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Semi-invariant conformal submersions with horizontal Reeb vector field

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Abstract. The present paper deals with the characterization of a new submersion named semi-invariant conformal submersions with horizontal Reeb vector field from almost contact metric manifolds onto Riemannian metric manifolds which is the generalization of some known submersions on Riemannian metric manifolds. We give important and adequate conditions for such submersions to be totally geodesic and harmonic. Also, a few examples are examined for such submersions endowed with horizontal Reeb vector field.

1. Introduction

The development of Riemannian metric manifolds along with sectional non-negative curvature is an iconic problem in Riemannian geometry. In this regard, the initial studies (O'Neill [26] and Gray [13]) connect Riemannian metric manifolds to the Riemannian submersions. The various theories are available in Mathematical physics based Riemannian submersions such as Yang–Mills theory [9, 39], Kaluza-Klein theory [10, 15], supergravity and superstring theories [16, 25]. Submersions were studied by many authors between differential manifolds like almost Hermitian manifolds, Kähler manifold, almost contact manifolds, nearly K-cosymplectic manifold, Sasakian type manifold etc. The authors studied semi-Riemannian submersion and Lorentzian submersion [12], anti-invariant submersion [22, 24, 30, 36], anti-invariant $\xi^{\perp}$-submersion [23], semi-invariant submersion [19], [20], [32], semi-invariant $\xi^{\perp}$-submersion [4] and many other type of submersions studied by several authors [11, 18, 21, 28, 29, 31, 38], hemi-slant submersion [34], anti-holomorphic semi-invariant submersion [35] etc. between differential manifolds with different structures.

In comparison of conformal submersions, Riemannian submersions are specific. The conformal maps don't preserve distance between points but they preserve angle between vector fields which allows us to transfer certain properties of manifolds to another manifolds by deforming such properties. Ornea [27] initiated the theory of conformal submersions between Riemannian metric manifolds. Later, it was studied by many authors [1–3, 5, 14].

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In the present manuscript, we establish the definition of semi-invariant conformal $\zeta^{\perp}$-Riemannian submersions (in short, sic $\zeta^{\perp}$-Rs) from almost contact metric (in short, acm) manifolds onto Riemannian metric manifolds, which is the generalization of conformal anti-invariant $\zeta^{\perp}$-submersions, semi-invariant conformal submersions and many others. In section 2, we give preliminaries about Sasakian manifolds, Riemannian submersions and conformal submersions. In section 3, we define sic $\zeta^{\perp}$-Rs from acm manifolds onto Riemannian metric manifolds with examples. We give the necessary and sufficient (briefly, ns) conditions for distributions to be integrable. After that, we establish the ns condition for sic $\zeta^{\perp}$-Rs to be homothetic map. Then we investigate the geometry of leaves of horizontal and vertical distributions and obtain ns condition for the distribution to be absolutely geodesic foliation on Sasakian manifolds. At last, we obtain a condition when the fibers of sic $\zeta^{\perp}$-Rs from Sasakian manifolds onto Riemannian metric manifolds to be locally product manifold. In section 4, we provide ns conditions for a sic $\zeta^{\perp}$-Rs to be harmonic. We also investigate the ns conditions for such submersions to be completely geodesic.

2. Preliminaries

The basic definitions have been restated here as we need them to establish the new results in this paper. An odd-dimensional acm manifold $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ is presented in terms of φ_* ((1,1)-tensor field), ζ (vector field), η_* (1-form) and $\mathfrak{g}_m$ (Riemannian metric). Then

$$\varphi_*^2 P = -P + \eta_*(P)\zeta, \tag{2.1}$$

$$\varphi_*\zeta = 0,\ \eta_* \circ \varphi_* = 0,\ \eta_*(\zeta) = 1,\ \mathfrak{g}_m(P,\zeta) = \eta_*(P), \tag{2.2}$$

and

$$\mathfrak{g}_m(\varphi_* P, \varphi_* Q) = \mathfrak{g}_m(P,Q) - \eta_*(P)\eta_*(Q), \quad \mathfrak{g}_m(\varphi_* P, Q) = -\mathfrak{g}_m(P, \varphi_* Q), \tag{2.3}$$

for any vector fields $P, Q \in \Gamma(T\Upsilon)$.

An acm manifold $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ is called Sasakian [33] if

$$(\nabla_P \varphi_*)Q = \mathfrak{g}_m(P,Q)\zeta - \eta_*(Q)P \tag{2.4}$$

and

$$\nabla_P \zeta = -\varphi_* P \tag{2.5}$$

for any vector fields $P, Q \in \Gamma(T\Upsilon)$, where ∇ is the Levi-Civita connection.

Now, we give some useful definitions of submersions to extend our definition of semi-invariant conformal Riemannian submersions s endowed with the horizontal Reeb vector field.

Definition 2.1. *[26] We consider two Riemannian metric manifolds* $(\Omega, \mathfrak{g}_n)$ *and* $(\Upsilon, \mathfrak{g}_m)$ *satisfying the condition that* $\dim(\Upsilon) > \dim(\Omega)$. *A Riemannian submersions* $\Psi : \Upsilon \to \Omega$ *is a map of* Υ *onto* Ω *satisfying the following conditions:*

(i) Ψ *has maximal rank.*

(ii) *The metric of horizontal vectors is preserved by the differential* Ψ_*.

For each $z \in \Omega$, $\Psi^{-1}(z)$ is an $(m-n)$-dimensional submanifold of Υ. The submanifolds $\Psi^{-1}(z)$, $z \in \Omega$, are called fibers. A vector field on Υ is called vertical and horizontal if it is always tangent to fibers and orthogonal to fibers, respectively. A vector field P on Υ is called basic if P is horizontal and Ψ-related to a vector field P' on Ω, that is, $\Psi_* P_x = P'_{\Psi_*(x)}$ for all $x \in M$. The projection morphisms on the distributions $\ker\Psi_*$ and $(\ker\Psi_*)^{\perp}$ are denoted by $\mathcal{V}_1$ and $\mathcal{H}_1$, respectively. The sections of $\mathcal{V}_1$ and $\mathcal{H}_1$ are called the vertical vector fields and horizontal vector fields, respectively. So

$$\mathcal{V}_{1p} = T_p\left(\Psi^{-1}(q)\right), \qquad \mathcal{H}_{1p} = T_p\left(\Psi^{-1}(q)\right)^{\perp}.$$

Definition 2.2. *[27] Let* (Υ, g_m) *and* (Ω, g_n) *be Riemannian metric manifolds, where* $\dim(\Upsilon) = m$, $\dim(\Omega) = n$ *and* $m > n$. *A (horizontally) conformal submersion* $\Psi : \Upsilon \to \Omega$ *is a map of* Υ *onto* Ω *satisfying the following axioms:*

(i) Ψ *has maximal rank.*

(ii) *The angle between the horizontal vectors is preserved by the differential* Ψ_*,

$$g_n(\Psi_* U, \Psi_* V) = \Lambda_*(p) g_m(U, V), \quad U, V \in \mathcal{H}_{1p_*}$$

where $\Lambda_*(p)$ *is a non-zero number and* $p \in \Upsilon$. *The number* $\Lambda_*(p)$ *is called the square dilation of* Ψ *at* p, *it is necessarily non-negative. Its square root* $\lambda_*(p) = \sqrt{\Lambda_*(p)}$ *is called the dilation of* Ψ *at* p.

Clearly, Riemannian submersions is a horizontally conformal submersion with $\lambda_* = 1$.

Definition 2.3. *[6] Let* $\Psi : (\Upsilon, g_m) \to (\Omega, g_n)$ *be a conformal submersion. A vector field* E_1 *on* Υ *is called projectiable if there exist a vector field* $\widehat{E_1}$ *on* N^* *such that* $\Psi_*(E_{1p}) = \widehat{E_1}_{\Psi_*(p)}$ *for any* $p \in \Upsilon$. *In this case,* E_1 *and* $\widehat{E_1}$ *are called* Ψ*-related. A horizontal vector field* V *on* Υ *is called basic, if it is projectiable.*

It is to be noted that if $\widehat{W}$ is a vector field on Ω, then there exists a unique basic vector field W on Υ which is called the horizontal lift of $\widehat{W}$.

Definition 2.4. *[6] A horizontally conformal submersion* $\Psi : (\Upsilon, g_m) \to (\Omega, g_n)$ *is called horizontally homothetic if the slope of its dilation* λ_* *is vertical, that is,*

$$\mathcal{H}_1(grad\lambda_*) = 0$$

at $p \in \Upsilon$, *where* $\mathcal{H}_1$ *is the projection on the horizontal space* $(\ker \Psi_*)^\perp$.

