathfrak{R}\varpi_{n+1}))) \\ &\leq \wp(\aleph(d(\varpi_n, \varpi_{n+1}))) + \wp(M(\varpi_n, \varpi_{n+1})), \end{aligned}$$

where

$$\begin{aligned} M(\varpi_n, \varpi_{n+1}) = \max\{&d(\varpi_n, \varpi_{n+1}), d(\varpi_n, \mathfrak{R}\varpi_n), d(\varpi_{n+1}, \mathfrak{R}\varpi_{n+1}), \\ &\frac{1}{2}[d(\varpi_n, \mathfrak{R}\varpi_{n+1}) + d(\varpi_{n+1}, \mathfrak{R}\varpi_n)]\} \leq \max\{d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+2})\}. \end{aligned}$$

If

$$\max\{d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+2})\} = d(\varpi_{n+1}, \varpi_{n+2}),$$

then

$$\wp(d(\varpi_{n+1}, \varpi_{n+2})) \leq \wp(\aleph(d(\varpi_n, \varpi_{n+1}))) + \wp(d(\varpi_{n+1}, \varpi_{n+2})) < \wp(d(\varpi_{n+1}, \varpi_{n+2}))$$

which is a contradiction. Thus

$$\max\{d(\varpi_n, \varpi_{n+1}), d(\varpi_{n+1}, \varpi_{n+2})\} = d(\varpi_n, \varpi_{n+1}).$$

Therefore,

$$\wp(d(\varpi_{n+1}, \varpi_{n+2})) \leq \wp(\aleph(d(\varpi_n, \varpi_{n+1}))) + \wp(d(\varpi_n, \varpi_{n+1})) \tag{12}$$

for each $n \geq 0$. Put $t_n := d(\varpi_n, \varpi_{n+1})$. From (12), we have

$$\wp(t_{n+1}) \leq \wp(\aleph(t_n)) + \wp(t_n), \quad for\ each\ n \geq 0. \tag{13}$$

Since $\aleph(t_n) < 1$ and $\wp$ is strictly increasing, we get $\wp(\aleph(t_n) < \wp(1) = 0$. Therefore, from (13), we have

$$\wp(t_{n+1}) \leq \wp(\aleph(t_n)) + \wp(t_n) < \wp(t_n), \quad for\ each\ n \geq 0. \tag{14}$$

Since $\wp$ is strictly increasing, $t_{n+1} < t_n$ and subsequently, for some $r \geq 0$, $t_n \to r^+$. Now, we illustrate that $r = 0$. Suppose to the contrary that $r > 0$. Passing to the limit throw (14), $\wp(r) \leq \wp(\limsup_{n\to\infty}(\aleph(t_n))) + \wp(r) < \wp(r)$, which is a contradiction. Accordingly, $\lim\limits_{n\to\infty} t_n = r = 0$. We shall show that $\{\varpi_n\}$ is Cauchy. If $\{\varpi_n\}$ is not Cauchy, then for some $\varepsilon > 0$ and for subsequences $\{\varpi_{m_i}\}$ and $\{\varpi_{n_i}\}$ of $\{\varpi_n\}$ one has

$$n_i > m_i > i,\ d(\varpi_{m_i}, \varpi_{n_i}) \geq \varepsilon \tag{15}$$

and

$$d(\varpi_{m_i}, \varpi_{n_i-1}) < \varepsilon. \tag{16}$$

Using (15), we get

$$\varepsilon \leq d(\varpi_{m_i}, \varpi_{n_i}) \leq d(\varpi_{m_i}, \varpi_{n_i-1}) + d(\varpi_{n_i-1}, \varpi_{n_i}) < \varepsilon + d(\varpi_{n_i-1}, \varpi_{n_i}). \tag{17}$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} d(\varpi_{m_i}, \varpi_{n_i}) = \varepsilon. \tag{18}$$

Also, we have

$$\begin{aligned} &d(\varpi_{m_i}, \varpi_{n_i}) - d(\varpi_{m_i}, \varpi_{m_i+1}) - d(\varpi_{n_i}, \varpi_{n_i+1}) \\ &\leq d(\varpi_{m_i+1}, \varpi_{n_i+1}) \\ &\leq d(\varpi_{m_i}, \varpi_{m_i+1}) + d(\varpi_{m_i}, \varpi_{n_i}) + d(\varpi_{n_i}, \varpi_{n_i+1}). \end{aligned}$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} d(\varpi_{m_i+1}, \varpi_{n_i+1}) = \varepsilon. \tag{19}$$