As per O'Neill [26], we can write

$$T_E F = \mathcal{H}_1 \nabla^M_{\mathcal{V}_1 E} \mathcal{V}_1 F + \mathcal{V}_1 \nabla^M_{\mathcal{V}_1 E} \mathcal{H}_1 F, \tag{2.6}$$

$$A_E F = \mathcal{H}_1 \nabla^M_{\mathcal{H}_1 E} \mathcal{V}_1 F + \mathcal{V}_1 \nabla^M_{\mathcal{H}_1 E} \mathcal{H}_1 F, \tag{2.7}$$

where $\mathcal{V}_1$ and $\mathcal{H}_1$ are projections the normal and tangential direction, respectively.
From equations (2.6) and (2.7), we obtain the following equations

$$\nabla_V W = T_V W + \widehat{\nabla}_V W, \tag{2.8}$$

$$\nabla_V P = \mathcal{H}_1 \nabla_V P + T_V P, \tag{2.9}$$

$$\nabla_P V = A_P V + \mathcal{V}_1 \nabla_P V, \tag{2.10}$$

$$\nabla_P Q = \mathcal{H}_1 \nabla_P Q + A_P Q, \tag{2.11}$$

for all $V, W \in \Gamma(\ker \Psi_*)$ and $P, Q \in \Gamma(\ker \Psi_*)^\perp$, where $\mathcal{V}_1 \nabla_V W = \widehat{\nabla}_V W$. If P is the basic vector field, then $A_P Q = \mathcal{H}_1 \nabla_Q P$.
Clearly, for $p \in \Upsilon$, $U \in \mathcal{V}_{1p}$ and $P \in \mathcal{H}_{1p}$, the linear operators

$$T_U, A_P : T_p \Upsilon \to T_p \Upsilon$$

are skew-symmetric, that is,

$$g_m(A_P E, F) = -g_m(E, A_P F) \text{ and } g_m(T_U E, F) = -g_m(E, T_U F), \tag{2.12}$$

for all $E, F \in T_p \Upsilon$. From (2.12), one can say that totally geodesic fibres Ψ and $T \equiv 0$ has bi-conditional relation.

Consider the smooth map $\Psi : (\Upsilon, g_m) \to (\Omega, g_n)$ between Riemannian metric manifolds. Then the differential Ψ_* of Ψ can be noticed as a section of the bundle $Hom(T\Upsilon, \Psi^{-1}T\Omega) \to \Upsilon$, where $\Psi^{-1}T\Omega$ is the bundle which has fibres $\left(\Psi^{-1}T\Omega\right)_x = T_{f(x)}\Omega$, $x \in \Upsilon$. $Hom(T\Upsilon, \Psi^{-1}T\Omega)$ has a connection ∇ induced from the Riemannian connection ∇^{Υ} and the pullback connection. The second fundamental form of Ψ is

$$(\nabla\Psi_*)(E, F) = \nabla^{\Omega}_E \Psi_* F - \Psi_*(\nabla^{\Upsilon}_E F), \quad \text{for all } E, F \in \Gamma(T\Upsilon), \tag{2.13}$$

where ∇^{Ω} is the pullback connection ([6, 7]). The map Ψ is said to be totally geodesic [6] if $(\nabla\Psi_*)(E, F) = 0$, for all $E, F \in \Gamma(T\Upsilon)$.

Lemma 2.5. *[37, Lemma 1.16, pp.129] Let $\Psi : (\Upsilon, g_m) \to (\Omega, g_n)$ be a smooth map between Riemannian metric manifolds (Υ, g_m) and (Ω, g_n). Then*

$$\Psi_*([P, Q]) = \nabla^{\Omega}_P \Psi_* Q - \nabla^{\Omega}_Q \Psi_* P, \tag{2.14}$$

for all $P, Q \in \Gamma(T\Upsilon)$.

From Lemma 2.5, we come to the conclusion

$$[P, Q] \in \Gamma(ker\Psi_*), \tag{2.15}$$

for $P \in \Gamma(ker\Psi_*)^{\perp}$ and $Q \in \Gamma(ker\Psi_*)$.

A smooth map $\Psi : (\Upsilon, g_m) \to (\Omega, g_n)$ is said to be harmonic [6] if and only if trace$(\nabla\Psi_*) = 0$. The tension field of Ψ is the section $\tau(\Psi)$ of $\Gamma(\Psi^{-1}TN)$ and defined by

$$\tau(\Psi) = div\Psi_* = \sum_{i=1}^{m} (\nabla\Psi_*)(e_i, e_i), \tag{2.16}$$

where $\{e_1, e_2, ..., e_m\}$ is the orthonormal basis on Υ. Then it follows that ns condition for Ψ to be harmonic is $\tau(\Psi) = 0$ [6].

Now, we recall an important Lemma from [6], which will be needed in the study of whole paper.

Lemma 2.6. *Let $\Psi : (\Upsilon, g_m) \to (\Omega, g_n)$ be a horizontal conformal submersion, P, Q are horizontal vector fields and V, W are vertical vector fields. Then*

(*a*) $(\nabla\Psi_*)(P, Q) = P(\ln\lambda_*)\Psi_*(Q) + Q(\ln\lambda_*)\Psi_*(P) - g_m(P, Q)\Psi_*(grad\ ln\lambda_*)$,
(*b*) $(\nabla\Psi_*)(V, W) = -\Psi_*(\mathcal{T}_V W)$,
(*c*) $(\nabla\Psi_*)(P, V) = -\Psi_*(\nabla^M_P V) = -\Psi_*(A_P V)$.

3. Semi-invariant conformal $\zeta^{\perp}$-Riemannian submersions

In this segment, we deal with the definition and examples of sic $\zeta^{\perp}$-Rs from acm manifolds onto Riemannian metric manifolds. We acquire the integrability of distributions and also geometry of leaves of $ker\Psi_*$ and $(ker\Psi_*)^{\perp}$.

Definition 3.1. *Let $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ be an acm manifold and (Ω, g_n) be a Riemannian metric manifold. A horizontally conformal submersion $\Psi : (\Upsilon, g_m) \to (\Omega, g_n)$ with slope λ_*, is said to be a sic $\zeta^{\perp}$-Rs if ζ is normal to $ker\Psi_*$ and there is a distribution $\mathfrak{D}^*_1 \subseteq ker\Psi_*$ such that*

$$ker\Psi_* = \mathfrak{D}^*_1 \oplus \mathfrak{D}^*_2, \tag{3.1}$$

$$\varphi_*(\mathfrak{D}^*_1) = \mathfrak{D}^*_1,\ \varphi_*(\mathfrak{D}^*_2) \subseteq (ker\Psi_*)^{\perp}, \tag{3.2}$$

where $\mathfrak{D}^*_2$ is orthogonal complementary to $\mathfrak{D}^*_1$ in $ker\Psi_*$.

It is to be noted that the distribution $ker\Psi_*$ is integrable.

Further to prove the consistency of sic $\zeta^{\perp}$-Rs in acm manifolds, we are putting some examples here.

Example 3.2. *Every anti-invariant $\zeta^{\perp}$-submersions from an acm manifold onto a Riemannian metric manifold is a sic $\zeta^{\perp}$-Rs with $\lambda_* = I$ and $\mathfrak{D}_1^* = \{0\}$, where I denotes the identity function.*

Example 3.3. *Every semi-invariant $\zeta^{\perp}$-submersions from an acm manifold onto a Riemannian metric manifold is a sic $\zeta^{\perp}$-Rs with $\lambda_* = I$.*

Example 3.4. *[8] Let $(\mathbb{R}^m, g_m, \varphi_*, \zeta, \eta_*)$, $(m = 2n + 1)$ be a Sasakian manifold given by*

$$\eta_* = \frac{1}{2}\left(dw - \sum_{i=1}^{n} v^i du^i\right), \quad \zeta = 2\frac{\partial}{\partial w},$$

$$g_m = \eta_* \otimes \eta_* + \frac{1}{4}\sum_{i=1}^{n}(du^i \otimes du^i + dv^i \otimes dv^i),$$

$$\varphi_* = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & v^j & 0 \end{pmatrix},$$

where $\{(u^i, v^i, w)|i = 1, \dots, n\}$ are the cartesian coordinates. The vector fields $X_i = 2\frac{\partial}{\partial v^i}$, $\quad X_{n+i} = 2\left(\frac{\partial}{\partial u^i} + v^i\frac{\partial}{\partial w}\right)$ and ζ form a φ_*-basis for the contact metric structure.
Let $\Psi : \left(\mathbb{R}^7, g_1\right) \to \left(\mathbb{R}^2, g_2\right)$ be a submersion defined by

$$\Psi(u^1, u^2, u^3, v^1, v^2, v^3, w) = (\sinh v^3 \cos w, \cosh v^3 \sin w).$$

Also, define $g_1 = \eta_* \otimes \eta_* + \sum_{i=1}^{3}(du^i \otimes du^i + dv^i \otimes dv^i)$ and $g_2 = (dx \otimes dx + dy \otimes dy)$, where (x, y) is coordinate system in $\mathbb{R}^2$, $\eta_* = dw$, $\zeta = \frac{\partial}{\partial w}$ and $v^j = 0$ in φ_*. Then it follows that

$$ker\Psi^* = span\{L_1 = \partial u^1, L_2 = \partial u^2, L_3 = \partial u^3, L_4 = \partial v^1, L_5 = \partial v^2\}$$

and

$$(ker\Psi_*)^{\perp} = span\left\{W_1 = \partial v^3, W_2 = \partial w\right\}.$$

Hence, we have $\varphi_*L_1 = -L_4, \varphi_*L_2 = -L_5, \varphi_*L_3 = -W_1$, $\varphi_*L_4 = L_1$ and $\varphi_*L_5 = L_2$. Thus it follows that $\mathfrak{D}_1^* = span\{L_1, L_2, L_4, L_5\}$, $\mathfrak{D}_2^* = span\{L_3\}$ and $\zeta \in (ker\Psi_*)^{\perp}$. We can easily compute that

$$\begin{aligned} g_2\left(\Psi_*W_1, \Psi_*W_1\right) &= \left(\sinh^2 v_3 \sin^2 w + \cosh^2 v_3 \cos^2 w\right) g_1\left(W_1, W_1\right), \\ g_2\left(\Psi_*W_2, \Psi_*W_2\right) &= \left(\sinh^2 v_3 \sin^2 w + \cosh^2 v_3 \cos^2 w\right) g_1\left(W_2, W_2\right). \end{aligned}$$

Therefore, Ψ is a sic $\zeta^{\perp}$-Rs with $\lambda_* = \sqrt{\left(\sinh^2 v_3 \sin^2 w + \cosh^2 v_3 \cos^2 w\right)}$.