Also,

$$d(\varpi_{m_i+1}, \varpi_{n_i+1})) \leq d(\varpi_{m_i+1}, \mathfrak{R}\varpi_{m_i})) + H(\mathfrak{R}\varpi_{m_i}, \mathfrak{R}\varpi_{n_i}) = H(\mathfrak{R}\varpi_{m_i}, \mathfrak{R}\varpi_{n_i}).$$

By (29), we find

$$\begin{aligned} \wp(d(\varpi_{m_i+1}, \varpi_{n_i+1})) &\leq \wp(H(\mathfrak{R}\varpi_{m_i}, \mathfrak{R}\varpi_{n_i})) \\ &\leq \wp(\aleph(d(\varpi_{m_i}, \varpi_{n_i})) + \wp(M(\varpi_{m_i}, \varpi_{n_i})). \end{aligned} \tag{20}$$

On the other hand,

$$\begin{aligned} &d(\varpi_{m_i}, \varpi_{n_i}) \leq M(\varpi_{m_i}, \varpi_{n_i}) \\ &\leq \max\{d(\varpi_{m_i}, \varpi_{n_i}), d(\varpi_{m_i}, \varpi_{m_i+1}), d(\varpi_{n_i}, \varpi_{n_i+1}), \frac{1}{2}[d(\varpi_{n_i}, \varpi_{m_i+1}) + d(\varpi_{m_i}, \varpi_{n_i+1})]\} \\ &\leq d(\varpi_{m_i}, \varpi_{n_i}) + d(\varpi_{m_i}, \varpi_{m_i+1}) + d(\varpi_{n_i}, \varpi_{n_i+1}). \end{aligned}$$

As $i \to \infty$, we find

$$\lim_{i\to\infty} M(\varpi_{m_i}, \varpi_{n_i}) = \varepsilon.$$

Taking limit in both sides of (20),

$$\wp(\varepsilon) \leq \wp(\limsup_{i\to\infty} \aleph(d(\varpi_{m_i}, \varpi_{n_i})) + \wp(\varepsilon). \tag{21}$$

Since $d(\varpi_{m_i}, \varpi_{n_i}) \to \varepsilon^+$, thus $\limsup_{i\to\infty} \aleph(d(\varpi_{m_i}, \varpi_{n_i})) < 1$. Therefore,

$$\wp(\limsup_{i\to\infty} \aleph(d(\varpi_{m_i}, \varpi_{n_i})) < 0.$$

Thus (21) leads to $\wp(\varepsilon) < \wp(\varepsilon)$, a contradiction.

Consequently, $\{\varpi_n\}$ is a Cauchy sequence in the c.m.s. (Λ, d). Hereafter, there is $z \in \Lambda$ so that

$$\lim_{n\longrightarrow\infty} \varpi_n = z. \tag{22}$$

We claim that $d(u, \Re u) = 0$. Suppose to the contrary that $d(u, \Re u) \neq 0$.

Since K is a (C)-graph, there exists a subsequence $\{\varsigma_{n_k}\}$ such that $(\varsigma_{n_k}, \varsigma) \in E(K)$ for all $k \in \mathbb{N}$.

There are two cases as follows:

Case (i): $\Re\varpi_{n_k} \neq \Re z$ for each $k \geq N$ where $N \in \mathbb{N}$.
Case (ii): $\Re\varpi_{n_i} = \Re z$ for each $i \geq 0$ where $\{\varpi_{n_i}\}$ is a subsequence of $\{\varpi_{n_k}\}$.
In the case (i), we have

$$\begin{aligned} \wp(d(\varpi_{n_k+1}, \Re z)) &\leq \wp(H(\Re\varpi_{n_k}, \Re z)) \\ &\leq \wp(\aleph(d(\varpi_{n_k}, z))) + \wp(M(\varpi_{n_k}, z)). \end{aligned} \tag{23}$$