Let Ψ be a sic $\zeta^{\perp}$-Rs from acm manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. Consider

$$(\ker \Psi_*)^{\perp} = \varphi_*\mathfrak{D}_2^* \oplus \mu,$$

where μ is the complementary distribution to $\varphi_*\mathfrak{D}_2^*$ in $(ker\Psi_*)^{\perp}$ and $\varphi_*\mu \subset \mu$. Therefore $\zeta \in \mu$.

For $V \in \Gamma(ker\Psi_*)$, we write

$$\varphi_* V = \phi V + \omega V, \tag{3.3}$$

where $\phi V \in \Gamma(\mathfrak{D}_1^*)$ and $\omega V \in \Gamma(\varphi_* \mathfrak{D}_2^*)$.

For $P \in \Gamma(ker\Psi_*)^\perp$, we have

$$\varphi_* P = \alpha P + \beta P, \tag{3.4}$$

where $\alpha P \in \Gamma(\mathfrak{D}_2^*)$ and $\beta P \in \Gamma(\mu)$.

Now, using (2.8), (2.9), (3.3) and (3.4), we get

$$(\nabla_V^M \phi)W = \alpha T_V W - T_V \omega W, \tag{3.5}$$

$$(\nabla_V^M \omega)W = \beta T_V W - T_V \phi W \tag{3.6}$$

for $V, W \in \Gamma(ker\Psi_*)$, where

$$(\nabla_V^M \phi)W = \hat{\nabla}_V \phi W - \phi \hat{\nabla}_V W \tag{3.7}$$

and

$$(\nabla_V^M \omega)W = \mathcal{H}_1 \nabla_V^M \omega W - \omega \hat{\nabla}_V W. \tag{3.8}$$

Lemma 3.5. *Let Ψ be sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ onto Riemannian metric manifolds (Ω, g_n). Then the ns condition for the distribution $\mathfrak{D}_1^*$ to be integrable is*

$$(\nabla\Psi_*)(V, \varphi_* U) - (\nabla\Psi_*)(U, \varphi_* V) \in \Gamma(\Psi_*(\mu))$$

for $U, V \in \Gamma(\mathfrak{D}_1^*)$.

Proof. Let $U, V \in \Gamma(\mathfrak{D}_1^*)$ and $W \in \Gamma(\mathfrak{D}_2^*)$. With the help of (2.2), we get

$$g_m(\nabla_U \varphi_* V, \varphi_* W) = g_m(\nabla_U V, W). \tag{3.9}$$

So

$$g_m([U, V], W) = g_m(\nabla_U V, W) - g_m(\nabla_V U, W).$$

Using (2.8) and (3.9), we obtain

$$\begin{aligned} g_m([U, V], W) &= g_m(\nabla_U \varphi_* V, \varphi_* W) - g_m(\nabla_V \varphi_* U, \varphi_* W) \\ &= g_m(T_U \varphi_* V - T_V \varphi_* U, \varphi_* W). \end{aligned}$$

The distribution $\mathfrak{D}_1^*$ is integrable if and only if $g_m([U, V], W) = g_m([U, V], Z) = 0$ for $U, V \in \Gamma(\mathfrak{D}_1^*)$, $W \in \Gamma(\mathfrak{D}_2^*)$ and $Z \in \Gamma((ker\Psi_*)^\perp)$. Since $ker\Psi_*$ is integrable, so we immediately have $g_m([U, V], Z) = 0$. Thus the ns condition for the distribution $\mathfrak{D}_1^*$ to be integrable is $g_m([U, V], W) = 0$. Since Ψ is a conformal submersion, in view of Lemma 2.6 result (b), we conclude that

$$g_m([U, V], W) = \frac{1}{\lambda_*^2} g_n((\nabla\Psi_*)(V, \varphi_* U) - (\nabla\Psi_*)(U, \varphi_* V), \Psi_* \varphi_* W).$$

Lemma 3.6. *Let Ψ be sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ onto Riemannian metric manifolds (Ω, g_n). Then the ns condition for the integrability of distribution $\mathfrak{D}_2^*$ is*

$$\hat{\nabla}_W \phi \xi + T_W \omega \xi - \hat{\nabla}_\xi \phi W - T_\xi \omega W \in \Gamma(\mathfrak{D}_2^*)$$

for $W, \xi \in \Gamma(\mathfrak{D}_2^*)$.

Proof. Let $U \in \Gamma(\mathfrak{D}_1^*)$ and $W, \xi \in \Gamma(\mathfrak{D}_2^*)$. Using (2.2), (2.3), (2.6), (2.8), (2.9) and (3.9), we get

$$\begin{aligned} g_m(\varphi_*[W,\xi],U) &= g_m(\nabla_W \varphi_* \xi, U) - g_m(\nabla_\xi \varphi_* W, U) \\ &= g_m(T_W \phi\xi + \hat{\nabla}_W \phi\xi + T_W \omega\xi + \mathcal{H}_1 \nabla_W \omega\xi \\ &\quad -T_\xi \phi W - \hat{\nabla}_\xi \phi W - T_\xi \omega W - \mathcal{H}_1 \nabla_\xi \omega W, U) \\ &= g_m(\hat{\nabla}_W \phi\xi + T_W \omega\xi - \hat{\nabla}_\xi \phi W - T_\xi \omega W, U). \end{aligned}$$

Theorem 3.7. *Let Ψ be a sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ onto Riemannian metric manifolds (Ω, g_n). Then the ns conditions for the distribution $(ker\Psi_*)^\perp$ to be integrable are*

$$A_Q \omega\alpha P - A_P \omega\alpha Q - \varphi_* A_P \beta Q + \varphi_* A_Q \beta P \notin \Gamma(\mathfrak{D}_1^*)$$

and

$$\begin{aligned} \frac{1}{\lambda_*^2} g_n(\nabla_V \Psi_* CU - \nabla_U \Psi_* CV, \Psi_* \varphi_* Z) = g_m(&A_V BU - A_U BV - CV(ln\lambda_*)U \\ &+ CU(ln\lambda_*)V + 2g_m(U, CV) grad\, ln\lambda_* \\ &+ \eta_*(U)V - \eta_*(V)U, \varphi_* Z) \end{aligned}$$

for $U, V \in \Gamma(ker\Psi_*)^\perp)$ and $Z \in \Gamma(\mathfrak{D}_2^*)$.

Proof. The ns condition for the distribution $(ker\Psi_*)^\perp$ to be integrable on manifold Υ is that

$$g_m([P,Q],U) = 0 \ \text{ and } \ g_m([P,Q],W) = 0,$$

for $P, Q \in \Gamma((ker\Psi_*)^\perp)$, $U \in \Gamma(\mathfrak{D}_1^*)$ and $W \in \Gamma(\mathfrak{D}_2^*)$.

Using (2.1), (2.2), (2.4) and (3.4), we infer

$$\begin{aligned} g_m([P,Q],U) &= g_m(\varphi_*[P,Q], \varphi_* U) \\ &= g_m(\varphi_* \nabla_P Q - \varphi_* \nabla_Q P, \varphi_* U) \\ &= g_m(\nabla_P \alpha Q, \varphi_* U) + g_m(\nabla_P \beta Q, \varphi_* U) \\ &- g_m(\nabla_Q \alpha P, \varphi_* U) - g_m(\nabla_Q \beta P, \varphi_* U) \\ &= g_m(\varphi_* \nabla_P \alpha Q, U) - g_m(\varphi_* \nabla_P \beta Q, U) \\ &+ g_m(\varphi_* \nabla_Q \alpha P, U) + g_m(\varphi_* \nabla_Q \beta P, U). \end{aligned}$$

By simple calculations, we have

$$\begin{aligned} g_m([P,Q],U) &= -g_m(\nabla_P \varphi_* \alpha Q, U) + g_m(\nabla_P \beta Q, \varphi_* U) \\ &+ g_m(\nabla_Q \varphi_* \alpha P, U) - g_m(\nabla_Q \beta P, \varphi_* U). \end{aligned}$$

By (3.3), we have $\varphi_* \alpha P = \phi\alpha P + \omega\alpha P$. Since $\varphi_* \alpha P, \omega\alpha P \in \Gamma((ker\Psi_*)^\perp)$, so $\phi\alpha P = 0$, $\forall P \in \Gamma((ker\Psi_*)^\perp)$. Using (2.4), (2.11) and (3.3), we get

$$g_m([P,Q],U) = g_m(A_Q \omega\alpha P - A_P \omega\alpha Q - \varphi_* A_P \beta Q + \varphi_* A_Q \beta P, U). \tag{3.10}$$

With the help of (2.1), (2.2), (2.4) and (3.4), we deduce

$$\begin{aligned} g_m([P,Q],W) &= g_m(\nabla_P \alpha Q, \varphi_* W) + g_m(\nabla_P \beta Q, \varphi_* W) \\ &- g_m(\nabla_Q \alpha P, \varphi_* W) - g_m(\nabla_Q \beta P, \varphi_* W) \\ &+ \eta_*(P) g_m(Q, \varphi_* W) - \eta_*(Q) g_m(P, \varphi_* W) \end{aligned}$$