Also,

$$\lim_{n\longrightarrow\infty} M(\varpi_{n_k}, z) = \lim_{n\longrightarrow\infty} \max\{d(\varpi_{n_k}, z), d(\varpi_{n_k}, \varpi_{n_k+1}), d(z, \Re z), \frac{1}{2}[d(\varpi_{n_k+1}, z) + d(\varpi_{n_k}, \Re z)]\} = d(z, \Re z).$$

Passing to the limit throw (23), we obtain that $\wp(d(z, \Re z)) < \wp(d(z, \Re z))$ which is a contradiction. Thus, $d(z, \Re z) = 0$.

In the case (ii),

$$d(z, \Re z) = \lim_{i\to\infty} d(\varpi_{n_i+1}, \Re z) = \lim_{i\to\infty} H(\Re\varpi_{n_i}, \Re z) = 0.$$

So, $d(z, \Re z) = 0$. Therefore in all cases, we have $d(z, \Re z) = 0$. Now since K admits comparable approximate valued property, there exists $u \in \Lambda$ such that $u \in \Re z$, $(z, u) \in E(K)$ and $d(z, u) = d(z, \Re z)$. Consequently, $d(z, u) = 0$ and so $z = u \in \Re z$. The proof is completed. □

We gather all nonempty compact subsets of Λ in $\mathcal{P}_{cp}(\Lambda)$.

Corollary 3.3. *In a c.m.s. (Λ, d), and for a directed graph K on Λ, assume that $\Re : \Lambda \to \mathcal{P}_{cp}(\Lambda)$ be a weakly generalized Wardowski-Mizoguchi-Takahashi multi-valued contraction. Moreover, assume that Graph$(\Re) = \{(\varsigma, \ell) : \ell \in \Re\varsigma\} \subseteq E(K)$. If K be a (C)-graph, then $\Re$ admits a fixed point.*

We are now ready to present and demonstrate the key outcomes of this study. From now on, assume that $\Lambda = C([a, b], \mathbb{R})$ is the Banach space of all continuous functions from $[a, b]$ to $\mathbb{R}$ with the supremum norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in [a, b]\}.$$

In [11] we have the subsequent supplementary lemmas:

Lemma 3.4. *[11] Let $\vartheta, \rho_1, \rho_2$ be real continuous functions on $[a,b]$, $1 < r \leq 2$, $0 < \theta, \mu \leq 2$, $p, q \in [a,b]$ and c_i $(i = 1,2,3,4)$ are some constants. Then $\jmath \in A_{c_g}^2([a,b],\mathbb{R})$ is a solution of the following fractional boundary value problem*

$$\begin{cases} {}^cD^r_{a^+;g}\jmath(\ell) = \vartheta(\ell), \quad \ell \in [a,b], 1 < r \leq 2, \\ c_1\jmath(a) + c_2\delta_g\jmath(a) = \mathcal{I}^{\theta}_{a^+;g}\rho_1(p), \\ c_3\jmath(b) + c_4\delta_g\jmath(b) = \mathcal{I}^{\mu}_{a^+;g}\rho_2(q), \end{cases} \tag{24}$$

if and only if $\jmath$ be faitful in the following fractional order integral equation

$$\jmath(\ell) = L_\jmath(\ell) + \int_a^b K_g(\ell,\hbar)\vartheta(\hbar)d\hbar, \tag{25}$$

where

$$K_g(\ell,\hbar) = g'(\hbar)\begin{cases} \frac{(g(\ell)-g(\hbar))^{r-1}}{\Gamma(r)} + \frac{c_3(-c_1(g(\ell)-g(a))+c_2)}{\mathcal{G}\Gamma(r)}(g(b)-g(\hbar))^{r-1} \\ +\frac{c_4(-c_1(g(\ell)-g(a))+c_2)}{\mathcal{G}\Gamma(r-1)}(g(b)-g(\hbar))^{r-2}; \ a \leq \hbar \leq \ell, \\ \frac{c_3(-c_1(g(\ell)-g(a))+c_2)}{\mathcal{G}\Gamma(r)}(g(b)-g(\hbar))^{r-1} \\ +\frac{c_4(-c_1(g(\ell)-g(a))+c_2)}{\mathcal{G}\Gamma(r-1)}(g(b)-g(\hbar))^{r-2}; \ \ell \leq \hbar \leq b \end{cases}$$

and

$$L_\jmath(\ell) = \frac{1}{\mathcal{G}}\Big\{\Big[c_3(g(\ell)-g(a)) + c_4\Big]\mathcal{I}^{\theta}_{a^+;g}\rho_1(p) + \Big[c_1(g(\ell)-g(a)) - c_2\Big]\mathcal{I}^{\mu}_{a^+;g}\rho_2(q)\Big\},$$

with

$$\mathcal{G} = c_1c_3(g(b)-g(a)) + det(C),$$

$$C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}.$$

Lemma 3.5. *[11] Take*

$$\tilde{K}_g = \sup_{\ell\in[a,b]}\int_a^b |K_g(\ell,\hbar)|d\hbar$$

and

$$M_g = \frac{(g(b)-g(a))^{r-1}}{\Gamma(r)}\Big\{\frac{g(b)-g(a)}{r} + \Big[\frac{|c_3|}{r}(g(b)-g(a)) + |c_4|\Big]\frac{C_{1,2}}{|\mathcal{G}|}\Big\},$$

where

$$C_{i,j} = |c_i|(g(b)-g(a)) + |c_j|; \quad i,j \in \{1,2,3,4\}.$$

Then $\tilde{K}_g \leq M_g$.

Definition 3.6. *A function $\jmath \in A_{c_g}^2([a,b],\mathbb{R})$ is a solution of the inclusion problem (1) provided that for some function $\rho \in L^1([a,b],\mathbb{R})$ with $\rho(\ell) \in \mathfrak{R}(\ell,\jmath(\ell))$, $\jmath$ satisfies the conditions (2), (3) and ${}^cD^r_{a^+;g}\jmath(\ell) = \rho(\ell)$, a.e $\ell \in [a,b]$, $1 < r \leq 2$, where $g \in C_g^2([a,b],\mathbb{R})$ with $g' > 0$ on $[a,b]$.*

For any $\jmath \in E$, define

$$S_{\mathfrak{R},\jmath} = \Big\{\rho \in L^1([a,b],\mathbb{R}) : \rho(\ell) \in \mathfrak{R}(\ell,\jmath(\ell)) \ a.e \ \ell \in [a,b]\Big\}$$

and the operator $K : E \to \mathcal{P}_{cp}(E)$ associated with the problem (1)-(3) by

$$K(\jmath) = \Big\{ f \in E : f(\ell) = L_\jmath(\ell) + \int_a^b K_g(\ell,\hbar)\varrho(\hbar)d\hbar,\ \varrho \in S_{\mathfrak{R},\jmath} \Big\}, \tag{26}$$

where

$$L_\jmath(\ell) = \frac{1}{\mathcal{G}}\Big\{\Big[c_3(g(\ell)-g(a)) + c_4\Big]\mathcal{I}^{\theta}_{a^+;g}\mathcal{K}(p,\jmath(p)) + \Big[c_1(g(\ell)-g(a)) - c_2\Big]\mathcal{I}^{\mu}_{a^+;g}\chi(q,\jmath(q))\Big\}.$$

Theorem 3.7. *Suppose that*

(i) $\mathfrak{R} : [a,b]\times\mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$, $\mathcal{K}, \chi : [a,b]\times\mathbb{R}\to\mathbb{R}$ *are continuous functions.*

(ii) there are functions $\wp \in \Xi$ *and* $\aleph \in \mathfrak{B}$ *satisfying*

$$H(\mathfrak{R}(\ell,u),\mathfrak{R}(\ell,v)) \leq \frac{\alpha}{M_g}\wp^{-1}\Big(\wp(\aleph(|u-v|)) + \wp(|u-v|)\Big),$$

$$|\mathcal{K}(\ell,u) - \mathcal{K}(\ell,v)| \leq \frac{\beta|\mathcal{G}|\Gamma(\theta+1)}{C_{3,4}(g(p)-g(a))^{\theta}}\wp^{-1}\Big(\wp(\aleph(|u-v|)) + \wp(|u-v|)\Big),$$

$$|\chi(\ell,u) - \chi(\ell,v)| \leq \frac{\gamma|\mathcal{G}|\Gamma(\mu+1)}{C_{1,2}(g(q)-g(a))^{\mu}}\wp^{-1}\Big(\wp(\aleph(|u-v|)) + \wp(|u-v|)\Big),$$

for all $\ell \in [a,b]$ *and* $u, v \in \mathbb{R}$ *with* $u \neq v$, *where* $\alpha,\beta,\gamma \geq 0$ *and* $\alpha+\beta+\gamma \leq 1$.