Since Ψ is sic $\zeta^{\perp}$-Rs, using (2.10), (2.11), (2.13) and Lemma 2.5 and 2.6, we obtain

$$\begin{aligned}\mathfrak{g}_m([P,Q],W) &= -\eta_*(Q)\mathfrak{g}_m(P,\varphi_*W) + \eta_*(P)\mathfrak{g}_m(Q,\varphi_*W)\\ &- \frac{1}{\lambda_*^2}\mathfrak{g}_n((\nabla\Psi_*)(P,\alpha Q),\Psi_*\varphi_*W) + \frac{1}{\lambda_*^2}\mathfrak{g}_n((\nabla\Psi_*)(Q,\alpha P),\Psi_*\varphi_*W)\\ &+ \frac{1}{\lambda_*^2}\mathfrak{g}_n\{-P(ln\lambda_*)\Psi_*\beta Q - \beta Q(ln\lambda_*)\Psi_*P\\ &+ \mathfrak{g}_m(P,\beta Q)\Psi_*(grad\ ln\lambda_*) + \nabla_P^{\Psi}\Psi_*\beta Q,\Psi_*\varphi_*W\}\\ &- \frac{1}{\lambda_*^2}\mathfrak{g}_n\{-Q(ln\lambda_*)\Psi_*\beta P - \beta P(ln\lambda_*)\Psi_*Q\\ &+ \mathfrak{g}_m(Q,\beta P)\Psi_*(grad\ ln\lambda_*) + \nabla_Q^{\Psi}\Psi_*\beta P,\Psi_*\varphi_*W\},\end{aligned}$$

$$\begin{aligned}\mathfrak{g}_m([P,Q],W) &= -\eta_*(Q)\mathfrak{g}_m(P,\varphi_*W) + \eta_*(P)\mathfrak{g}_m(Q,\varphi_*W) - \mathfrak{g}_m(A_P\alpha Q,\varphi_*W)\\ &+ \mathfrak{g}_m(A_Q\alpha P,\varphi_*W) - \frac{1}{\lambda_*^2}\{\mathfrak{g}_m(grad\ ln\lambda_*,P)\mathfrak{g}_n(\Psi_*\beta Q,\Psi_*\varphi_*W)\\ &+ \mathfrak{g}_m(grad\ ln\lambda_*,\beta Q)\mathfrak{g}_n(\Psi_*P,\Psi_*\varphi_*W)\\ &- \mathfrak{g}_m(P,\beta Q)\mathfrak{g}_n(\Psi_*(grad\ ln\lambda_*),\Psi_*\varphi_*W)\\ &- \mathfrak{g}_m(grad\ ln\lambda_*,Q)\mathfrak{g}_n(\Psi_*\beta P,\Psi_*\varphi_*W)\\ &- \mathfrak{g}_m(grad\ ln\lambda_*,\beta P)\mathfrak{g}_n(\Psi_*Q,\Psi_*\varphi_*W)\\ &+ \mathfrak{g}_m(Q,\beta P)\mathfrak{g}_n(\Psi_*(grad\ ln\lambda_*),\Psi_*\varphi_*W)\\ &- \mathfrak{g}_n(\nabla_P^{\Psi}\Psi_*\beta Q,\Psi_*\varphi_*W) + \mathfrak{g}_n(\nabla_Q^{\Psi}\Psi_*\beta P,\Psi_*\varphi_*W)\}.\end{aligned}$$

Using Definition 3.1, we have

$$\begin{aligned}\mathfrak{g}_m([P,Q],W) \quad = \quad & -\eta_*(Q)\mathfrak{g}_m(P,\varphi_*W) + \eta_*(P)\mathfrak{g}_m(Q,\varphi_*W)\\ & +\mathfrak{g}_m(A_Q\alpha P - A_P\alpha Q - (\beta Q)(P\ln\lambda_*)\\ & + (\beta P)(Q\ln\lambda_*) + 2\mathfrak{g}_m(P,\beta Q)grad\ ln\lambda_*,\varphi_*W)\\ & +\mathfrak{g}_m((\beta P ln\lambda_*)Q - (\beta Q ln\lambda_*)P,\varphi_*W)\\ & +\frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla_P^{\Psi}\Psi_*\beta Q - \nabla_Q^{\Psi}\Psi_*\beta P,\Psi_*\varphi_*W).\end{aligned} \tag{3.11}$$

From the above expression, we entail that

$$\begin{aligned}\frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla_Q^{\Psi}\Psi_*\beta P - \nabla_P^{\Psi}\Psi_*\beta Q,\Psi_*\varphi_*W) = \ & \mathfrak{g}_m(A_Q\alpha P - A_P\alpha Q - (\beta Q)(P\ln\lambda_*)\\ & + (\beta P)(Q\ln\lambda_*) + 2\mathfrak{g}_m(P,\beta Q)grad\ ln\lambda_*\\ & + (\beta P ln\lambda_*)Q - (\beta Q ln\lambda_*)P + \eta_*(P)Q\\ & - \eta_*(Q)P,\varphi_*W).\end{aligned} \tag{3.12}$$

So from (3.10) and (3.12), we obtain the results.

Theorem 3.8. *Let Ψ be a sic $\zeta^{\perp}$-Rs from Sasakian manifolds $(\Upsilon,\varphi_*,\zeta,\eta_*,\mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega,\mathfrak{g}_n)$ with integrable distribution $(ker\Psi_*)^{\perp}$. Then the ns condition for the Ψ to be homothetic map is*

$$\begin{aligned}\mathfrak{g}_n(\nabla_Q^{\Psi}\Psi_*\beta P - \nabla_P^{\Psi}\Psi_*\beta Q,\Psi_*\varphi_*W) = \ & \lambda_*^2\mathfrak{g}_m(A_Q\alpha P - A_P\alpha Q\\ & + \eta_*(P)Q - \eta_*(Q)P,\varphi_*W)\end{aligned} \tag{3.13}$$

for $P,Q \in \Gamma((ker\Psi_*)^{\perp})$ and $W \in \Gamma(\mathfrak{D}_2^*)$.

Proof. Let $P, Q \in \Gamma((ker\Psi_*)^\perp)$ and $W \in \Gamma(\mathfrak{D}_2^*)$. Using (3.12), we infer

$$\begin{aligned}\mathfrak{g}_m([P,Q],W) = &-\eta_*(Q)\mathfrak{g}_m(P,\varphi_*W) + \eta_*(P)\mathfrak{g}_m(Q,\varphi_*W)\\ &+ \mathfrak{g}_m(A_Q\alpha P - A_P\alpha Q - (\beta Q)(P\ln\lambda_*)\\ &+ (\beta P)(Q\ln\lambda_*) + 2\mathfrak{g}_m(P,\beta Q)grad\ ln\lambda_*, \varphi_*W)\\ &+ \mathfrak{g}_m((\beta P ln\lambda_*)Q - (\beta Q ln\lambda_*)P, \varphi_*W)\\ &+ \frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla_P^\Psi\Psi_*\beta Q - \nabla_Q^\Psi\Psi_*\beta P, \Psi_*\varphi_*W).\end{aligned} \tag{3.14}$$

If Ψ is a parallel homothetic map, therefore we acquire (3.13). Conversely, if (3.13) holds, then we gain

$$\begin{aligned}0 = &\mathfrak{g}_m((\beta P ln\lambda_*)Q - (\beta Q ln\lambda_*)P, \varphi_*W)\\ &+ \mathfrak{g}_m(-(\beta Q)(P\ln\lambda_*) + (\beta P)(Q\ln\lambda_*) + 2\mathfrak{g}_m(P,\beta Q)grad\ ln\lambda_*, \varphi_*W).\end{aligned} \tag{3.15}$$

Replacing Q by φ_*W for $W \in \Gamma(\mathfrak{D}_2^*)$ in (3.15), we entail that

$$\mathfrak{g}_m(grad\ln\lambda_*, \beta P)\mathfrak{g}_m(\varphi_*W, \varphi_*W) = 0,$$

which implies that λ_* is a invariable on $\Gamma(\mu)$ since $\|\varphi_*W\| \neq 0$.
Now, again replace Q by βP for $P \in \Gamma(\mu)$ in (3.15), we have

$$\mathfrak{g}_m(grad\ ln\lambda_*, \varphi_*W)\mathfrak{g}_m(P, \beta^2 P) = \mathfrak{g}_m(grad\ ln\lambda_*, \varphi_*W)\mathfrak{g}_m(\beta P, \beta P) = 0.$$

Now, we see that the $\|\beta P\| \neq 0$, this concludes λ_* is a invariable on $\Gamma(\varphi_*\mathfrak{D}_2^*)$, which finalize the proof of our theorem.

Now, we come to the following results for the geometry of leaves of horizontal distributions.

Theorem 3.9. *Let Ψ be a sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. Then the ns conditions for the $(ker\Psi_*^\perp)$ to be completely geodesic foliation on Υ are*

$$A_P\varphi_*Q + \mathcal{V}_1\nabla_P^m\alpha Q \in \Gamma(\mathfrak{D}_2^*),$$

and

$$\begin{aligned}\mathfrak{g}_n(\nabla_P\Psi_*\varphi_*W, \Psi_*\beta Q) = &\lambda_*^2\mathfrak{g}_m(-A_P\alpha Q + \beta Q(ln\lambda_*)P - \mathfrak{g}_m(P,\beta Q)(grad\ln\lambda_*)\\ &- P(ln\lambda_*)\beta Q + \eta_*(Q)P, \varphi_*W)\end{aligned}$$

for $P, Q \in \Gamma(ker\Psi_*)^\perp$, $U \in \Gamma(\mathfrak{D}_1^*)$ and $W \in \Gamma(\mathfrak{D}_2^*)$.