Then, the problem 1 admits a solution.

Proof. We shall show that the operator $K : E \to \mathcal{P}_{cp}(E)$ defined in 26 admits at least one fixed point. We prove that K is a weakly generalized Wardowski-Mizoguchi-Takahashi multi-valued contraction, i.e.

$$\wp(H(K\jmath_1, K\jmath_2)) \leq \wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|) \tag{27}$$

for all $\jmath_1,\jmath_2 \in A^2_{c_g}([a,b],\mathbb{R})$. Now let $\jmath_1,\jmath_2 \in A^2_{c_g}([a,b],\underline{\mathbb{R}})$ and $f_1 \in K\jmath_1$. Then there exists $\varrho_1 \in S_{\mathfrak{R},\jmath_1}$ such that for any $\ell \in [a,b]$, $f_1(\ell) = L_{\jmath_1}(\ell) + \int_a^b K_g(\ell,\hbar)\varrho_1(\hbar)d\hbar$. By hypothesis (ii) we have

$$H(\mathfrak{R}(\ell,\jmath_1(\ell)),\mathfrak{R}(\ell,\jmath_2(\ell))) \leq \frac{\alpha}{M_g}\wp^{-1}\Big(\wp(\aleph(|\jmath_1(\ell)-\jmath_2(\ell)|)) + \wp(|\jmath_1(\ell)-\jmath_2(\ell)|)\Big).$$

Then there exists $z \in \mathfrak{R}(\ell,\jmath_2(\ell))$ such that

$$|\varrho_1(\ell) - z| \leq \frac{\alpha}{M_g}\wp^{-1}\Big(\wp(\aleph(|\jmath_1(\ell)-\jmath_2(\ell)|)) + \wp(|\jmath_1(\ell)-\jmath_2(\ell)|)\Big), \ell \in [a,b].$$

Define $U : [a,b] \to \mathcal{P}(\mathbb{R})$ by

$$U(\ell) = \Big\{ z \in \mathbb{R} : |\jmath_1(\ell) - z| \leq \frac{\alpha}{M_g}\wp^{-1}\Big(\wp(\aleph(|\jmath_1(\ell)-\jmath_2(\ell)|)) + \wp(|\jmath_1(\ell)-\jmath_2(\ell)|)\Big)\Big\}.$$

Measurability of $U(\ell) \cap \mathfrak{R}(\ell,\jmath_2(\ell))$ unable us to find a measurable selection $\varrho_2(\ell)$ for $U(\ell)\cap\mathfrak{R}(\ell,\jmath_2(\ell))$. Thus, $\varrho_2 \in L^1([a,b],\mathbb{R})$, $\varrho_2 \in \mathfrak{R}(\ell,\jmath_2(\ell))$ and

$$|\varrho_1(\ell) - z| \leq \frac{\alpha}{M_g}\wp^{-1}\Big(\wp(\aleph(|\jmath_1(\ell)-\jmath_2(\ell)|)) + \wp(|\jmath_1(\ell)-\jmath_2(\ell)|)\Big), \ell \in [a,b].$$

We define $f_2(\ell) = L_{\jmath_2}(\ell) + \int_a^b K_g(\ell,\hbar)\varrho_2(\hbar)d\hbar$, $\ell \in [a,b]$. Then for each $\ell \in [a,b]$,

$$\begin{aligned}
&\left|f_1(\ell) - f_2(\ell)\right| \le \left|L_{\jmath_1}(\ell) - L_{\jmath_2}(\ell)\right| + \int_a^b |K_g(\ell,\hbar)|\left|\mathfrak{R}(\ell,\jmath_1(\hbar)) - \mathfrak{R}(\ell,\jmath_2(\hbar))\right|d\hbar \\
&\le \frac{1}{|\mathcal{G}|}\Big[|c_3|(g(b)-g(a)) + |c_4|\Big]\left|\mathcal{I}^{\theta}_{a^+;g}\mathcal{K}(p,\jmath_1(p)) - \mathcal{I}^{\theta}_{a^+;g}\mathcal{K}(p,\jmath_2(p))\right| \\
&+\frac{1}{|\mathcal{G}|}\Big[|c_1|(g(b)-g(a)) + |c_2|\Big]\left|\mathcal{I}^{\mu}_{a^+;g}\chi(q,\jmath_1(q)) - \mathcal{I}^{\mu}_{a^+;g}\chi(q,\jmath_2(q))\right| \\
&+ \int_a^b |K_g(\ell,\hbar)|\left|\mathfrak{R}(\ell,\jmath_1(\hbar)) - \mathfrak{R}(\ell,\jmath_2(\hbar))\right|d\hbar \\
&= \frac{C_{3,4}}{|\mathcal{G}|}\left|\mathcal{I}^{\theta}_{a^+;g}\mathcal{K}(p,\jmath_1(p)) - \mathcal{I}^{\theta}_{a^+;g}\mathcal{K}(p,\jmath_2(p))\right| \\
&+\frac{C_{1,2}}{|\mathcal{G}|}\left|\mathcal{I}^{\mu}_{a^+;g}\chi(q,\jmath_1(q)) - \mathcal{I}^{\mu}_{a^+;g}\chi(q,\jmath_2(q))\right| \\
&+ \int_a^b |K_g(\ell,\hbar)|\left|\mathfrak{R}(\ell,\jmath_1(\hbar)) - \mathfrak{R}(\ell,\jmath_2(\hbar))\right|d\hbar \\
&\le \frac{C_{3,4}}{|\mathcal{G}|}\frac{1}{\Gamma(\theta)}\int_a^p g'(\hbar)(g(p)-g(\hbar))^{\theta-1}\left|\mathcal{K}(\hbar,\jmath_1(\hbar)) - \mathcal{K}(\hbar,\jmath_2(\hbar))\right|d\hbar \\
&+\frac{C_{1,2}}{|\mathcal{G}|}\frac{1}{\Gamma(\mu)}\int_a^q g'(\hbar)(g(q)-g(\hbar))^{\mu-1}\left|\chi(\hbar,\jmath_1(\hbar)) - \chi(\hbar,\jmath_2(\hbar))\right|d\hbar \\
&+ \int_a^b |K_g(\ell,\hbar)|\left|\mathfrak{R}(\ell,\jmath_1(\hbar)) - \mathfrak{R}(\ell,\jmath_2(\hbar))\right|d\hbar \\
&\le \frac{C_{3,4}}{|\mathcal{G}|}\frac{1}{\Gamma(\theta)}\frac{\aleph|\mathcal{G}|\Gamma(\theta+1)}{C_{3,4}(g(p)-g(a))^{\theta}}\wp^{-1}\left\{\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)\right\}\int_a^p g'(\hbar)(g(p)-g(\hbar))^{\theta-1}d\hbar \\
&+\frac{C_{1,2}}{|\mathcal{G}|}\frac{1}{\Gamma(\mu)}\frac{\gamma|\mathcal{G}|\Gamma(\mu+1)}{C_{1,2}(g(q)-g(a))^{\mu}}\wp^{-1}\left\{\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)\right\}\int_a^q g'(\hbar)(g(q)-g(\hbar))^{\mu-1}d\hbar \\
&+\frac{\alpha}{M_g}\wp^{-1}\left\{\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)\right\}\int_a^b |K_g(\ell,\hbar)|d\hbar \\
&\le \aleph\wp^{-1}\left\{\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)\right\} \\
&+\gamma\wp^{-1}\left\{\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)\right\} \\
&+\alpha\wp^{-1}\left\{\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)\right\} \\
&= (\alpha+\beta+\gamma)\wp^{-1}\left\{\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)\right\} \\
&\le \wp^{-1}\left\{\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)\right\}.
\end{aligned} \tag{28}$$