Proof. Here, it is to be noted that the distribution $(ker\Psi_*)^\perp$ characterizes a completely geodesic foliation on Υ if and only if $\mathfrak{g}_m(\nabla_P^m Q, U) = 0$ and $\mathfrak{g}_m(\nabla_P^m Q, W) = 0$ for $P, Q \in \Gamma(ker\Psi_*)^\perp$, $U \in \Gamma(\mathfrak{D}_1^*)$ and $W \in \Gamma(\mathfrak{D}_2^*)$. Then with the help of (2.1),(2.2), (2.4), (2.10), (2.11) and (3.4), we infer

$$\begin{aligned}\mathfrak{g}_m(\nabla_P^m Q, U) &= \mathfrak{g}_m(\nabla_P^m\varphi_*Q, \varphi_*U) - \eta_*(Q)\mathfrak{g}_m(P, \varphi_*U)\\ &= -\mathfrak{g}_m(\varphi_*\left(A_P\varphi_*Q + \mathcal{V}_1\nabla_P^m\alpha Q\right), U),\end{aligned} \tag{3.16}$$

which implies that $A_P\varphi_*Q + \mathcal{V}_1\nabla_P^m\alpha Q \in \Gamma(\mathfrak{D}_2^*)$.
Using (2.1),(2.2), (2.4) and (3.4), we observe that

$$\begin{aligned}\mathfrak{g}_m(\nabla_P^m Q, W) &= \mathfrak{g}_m(\nabla_P^m\alpha Q + \nabla_P^m\beta Q, \varphi_*W)\\ &\quad - \eta_*(Q)\mathfrak{g}_m(P, \varphi_*W),\\ &= -\mathfrak{g}_m(\alpha Q, \nabla_P^m\varphi_*W) - \mathfrak{g}_m(\beta Q, \nabla_P^m\varphi_*W)\\ &\quad - \eta_*(Q)\mathfrak{g}_m(P, \varphi_*W).\end{aligned}$$

Since Ψ is a conformal submersion, in view of (2.11), (2.13) and Lemma 2.5, we gain

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_P Q, U) &= -\mathfrak{g}_m(\alpha Q, A_P \varphi_* W) + \frac{1}{\lambda_*^2}\mathfrak{g}_m(grad\ln\lambda_*, \varphi_* W)\mathfrak{g}_n(\Psi_* P, \Psi_*\beta Q)\\ &+ \frac{1}{\lambda_*^2}\mathfrak{g}_m(grad\ln\lambda_*, P)\mathfrak{g}_n(\Psi_*\varphi_* W, \Psi_*\beta Q,)\\ &- \frac{1}{\lambda_*^2}\mathfrak{g}_n(\Psi_*(grad\ln\lambda_*), \Psi_*\beta Q)\mathfrak{g}_m(P, \varphi_* W)\\ &+ \frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla^\Psi_P \Psi_*\varphi_* W, \Psi_*\beta Q) - \eta_*(Q)\mathfrak{g}_m(P, \varphi_* W).\end{aligned}$$

Using the definition of sic $\zeta^\perp$-Rs, we obtain

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_P Q, U) &= \mathfrak{g}_m(A_P\alpha Q - \beta Q(ln\lambda_*)P + \mathfrak{g}_m(P, \beta Q)(grad\ln\lambda_*)\\ &+ P(ln\lambda_*)\beta Q - \eta_*(Q)P, \varphi_* W) + \frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla_P \Psi_*\varphi_* W, \Psi_*\beta Q).\end{aligned} \tag{3.17}$$

Thus, we obtain the proof from (3.16) and (3.17).

Now, we state the following definition for further result.

Definition 3.10. *Let Ψ be a sic $\zeta^\perp$-Rs from acm manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. The distribution $\mathfrak{D}_2^*$ is parallel along $(ker\Psi_*)^\perp$ if $\nabla^m_P W \in \Gamma(\mathfrak{D}_2^*)$ for $P \in \Gamma((ker\Psi_*)^\perp)$ and $W \in \Gamma(\mathfrak{D}_2^*)$.*

Corollary 3.11. *Let Ψ be a sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$ such that $\mathfrak{D}_2^*$ is parallel along $(ker\Psi_*)^\perp$. Then the ns condition for the Ψ to be horizontally homothetic map is*

$$\lambda_*^2\mathfrak{g}_m(A_P\alpha Q, \varphi_* W) = \mathfrak{g}_n(\nabla_P \Psi_*\varphi_* W, \Psi_*\beta Q) \tag{3.18}$$

for $P, Q \in \Gamma(ker\Psi_*)^\perp$ and $W \in \Gamma(\mathfrak{D}_2^*)$.

Proof. Using (2.11), (3.18) and Lemma 2.6, we obtain

$$-\mathfrak{g}_m(grad\ ln\lambda_*, \beta Q)\mathfrak{g}_m(P, \varphi_* W) + \mathfrak{g}_m(P, \beta Q)\mathfrak{g}_m(grad\ ln\lambda_*, \varphi_* W) = 0. \tag{3.19}$$

Now, replacing P by $\varphi_* W$ for $W \in \Gamma(\mathfrak{D}_2^*)$ in (3.19), we get

$$\mathfrak{g}_m(grad\ ln\lambda_*, \beta Q)\mathfrak{g}_m(\varphi_* W, \varphi_* W) = 0.$$

Since $||\varphi_* W|| \neq 0$, Thus λ_* is a invariable on $\Gamma(\mu)$.
Now again replace P by βQ for $Q \in \Gamma(ker\Psi_*)^\perp$ in (3.19), we get

$$\mathfrak{g}_m(\beta Q, \beta Q)\mathfrak{g}_m(grad\ ln\lambda_*, \varphi_* W) = 0.$$

Since $||\beta Q|| \neq 0$, Thus λ_* is a invariable on $\Gamma(\varphi_*\mathfrak{D}_2^*)$. Therefore λ_* is invariable on $\Gamma(ker\Psi_*)^\perp$.
The converse easily follows from (3.19).

Now, we explore the geometric investigation of leaves of the distribution $(ker\Psi_*)$.

Theorem 3.12. *Let Ψ be a sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. Then the ns conditions for the $(ker\Psi_*)$ to be completely geodesic foliation on Υ are*

$$\begin{aligned}\mathfrak{g}_n(\nabla_{\omega V}\Psi_* Z, \Psi_*\omega U) = \lambda_*^2\{&\mathfrak{g}_m(\beta T_U\phi V + A_{\omega V}\phi U + \mathfrak{g}_m(\omega V, \omega U)grad\ ln\lambda_*\\ &- \mathfrak{g}_m(\phi U, V)\zeta, Z)\},\end{aligned}$$

and

$$T_U\omega V + \hat{\nabla}_U\phi V \in \Gamma(\mathfrak{D}_1^*).$$

for $U, V \in \Gamma(ker\Psi_*)$, $Z \in \Gamma(\mu)$ and $W \in \Gamma(\mathfrak{D}_2^*)$.

Proof. It is to be noted that the ns conditions for the $(ker\Psi_*)$ to be completely geodesic foliation on Υ are $\mathfrak{g}_m(\nabla^m_U V, Z) = 0$ and $\mathfrak{g}_m(\nabla^m_U V, \varphi_* W) = 0$ for $U, V \in \Gamma(ker\Psi_*)$, $Z \in \Gamma(\mu)$ and $W \in \Gamma(\mathfrak{D}_2^*)$. Then with the help of (2.1), (2.2), (2.4) and (3.3), we infer

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_U V, Z) &= \mathfrak{g}_m(\nabla^m_U \phi V, \varphi_* Z) + \mathfrak{g}_m(\phi U, \nabla^m_{\omega V} Z) + \mathfrak{g}_m(\omega U, \nabla^m_{\omega V} Z)\\ &\quad + \eta_*(Z)\mathfrak{g}_m(\varphi_* U, V).\end{aligned}$$

Since Ψ is a sic $\zeta^\perp$-Rs, from (2.8), (2.11), (2.13) and Lemma 2.6, we get

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_U V, Z) &= \mathfrak{g}_m(T_U \phi V, \varphi_* Z) + \mathfrak{g}_m(\phi U, A_{\omega V} Z)\\ &\quad - \frac{1}{\lambda_*^2}\mathfrak{g}_m(grad\ ln\lambda_*, Z)\mathfrak{g}_n(\Psi_*\omega V, \Psi_*\omega U)\\ &\quad + \frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla^\Psi_{\omega V}\Psi_* Z, \Psi_*\omega U) + \eta_*(Z)\mathfrak{g}_m(\varphi_* U, V).\end{aligned}$$

From the previous equation, we have

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_U V, Z) &= \mathfrak{g}_m(-\beta T_U \phi V - A_{\omega V}\phi U - \mathfrak{g}_m(\omega V, \omega U)(grad\ ln\lambda_*), Z)\\ &\quad + \mathfrak{g}_m(\phi U, V)\zeta, Z) + \frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla^\Psi_{\omega V}\Psi_* Z, \Psi_*\omega U).\end{aligned} \tag{3.20}$$

Now, using (2.1), (2.2) (2.8), (2.9) and (3.3), we infer

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_U V, \varphi_* W) &= -\mathfrak{g}_m(\varphi_*(T_U \omega V + \hat{\nabla}_U \phi V), \varphi_* W)\\ &= -\mathfrak{g}_m(\omega(T_U \omega V + \hat{\nabla}_U \phi V), \varphi_* W)\end{aligned} \tag{3.21}$$

So the results follows from (3.20) and (3.21).