Therefore

$$\|f_1 - f_2\| \le \wp^{-1}[\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)]$$

and so

$$H(K\jmath_1, K\jmath_2) \le \wp^{-1}[\wp(\aleph(\|\jmath_1-\jmath_2\|)) + \wp(\|\jmath_1-\jmath_2\|)] \tag{29}$$

which yields that

$$\wp(\|\mathfrak{R}\jmath_1 - \mathfrak{R}\jmath_2\|) \le \wp(\aleph(\|\jmath_1-\jmath_2\|) + \wp(\|\jmath_1-\jmath_2\|).$$

Therefore, by Corollary 3.3 , $\mathfrak{R}$ possesses a fixed point and so the problem 1 possesses a solution in $A^2_{c_g}([a,b],\mathbb{R})$. □

Example 3.8. *Consider the differential inclusion of fractional order*

$$\begin{cases} {}^cD^{1.5}_{1^+;g}\jmath(\ell) \in [0, \frac{9}{160}\sqrt{\frac{\pi}{2}}\frac{2|\jmath(\ell)|}{6+3|\jmath(\ell)|}], \quad \ell \in [1,2], g(\ell) = 2\ell, \\ \jmath(1) + 2\delta_g\jmath(1) = \mathcal{I}^{\frac{1}{2}}_{1^+;g}\mathcal{K}(\frac{4}{3},\jmath(\frac{4}{3})),\ \mathcal{K}(\hbar,u) = \frac{1}{14}\sqrt{\frac{3\pi}{2}}\frac{e^{-\hbar}sin(\hbar^2+1)|u|}{1+\frac{1}{2}|u|} \\ 2\jmath(2) - 3\delta_g\jmath(2) = \mathcal{I}^{\frac{3}{2}}_{1^+;g}\chi(\frac{5}{3},\jmath(\frac{5}{3})),\ \chi(\hbar,u) = \frac{9\sqrt{2\pi}}{128}\frac{e^{-\hbar}cos(3\hbar+1)|u|}{1+\frac{1}{2}|u|}. \end{cases} \tag{30}$$

Note that,

$$\mathfrak{R}(\hbar, u) = [0, \frac{9}{160}\sqrt{\frac{\pi}{2}}\frac{2|u|}{6+3|u|}].$$

Here, $r = 1.5$, $c_1 = 1$, $c_2 = 2$, $c_3 = 2$, $c_4 = -3$. $a = 1$, $b = 2$, $p = \frac{4}{3}$, $q = \frac{5}{3}$, $\theta = \frac{1}{2}$, *and* $\mu = \frac{3}{2}$. *Thus*

$$C = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$$

and

$$\mathcal{G} = c_1c_3(g(b) - g(a)) + det(C) = (1)(2)(2b - 2a) - 7 = -3,$$

$$C_{1,2} = |c_1|(g(b) - g(a)) + |c_2| = (2b - 2a) + 2 = 4,$$

$$C_{3,4} = |c_3|(g(b) - g(a)) + |c_4| = 2(2b - 2a) + 3 = 7.$$

$$M_g = \frac{(2)^{.5}}{\Gamma(1.5)}\Big\{\frac{2}{1.5} + \Big[\frac{2}{1.5}(2) + 3\Big]\frac{4}{3}\Big\} = \frac{\sqrt{2}}{\frac{1}{2}\sqrt{\pi}}\{\frac{80}{9}\} = 2\sqrt{\frac{2}{\pi}}\frac{80}{9} = \frac{160}{9}\sqrt{\frac{2}{\pi}}.$$

Take $\alpha = \beta = \gamma = \frac{1}{3}$. *Then*

$$\frac{\beta|\mathcal{G}|\Gamma(\theta+1)}{C_{3,4}(g(p) - g(a))^{\theta}} = \frac{(\frac{1}{3})(3)\Gamma(\frac{3}{2})}{7(\frac{8}{3}-2)^{\frac{1}{2}}} = \frac{\frac{1}{2}\sqrt{\pi}}{7(\frac{2}{3})^{\frac{1}{2}}} = \frac{1}{14}\sqrt{\frac{3\pi}{2}},$$

$$\frac{\gamma|\mathcal{G}|\Gamma(\mu+1)}{C_{1,2}(g(q) - g(a))^{\mu}} = \frac{(\frac{1}{3})(3)\Gamma(\frac{5}{2})}{4(\frac{10}{3}-2)^{\frac{3}{2}}} = \frac{\frac{3}{4}\sqrt{\pi}}{4(\frac{4}{3})^{\frac{3}{2}}} = \frac{9}{128}\sqrt{2\pi}$$

Now, for any $\hbar \in [a, b] = [1, 2]$ *and* $u, v \in \mathbb{R}$, *we have*

$$\begin{aligned} H(\mathfrak{R}(\hbar, u) - \mathfrak{R}(\hbar, v)) &= \frac{9}{160}\sqrt{\frac{\pi}{2}}\Big|\frac{2|u|}{6+3|u|} - \frac{2|v|}{6+3|v|}\Big| \\ &= \frac{1}{3}\frac{9}{160}\sqrt{\frac{\pi}{2}}\Big|\frac{|u|}{1+\frac{1}{2}|u|} - \frac{|v|}{1+\frac{1}{2}|v|}\Big| \\ &\leq \frac{1}{3}\frac{9}{160}\sqrt{\frac{\pi}{2}}\frac{||u| - |v||}{(1+\frac{1}{2}|u|)(1+\frac{1}{2}|v|)} \\ &\leq \frac{1}{3}\frac{9}{160}\sqrt{\frac{\pi}{2}}\frac{||u| - |v||}{1+\frac{1}{2}(||u| - |v||)} \\ &\leq \frac{1}{3}\frac{9}{160}\sqrt{\frac{\pi}{2}}\frac{|u - v|}{1+\frac{1}{2}|u - v|} \\ &= \frac{\alpha}{M_g}\wp^{-1}\Big(\wp(\aleph(|u - v|)) + \wp(|u - v|)\Big), \end{aligned}$$

where $\aleph(t) = \frac{2}{3}$ *and* $\Gamma(t) = \begin{cases} \frac{-1}{t} + 1, & t \in (0, \infty), \\ -\infty, & t = 0, \\ 1, & t = \infty. \end{cases}$

On the other hand,

$$\begin{aligned}
\Big|\mathcal{K}(\hbar,u)-\mathcal{K}(\hbar,v)\Big| &\leq \frac{1}{14}\sqrt{\frac{3\pi}{2}}\Big|\frac{|u|}{1+\frac{1}{2}|u|}-\frac{|v|}{1+\frac{1}{2}|v|}\Big| \\
&\leq \frac{1}{14}\sqrt{\frac{3\pi}{2}}\frac{||u|-|v||}{1+\frac{1}{2}(||u|-|v||)} \\
&\leq \frac{1}{14}\sqrt{\frac{3\pi}{2}}\frac{|u-v|}{1+\frac{1}{2}|u-v|} \\
&= \frac{\beta|\mathcal{G}|\Gamma(\theta+1)}{C_{3,4}(g(p)-g(a))^{\theta}}\wp^{-1}\Big(\wp(\aleph(|u-v|))+\wp(|u-v|)\Big),
\end{aligned}$$

and

$$\begin{aligned}
\Big|\chi(\hbar,u)-\chi(\hbar,v)\Big| &\leq \frac{9\sqrt{2\pi}}{128}\Big|\frac{|u|}{1+\frac{1}{2}|u|}-\frac{|v|}{1+\frac{1}{2}|v|}\Big| \\
&\leq \frac{9\sqrt{2\pi}}{128}\frac{||u|-|v||}{1+\frac{1}{2}(||u|-|v||)} \\
&\leq \frac{9\sqrt{2\pi}}{128}\frac{|u-v|}{1+\frac{1}{2}|u-v|} \\
&= \frac{\gamma|\mathcal{G}|\Gamma(\mu+1)}{C_{1,2}(g(q)-g(a))^{\mu}}\wp^{-1}\Big(\wp(\aleph(|u-v|))+\wp(|u-v|)\Big).
\end{aligned}$$

Also, $\alpha+\beta+\gamma=1$. Therefore, all the conditions of Theorem 3.7 are fulfilled. Thus, by this theorem the problem 30 possesses a unique solution.

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