Definition 3.13. *Let Ψ be a sic $\zeta^\perp$-Rs from acm manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. Then we call that μ is horizontal along $(ker\Psi_*)$ if $\nabla^m_U Z \in \Gamma(\mu)$ for $Z \in \Gamma(\mu)$ and $U \in \Gamma(ker\Psi_*)$.*

Corollary 3.14. *Let Ψ be a sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$ such that μ is horizontal along $(ker\Psi_*)$. Then the ns condition for the Ψ to be invariable on μ is*

$$\lambda_*^2\mathfrak{g}_m(\beta T_U \phi V + A_{\omega V}\phi U - \mathfrak{g}_m(\phi U, V)\zeta, Z) = \mathfrak{g}_n(\nabla_{\omega V}\Psi_* Z, \Psi_*\omega U), \tag{3.22}$$

for $U, V \in (ker\Psi_*)$ and $Z \in \Gamma(\mu)$.

Proof. Let $U, V \in \Gamma(ker\Psi_*)$, $Z \in \Gamma(\mu)$. Then $\mathfrak{g}_m(\nabla^m_U V, Z) = -\mathfrak{g}_m(V, \nabla^m_U Z)$. Since μ is horizontal along $(ker\Psi_*)$, so $\mathfrak{g}_m(V, \nabla^m_U Z) = 0$.

Using (3.20) and (3.22), we infer

$$\mathfrak{g}_m(\omega V, \omega U)\mathfrak{g}_m(grad\ ln\lambda_*, Z) = 0. \tag{3.23}$$

Replacing U by V in the foregoing equation, we may say that λ_* is a invariable on $\Gamma(\mu)$.
The converse follows from (3.20).

Theorem 3.15. *Let Ψ be a sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. Then the ns conditions for Υ to be a locally product manifold of the form $\Upsilon_{\ker\Psi_*} \times_{\lambda_*} \Upsilon_{\ker\Psi_*^\perp}$ are*

$$\begin{aligned}\mathfrak{g}_n(\nabla_{\omega V}\Psi_* Z, \Psi_*\omega U) &= \lambda_*^2\{\mathfrak{g}_m(\beta T_U \phi V + A_{\omega V}\phi U + \mathfrak{g}_m(\omega V, \omega U)grad\ ln\lambda_*\\ &\quad - \mathfrak{g}_m(\phi U, V)\zeta, Z)\},\end{aligned}$$

$$T_U \omega V + \hat{\nabla}_U \phi V \in \Gamma(\mathfrak{D}_1^*).$$

and

$$A_P\varphi_*Q + \mathcal{V}_1\nabla^m_P\alpha Q \in \Gamma(\mathfrak{D}_2^*),$$

$$\begin{aligned}\mathfrak{g}_n(\nabla_P\Psi_*\varphi_*W, \Psi_*\beta Q) =& \lambda_*^2\mathfrak{g}_m(-A_P\alpha Q + \beta Q(ln\lambda_*)P - \mathfrak{g}_m(P,\beta Q)(grad\ln\lambda_*)\\ &- P(ln\lambda_*)\beta Q + \eta_*(Q)P, \varphi_*W)\end{aligned}$$

for $P, Q \in \Gamma(ker\Psi_*)^\perp$, $U, V, W \in \Gamma(ker\Psi_*)$.

Let Ψ be a sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. For $U \in \Gamma(\mathfrak{D}_2^*)$ and $Q \in \Gamma(\mu)$, we have

$$\frac{1}{\lambda_*^2}\mathfrak{g}_n(\Psi_*\varphi_*U, \Psi_*Q) = \mathfrak{g}_m(\varphi_*U, Q) = 0,$$

which shows that the distributions $\Psi_*(\varphi_*\mathfrak{D}_2^*)$ and $\Psi_*(\mu)$ are orthogonal.W

Now, we explore the geometric properties of the leaves of the distributions $\mathfrak{D}_1^*$ and $\mathfrak{D}_2^*$.

Theorem 3.16. *Let Ψ be a sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. Then the ns conditions for the distribution $\mathfrak{D}_1^*$ to be completely geodesic foliations on Υ are*

$$(\nabla\Psi_*)(U, \varphi_*V) \in \Gamma(\Psi_*\mu)$$

and

$$\frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla\Psi_*)(U, \varphi_*V), \Psi_*\beta Z) = \mathfrak{g}_m(V, T_U\omega\alpha Z + \hat{\nabla}_U\phi\alpha Z) - \eta_*(\beta Z)\mathfrak{g}_m(U, V)$$

for $U, V \in \Gamma(\mathfrak{D}_1^*)$ *and* $Z \in \Gamma(ker\Psi_*)^\perp$.

Proof. The ns conditions for the distribution $\mathfrak{D}_1^*$ to be completely geodesic foliations on Υ are $\mathfrak{g}_m(\nabla^m_U V, W) = 0$ and $\mathfrak{g}_m(\nabla^m_U V, Z) = 0$ for $U, V \in \Gamma(\mathfrak{D}_1^*)$, $W \in \Gamma(\mathfrak{D}_2^*)$ and $Z \in \Gamma(\ker\Psi_*)^\perp$. Using (2.1), (2.2), (2.4) and (3.3), we obtain

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_U V, W) &= \mathfrak{g}_m(\varphi_*\nabla^m_U V, \varphi_*W) - \eta_*(\nabla^m_U V)\eta_*(W)\\ &= \mathfrak{g}_m(\nabla^m_U\varphi_*V, \varphi_*W)\end{aligned}$$

From (2.13) and using the fact that Ψ is a sic $\zeta^\perp$-Rs, we have

$$\mathfrak{g}_m(\nabla^m_U V, W)) = -\frac{1}{\lambda_*^2}\mathfrak{g}_n((\nabla\Psi_*)(U, \varphi_*V), \Psi_*\varphi_*W) \tag{3.24}$$

Using (2.1), (2.2), (2.4), (2.8) and (3.4), we have

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_U V, Z) &= -\mathfrak{g}_m(V, \nabla^m_U Z) = \mathfrak{g}_m(V, \nabla^m_U\varphi_*^2 Z)\\ &= \mathfrak{g}_m(V, \nabla^m_U\varphi_*\alpha Z) + \mathfrak{g}_m(V, \nabla^m_U\varphi_*\beta Z)\\ &= \mathfrak{g}_m(V, \nabla^m_U\varphi_*\alpha Z) + \mathfrak{g}_m(\nabla^m_U\varphi_*V, \beta Z) - \eta_*(\beta Z)\mathfrak{g}_m(U, V)\end{aligned}$$

Now, using (2.9), (2.13) and (3.3), we get

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_U V, Z) =& \mathfrak{g}_m(V, T_U\omega\alpha Z) + \mathfrak{g}_m(V, \hat{\nabla}_U\phi\alpha Z)\\ &-\tfrac{1}{\lambda_*^2}\mathfrak{g}_m((\nabla\Psi_*)(U, \varphi_*V), \Psi_*\beta Z)\\ &-\eta_*(\beta Z)\mathfrak{g}_m(U, V).\end{aligned} \tag{3.25}$$

The proof of the theorem comes from (3.24) and (3.25).

Theorem 3.17. *Let Ψ be a sic $\zeta^{\perp}$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. Then the ns conditions for the distribution $\mathfrak{D}_2^*$ to be a completely geodesic foliations on Υ are*

$$(\nabla\Psi_*)(W, \varphi_*U) \in \Gamma(\Psi_*\mu)$$

and

$$\begin{aligned}\frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla^{\Psi}_{\varphi_*\xi}\Psi_*\varphi_*W, \Psi_*\varphi_*\beta Z) &= \mathfrak{g}_m(\varphi_*\xi, T_W\alpha Z) - \mathfrak{g}_m(W,\xi)\mathfrak{g}_m(\mathcal{H}_1 grad\ ln\lambda_*, \varphi_*\beta Z)\\ &\quad +\eta_*(\beta Z)\mathfrak{g}_m(W,\xi).\end{aligned}$$

for $W, \xi \in \Gamma(\mathfrak{D}_2^*)$, $U \in \Gamma(\mathfrak{D}_1^*)$ and $Z \in \Gamma(ker\Psi_*^{\perp})$.

Proof. The ns conditions for the distribution $\mathfrak{D}_2^*$ to be a completely geodesic foliations on Υ are $\mathfrak{g}_m(\nabla^m_W\xi, U) = 0$ and $\mathfrak{g}_m(\nabla^m_W\xi, Z) = 0$ for $W, \xi \in \Gamma(\mathfrak{D}_2^*)$, $U \in \Gamma(\mathfrak{D}_1^*)$ and $Z \in \Gamma((\ker\Psi_*)^{\perp})$. Using (2.1), (2.2), (2.4) and (3.3), we come to the next equation

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_W\xi, U)) &= \mathfrak{g}_m(\varphi_*\nabla^m_W\xi, \varphi_*U) + \eta_*(\nabla^m_W\xi)\eta_*(U)\\ &= \mathfrak{g}_m(\nabla^m_W\varphi_*\xi, \varphi_*U) = -\mathfrak{g}_m(\varphi_*\xi, \nabla^m_W\varphi_*U)\end{aligned}$$

From (2.13) and definition of sic $\zeta^{\perp}$-Rs, we get

$$\mathfrak{g}_m(\nabla^m_W\xi, U)) = \frac{1}{\lambda_*^2}\mathfrak{g}_m((\nabla\Psi_*)(W, \varphi_*U), \Psi_*\varphi_*\xi). \tag{3.26}$$

With the help of (2.1), (2.2), (2.4), (2.8) and (3.4), we obtain

$$\mathfrak{g}_m(\nabla^m_W\xi, Z)) = -\mathfrak{g}_m(\varphi_*\xi, T_W\alpha Z) + \mathfrak{g}_m(\nabla^m_{\varphi_*\xi}W, \beta Z).$$

$$\mathfrak{g}_m(\nabla^m_W\xi, Z) = -\mathfrak{g}_m(\varphi_*\xi, T_W\alpha Z) + \mathfrak{g}_m(\nabla^m_{\varphi_*\xi}\varphi_*W, \varphi_*\beta Z) - \eta_*(\beta Z)\mathfrak{g}_m(W,\xi).$$

Then from (2.13), (3.4) and Lemma 2.6, we obtain

$$\begin{aligned}\mathfrak{g}_m(\nabla^m_W\xi, Z) &= -\mathfrak{g}_m(\varphi_*\xi, T_W\alpha Z) + \mathfrak{g}_m(W,\xi)\mathfrak{g}_m(\mathcal{H}_1 grad\ ln\lambda_*, \varphi_*\beta Z)\\ &\quad + \frac{1}{\lambda_*^2}\mathfrak{g}_n(\nabla^{\Psi}_{\varphi_*\xi}\Psi_*\varphi_*W, \Psi_*\varphi_*\beta Z) - \eta_*(\beta Z)\mathfrak{g}_m(W,\xi).\end{aligned} \tag{3.27}$$

The proof of the theorem comes from (3.26) and (3.27).

From Theorem 3.16 and 3.17, we have the following theorem:

Theorem 3.18. *Let Ψ be a sic $\zeta^{\perp}$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, \mathfrak{g}_m)$ onto Riemannian metric manifolds $(\Omega, \mathfrak{g}_n)$. Then the ns conditions for the fibers of Ψ to be locally product manifold are*

$$(\nabla\Psi_*)(U, \varphi_*V) \in \Gamma(\Psi_*\mu)$$

and

$$(\nabla\Psi_*)(W, \varphi_*U) \in \Gamma(\Psi_*\mu)$$

for any $U, V \in \Gamma(\mathfrak{D}_1^*)$ and $W \in \Gamma(\mathfrak{D}_2^*)$.

4. Harmonicity of semi-invariant conformal $\zeta^{\perp}$-Riemannian submersions

In this part, we provide the ns conditions for a sic $\zeta^{\perp}$-Rs to be harmonic. We also look into the ns conditions for such submersions to be completely geodesic. By decomposition of all over space of sic $\zeta^{\perp}$-Rs, we have the following Lemma.

Lemma 4.1. *Let $\Psi : (\Upsilon^{2(p+q+r)+1}, g_m, \varphi_*) \to (\Omega^{q+2r+1}, g_n)$ be a sic $\zeta^{\perp}$-Rs from acm manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ onto Riemannian metric manifolds (Ω, g_n). Then the tension field τ of Ψ is*

$$\tau(\Psi) = -(2p+q)\Psi_*(\mu^{\ker\Psi_*}) + (2-q-2r)\Psi_*(grad \ln \lambda_*), \tag{4.1}$$

where $\mu^{\ker\Psi_*}$ is the mean curvature vector field of the distribution of $\ker\Psi_*$.

Proof: It is easy to prove this lemma with the help of the method [1].

In view of Lemma 4.1, we have the following result.

Theorem 4.2. *Let $\Psi : (\Upsilon^{2(p+q+r)+1}, g_m, \varphi_*) \to (\Omega^{q+2r+1}, g_n)$ be a sic $\zeta^{\perp}$-Rs from acm manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ onto Riemannian metric manifolds (Ω, g_n) such that $q + 2r \neq 2$. Then any two of the conditions imply the third:*

(i) Ψ is harmonic

(ii) The fibers are minimal.

(iii) Ψ is a horizontally homothetic map.

Proof: Using (4.1), we can easily find the results.

Corollary 4.3. *Let $\Psi : (\Upsilon^{2(p+q+r)+1}, g_m, \varphi_*) \to (\Omega^{q+2r+1}, g_n)$ be a sic $\zeta^{\perp}$-Rs from acm manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ onto Riemannian metric manifolds (Ω, g_n). If $q + 2r = 2$ then the ns condition for the Ψ to be harmonic is the minimal fiber.*

A map is claimed to be totally geodesic map if it maps every geodesic in the total manifold into a geodesic within the base manifold in proportion to arc lengths. We recall that a differentiable map Ψ between two Riemannian metric manifolds is called completely geodesic if $(\nabla\Psi_*)(U, V) = 0 \;\; \forall U, V \in \Gamma(T\Upsilon)$.

We now, present the subsequent definition:

Definition 4.4. *Let Ψ be a sic $\zeta^{\perp}$-Rs from acm manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ onto Riemannian metric manifolds (Ω, g_n). Then Ψ is called a $(\varphi_*\mathfrak{D}_2^*, \mu)$-completely geodesic map if*

$$(\nabla\Psi_*)(\varphi_*W, Q) = 0 \;\; \textit{for } W \in \Gamma(\mathfrak{D}_2^*) \textit{ and } Q \in \Gamma(\mu).$$

In this sequence, we have the following important result.

Theorem 4.5. *Let Ψ be a sic $\zeta^{\perp}$-Rs from acm manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ onto Riemannian metric manifolds (Ω, g_n). The ns condition for Ψ to be a $(\varphi_*\mathfrak{D}_2^*, \mu)$-completely geodesic map if Ψ is a horizontally homothetic map.*

Proof. For $W \in \Gamma(\mathfrak{D}_2^*)$ and $Q \in \Gamma(\mu)$, from Lemma 2.6, we infer

$$(\nabla\Psi_*)(\varphi_*W, Q) = \varphi_*W(\ln\lambda_*)\Psi_*Q + Q(ln\lambda_*)\Psi_*\varphi_*W - g_m(\varphi_*W, Q)\Psi_*(grad \ln\lambda_*).$$

Since Ψ is a horizontally homothetic therefore $(\nabla\Psi_*)(\varphi_*W, Q) = 0$.
Conversely, if $(\nabla\Psi_*)(\varphi_*W, Q) = 0$, we have

$$\varphi_*W(\ln\lambda_*)\Psi_*Q + Q(\ln\lambda_*)\Psi_*\varphi_*W = 0 \tag{4.2}$$

Taking scaler product in (4.2) with $\Psi_*\varphi_*W$ and since Ψ is a sic $\zeta^\perp$-Rs, we infer

$$g_m(grad \ln \lambda_*, \varphi_*W)g_n(\Psi_*Q, \Psi_*\varphi_*W) + g_m(grad \ln \lambda_*, Q)g_n(\Psi_*\varphi_*W, \Psi_*\varphi_*W) = 0.$$

Since $\|\varphi_*W\| \neq 0$, then the above equation reduces to

$$g_m(\mathcal{H}_1 grad \ln \lambda_*, Q) = 0,$$

which implies that λ_* is a invariable on $\Gamma(\mu)$.
Taking scaler product in (4.2) with Ψ_*Q ,we get

$$g_m(grad \ln \lambda_*, \varphi_*W)g_n(\Psi_*Q, \Psi_*Q) + g_m(grad \ln \lambda_*, Q)g_n(\Psi_*\varphi_*W, \Psi_*Q) = 0.$$

Since $\|Q\| \neq 0$, then the above equation reduces to

$$g_m(grad \ln \lambda_*, \varphi_*W) = 0,$$

it follows that λ_* is a invariable on $\Gamma(\varphi_*\mathfrak{D}_2^*)$. Thus λ_* is a invariable on $\Gamma((ker\Psi_*)^\perp)$, which over the proof.
At last, we infer the ns conditions for sic $\zeta^\perp$-Rs to be completely geodesic.

Theorem 4.6. *Let Ψ be a sic $\zeta^\perp$-Rs from Sasakian manifolds $(\Upsilon, \varphi_*, \zeta, \eta_*, g_m)$ onto Riemannian metric manifolds (Ω, g_n). Then the ns conditions for the Ψ to be completely geodesic map are*

(a) $\beta T_U\varphi_*V + \omega\hat{\nabla}_U\varphi_*V - \eta_*(T_UV)\zeta = 0,\ U, V \in \Gamma(\mathfrak{D}_1^*)$,

(b) $\omega T_U\varphi_*W + \beta H\hat{\nabla}_U\varphi_*W - \eta_*(T_U^m W)\zeta = 0,\ U \in \Gamma(ker\Psi_*),\ W \in \Gamma(\mathfrak{D}_2^*)$,

(c) Ψ is a horizontally homothetic map,

(d) $\omega\mathcal{V}_1\nabla_P\phi U + \beta A_P\phi U + \beta\mathcal{H}_1\nabla_P\omega U + \omega A_P\omega U - \eta_*(A_PU)\zeta = 0$, $U \in \Gamma(ker\Psi_*)$, $P \in \Gamma((ker\Psi_*)^\perp)$.

Proof. (a) For $U, V \in \Gamma(\mathfrak{D}_1^*)$, using (2.1), (2.2), (2.4), (2.8) and (2.13), we have

$$\begin{aligned}(\nabla\Psi_*)(U, V) &= \nabla_U\Psi_*V - \Psi_*(\nabla_U^m V)\\ &= \Psi_*(\varphi_*^2\nabla_U^m V - \eta_*(\nabla_U^m V)\zeta)\\ &= \Psi_*(\varphi_*\nabla_U^m\varphi_*V - \eta_*(\nabla_U^m V)\zeta)\\ &= \Psi_*(\varphi_*(T_U\varphi_*V + \hat{\nabla}_U\varphi_*V) - \eta_*(T_UV)\zeta).\end{aligned}$$

With the help of (3.3) and (3.4) in the above expression turns into

$$(\nabla\Psi_*)(U, V) = \Psi_*(\alpha T_U\varphi_*V + \beta T_U\varphi_*V + \phi\hat{\nabla}_U\varphi_*V + \omega\hat{\nabla}_U\varphi_*V - \eta_*(\nabla_U^m V)\zeta),$$

since $\alpha T_U\varphi_*V + \phi\hat{\nabla}_U\varphi_*V \in \Gamma((ker\Psi_*)$, we infer

$$(\nabla\Psi_*)(U, V) = \Psi_*(\beta T_U\varphi_*V + \omega\hat{\nabla}_U\varphi_*V - \eta_*(\nabla_U^m V)\zeta).$$

Since Ψ is a linear isomorphism between $(ker\Psi_*)^\perp$ and $\Gamma(TN)$, so $(\nabla\Psi_*)(U, V) = 0$ if and only if $\beta T_U\varphi_*V + \omega\hat{\nabla}_U\varphi_*V - \eta_*(\nabla_U^m V)\zeta = 0$.
(b) For $U \in \Gamma(ker\Psi_*)$ and $W \in \Gamma(\mathfrak{D}_2^*)$, using (2.1), (2.2), (2.4) and 2.13), we come to the next equation

$$\begin{aligned}(\nabla\Psi_*)(U, W) &= \nabla_U\Psi_*W - \Psi_*(\nabla_U^m W)\\ &= \Psi_*(\varphi_*^2\nabla_U^m W - \eta_*(\nabla_U^m W)\zeta)\\ &= \Psi_*(\varphi_*\nabla_U^m\varphi_*W - \eta_*(T_U^m W)\zeta)\\ &= \Psi_*(\varphi_*(T_U\varphi_*W + \mathcal{H}_1\nabla_U\varphi_*W) - \eta_*(T_U^m W)\zeta).\end{aligned}$$

In view of (3.3) and (3.4) in the foregoing equation, we gain

$$(\nabla\Psi_*)(U,W) = \Psi_*(\phi T_U\varphi_*W + \omega T_U\varphi_*W + \alpha\mathcal{H}_1\nabla_U\varphi_*W + \beta\mathcal{H}_1\nabla_U\varphi_*W - \eta_*(T^m_U W)\zeta)$$

Since $\phi T_U\varphi_*W + \alpha\mathcal{H}_1\nabla_U\varphi_*W \in \Gamma(\ker\Psi_*)$, we infer

$$(\nabla\Psi_*)(U,W) = \Psi_*(\omega T_U\varphi_*W + \beta\mathcal{H}_1\nabla_U\varphi_*W - \eta_*(T^m_U W)\zeta).$$

Since Ψ is a linear isomorphism between $(ker\Psi_*)^\perp$ and $\Gamma(TN)$, so $(\nabla\Psi_*)(U,W) = 0$ if and only if $\omega T_U\varphi_*W + \beta H\hat{\nabla}_U\varphi_*W - \eta_*(T^m_U W)\zeta = 0$.
(c) For $U,V \in \Gamma(\mu)$, from Lemma 2.6, we get

$$(\nabla\Psi_*)(U,V) = U(\ln\lambda_*)\Psi_*V + V(\ln\lambda_*)\Psi_*U - \mathfrak{g}_m(U,V)\Psi_*(grad\ln\lambda_*).$$

In the above equation, replacing V by φ_*U, we infer

$$\begin{aligned}(\nabla\Psi_*)(U,\varphi_*U) &= U(\ln\lambda_*)\Psi_*\varphi_*U + \varphi_*U(\ln\lambda_*)\Psi_*U - \mathfrak{g}_m(U,\varphi_*U)\Psi_*(grad\ln\lambda_*).\\ &= U(\ln\lambda_*)\Psi_*\varphi_*U + \varphi_*U(\ln\lambda_*)\Psi_*U.\end{aligned}$$

If $(\nabla\Psi_*)(U,\varphi_*U) = 0$, we have

$$U(\ln\lambda_*)\Psi_*\varphi_*U + \varphi_*U(\ln\lambda_*)\Psi_*U = 0. \tag{4.3}$$

Taking scaler product of the above equation with $\Psi_*\varphi_*U$, we infer

$$\mathfrak{g}_m(grad\ln\lambda_*,U)\mathfrak{g}_n(\Psi_*\varphi_*U,\Psi_*\varphi_*U) + \mathfrak{g}_m(gradln\lambda_*,\varphi_*U)\mathfrak{g}_n(\Psi_*U,\Psi_*\varphi_*U) = 0,$$

which shows that λ_* is a invariable $\Gamma(\mu)$.
In a similar pattern, for $U,V \in \Gamma(\mathfrak{D}^*_2)$, using Lemma 2.6, we get

$$\begin{aligned}(\nabla\Psi_*)(\varphi_*U,\varphi_*V) &= \varphi_*U(ln\lambda_*)\Psi_*\varphi_*V + \varphi_*V(ln\lambda_*)\Psi_*\varphi_*U\\ &\quad - \mathfrak{g}_m(\varphi_*U,\varphi_*V)\Psi_*(grad\ ln\lambda_*).\end{aligned}$$

In the above equation, replacing V by U, we get

$$\begin{aligned}(\nabla\Psi_*)(\varphi_*U,\varphi_*U) &= \varphi_*U(ln\lambda_*)\Psi_*\varphi_*U + \varphi_*U(ln\lambda_*)\Psi_*\varphi_*U\\ &\quad - \mathfrak{g}_m(\varphi_*U,\varphi_*U)\Psi_*(grad\ ln\lambda_*)\\ &= 2\varphi_*U(ln\lambda_*)\Psi_*\varphi_*U - \mathfrak{g}_m(\varphi_*U,\varphi_*U)\Psi_*(grad\ ln\lambda_*).\end{aligned} \tag{4.4}$$

Consider $(\nabla\Psi_*)(\varphi_*U,\varphi_*U) = 0$ and taking scalar product of above equation with $\Psi_*\varphi_*U$ and using the fact that Ψ is sic $\zeta^\perp$-Rs, we infer

$$2\mathfrak{g}_m(grad\ ln\lambda_*,\varphi_*U)\mathfrak{g}_n(\Psi_*\varphi_*U,\Psi_*\varphi_*U) - \mathfrak{g}_m(\varphi_*U,\varphi_*U)\mathfrak{g}_n(\Psi_*(grad\ ln\lambda_*),\Psi_*\varphi_*U).$$

From the above expression, it follows that λ_* is a invariable on $\Gamma(\varphi_*\mathfrak{D}^*_2)$.
Thus λ_* is a invariable on $\Gamma((ker\Psi_*)^\perp)$.

If Ψ is a horizontally homothetic map, then $\mathcal{H}_1 grad\ ln\lambda_*$ vanishes.
Therefore $(\nabla\Psi_*)(U,V) = 0$ for $U,V \in \Gamma((ker\Psi_*)^\perp)$.
(d) For $U \in \Gamma(ker\Psi_*)$ and $P \in \Gamma((ker\Psi_*)^\perp)$, using (2.1), (2.2), (2.4), (2.9) and (2.13), we acquire

$$\begin{aligned}(\nabla\Psi_*)(P,U) &= \nabla_P\Psi_*U - \Psi_*(\nabla^m_P U)\\ &= \Psi_*(\varphi^2_*\nabla^m_P U - \eta_*(\nabla^m_P U)\zeta)\\ &= \Psi_*(\varphi_*\nabla^m_P\varphi_*U - \eta_*(\nabla^m_P U)\zeta).\end{aligned}$$

Then from (2.8), (2.9) and (3.4), we arrive at

$$
\begin{aligned}
(\nabla\Psi_*)(P,U) &= \Psi_*(\varphi_*\nabla^m_P(\phi+\omega)U - \eta_*(A_PU+\mathcal{V}_1\nabla_PU)\,\zeta)\\
&= \Psi_*(\varphi_*(A_P\phi U+\mathcal{V}_1\nabla_P\phi U)+\varphi_*(\mathcal{H}_1\nabla_P\omega U+A_P\omega U))\\
&\quad -\Psi_*(\eta_*(A_PU)\,\zeta).
\end{aligned}
$$

Using (3.3) and (3.4) in the above equation, we have

$$
\begin{aligned}
(\nabla\Psi_*)(P,U) &= \Psi_*(\phi\mathcal{V}_1\nabla_P\phi U+\omega\mathcal{V}_1\nabla_P\phi U+\alpha A_P\phi U+\beta A_P\phi U\\
&\quad +\alpha\mathcal{H}_1\nabla_P\omega U+\beta\mathcal{H}_1\nabla_P\omega U+\phi A_P\omega U+\omega A_P\omega U-\eta_*(A_PU)\,\zeta)
\end{aligned}
$$

Since $\phi\mathcal{V}_1\nabla_P\phi U+\alpha A_P\phi U+\alpha\mathcal{H}_1\nabla_P\omega U+\phi A_P\omega U\in\Gamma(ker\Psi_*)$, we infer

$$(\nabla\Psi_*)(P,U)=\Psi_*(\omega\mathcal{V}_1\nabla_P\phi U+\beta A_P\phi U+\beta\mathcal{H}_1\nabla_P\omega U+\omega A_P\omega U-\eta_*(A_PU)\,\zeta).$$

Since Ψ is a linear isomorphism between $(ker\Psi_*)^\perp$ and $\Gamma(TN)$, so $(\nabla\Psi_*)(P,U)=0$ if and only if $\omega\mathcal{V}_1\nabla_P\phi U+\beta A_P\phi U+\beta\mathcal{H}_1\nabla_P\omega U+\omega A_P\omega U-\eta_*(A_PU)\,\zeta=0$.

Conflict of interest: The authors declare that they have no conflict of interest.

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