The g-Drazin invertibility in a Banach algebra

Huanyin Chen, Marjan Sheibani Abdolyousef

Abstract. We present necessary and sufficient conditions under which the anti-triangular matrix \( \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \) over a Banach algebra has g-Drazin inverse. New additive results for g-Drazin inverse are obtained. Then we apply our results to \( 2 \times 2 \) operator matrices and generalize many known results, e.g., [5, Theorem 2.2], [13, Theorem 2.1] and [14, Theorem 4.1].

1. Introduction

Let \( \mathcal{A} \) be a Banach algebra with an identity. An element \( a \) in \( \mathcal{A} \) has g-Drazin inverse provided that there exists some \( b \in \mathcal{A} \) such that
\[
b = bab, ab = ba, \quad a - a^2 b \in \mathcal{A}^{\text{nil}}.
\]
Here, \( \mathcal{A}^{\text{nil}} = \{ a \in \mathcal{A} | 1 + ax \in \mathcal{A}^{-1} \text{ whenever } ax = xa \} \). That is, \( x \in \mathcal{A}^{\text{nil}} \iff \lim_{n \to \infty} \| x^n \| = 0 \). We say that \( a \in \mathcal{A} \) has Drazin inverse \( a^D \) if \( \mathcal{A}^{\text{nil}} \) is replaced by the set \( \mathcal{A}^{\text{nil}} \) of all nilpotents in \( \mathcal{A} \). As is well known, \( a \in \mathcal{A} \) has Drazin (resp. g-Drazin) inverse if and only if there exists an idempotent \( e \in \mathcal{A} \) such that \( ae = ea, a - e \) is invertible and \( ae \in \mathcal{A}^{\text{nil}} \) (resp. \( \mathcal{A}^{\text{nil}} \)). The Drazin and g-Drazin inverses play important roles in matrix and operator theory. They also were extensively studied in ring theory under strongly \( \pi \)-regularity and quasipolarity (see [3–5, 9, 12, 15, 16, 18]).

The solutions to singular systems of differential equations are determined by the Drazin (g-Drazin) inverses of certain anti-triangular block complex matrices (see [1]). This inspires to investigate the Drazin (g-Drazin) invertibility for the anti-triangular matrix \( M = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \in M_2(\mathcal{A}) \) with \( a, b \in \mathcal{A} \). In [8], Patrício and Hartwig considered the case \( ab = ba \) for the Drazin inverse of \( M \). For bounded linear operators on Banach spaces, a new expression of \( M^D \) was given under the same condition (see [12, Theorem 3.8]). Also Bu et al. gave the alternative representation of \( M^D \) for subblock complex matrices. In [16, Theorem 2.3], Zhang and Mosić presented the g-Drazin inverse of \( M \) under the condition \( bab^2 = 0 \). In [15, Theorem 2.6] the g-Drazin inverse \( M^d \) under the conditions \( b^2 ab^2 = 0, b^2 ba = 0 \) has been investigated. For the anti-triangular operator matrix \( M \) over a complex Hilbert space, Yu and Deng characterized its Drazin inverse under wider
conditions \(b^nab^D = 0\), \(b^nab = b^nba\) and \(b^nb^D = 0\), \(b^nab(b^n) = 0\), \((b^n)^D b^nab = 0\) (see [14, Theorem 4.1]). These conditions were also considered in [17, Theorem 2.12].

In Section 2, we present necessary and sufficient conditions under which the anti-triangular matrix 
\[
\begin{pmatrix}
a & 1 \\
0 & 0
\end{pmatrix}
\]
over a Banach algebra has g-Drazin inverse. [14, Theorem 4.1] and [17, Theorem 2.12] are thereby extended to a more general setting.

Let \(a, b \in A\). Many authors have studied when \(a+b \in A\) has Drazin (g-Drazin) inverse. In [14, Theorem 2.1], Yang and Liu considered the conditions \(ab^2 = 0\) and \(aba = 0\). In [2, Theorem 2.4], the authors extend to the conditions \(ab^2 = 0\) and \(b^2aba = 0\). In [9, Theorem 3.1], for the setting of complex matrices, Shakoor et al. investigated the Drazin inverse of \(a+b\) under the conditions \(ab^2 = 0\) and \(a^2ba = 0\). These conditions were also considered in [10, Theorem 3.1]. We refer the reader for more related papers, e.g., [11, 13, 16, 18].

In Section 3, we apply our results to establish some new additive results. Let \(a, b, ab, (ab)^n a \in A\). If \(ab^2 = 0\), \((ab)^n a(ab)^d = 0\) and \((ab)^n aba = 0\), we prove that \(a+b \in A\). This also extends the existing results above.

Let \(X, Y\) be Banach spaces and 
\[
M = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]
(\(^*\)), where \(A \in \mathcal{L}(X), B \in \mathcal{L}(X, Y), C \in \mathcal{L}(Y, X)\) and \(D \in \mathcal{L}(Y)\). Then \(M\) is a bounded linear operator on \(X \oplus Y\). Finally, in the last section, we split \(M\) into the sum of two block operator matrices. We then establish new results for the g-Drazin inverse of \(2 \times 2\) block operator matrix \(M\). These also recover some known results, e.g., [5, Theorem 2.2].

Throughout the paper, we use \(\mathcal{A}\) to denote the set of all g-Drazin invertible elements in \(A\). Let \(a \in \mathcal{A}\). The spectral idempotent \(1 - aa^D\) is denoted by \(a^D\). \(\mathcal{L}(X)\) denotes the Banach algebra of all bounded linear operators on the Banach space \(X\). \(\mathbb{C}^{n \times n}\) stands for the Banach algebra of all \(n \times n\) complex matrices.

2. Anti-triangular matrices over Banach algebra

The aim of this section is to investigate the g-Drazin invertibility of the operator matrix 
\[
\begin{pmatrix}
a & b \\
1 & 0
\end{pmatrix}
\]
over a Banach algebra \(A\). We begin with

Lemma 2.1. (see [16, Lemma 1.3]) Let \(a, b \in \mathcal{A}\). If \(ab = 0\), then \(a+b \in \mathcal{A}\) and
\[
(a+b)^d = \sum_{i=0}^{\infty} b^ib^\pi(a^D)^{i+1} + \sum_{i=0}^{\infty} (b^d)^{i+1} a^da^\pi.
\]

Theorem 2.2. Let \(a, b \in \mathcal{A}\) and \(b^\pi ab^d = 0\). Then the following are equivalent:

1. \(\begin{pmatrix}
a & 1 \\
0 & 0
\end{pmatrix}\) has g-Drazin inverse.
2. \(\begin{pmatrix}
b^\pi a & 1 \\
b^\pi b & 0
\end{pmatrix}\) has g-Drazin inverse.
3. \(\begin{pmatrix}
ab^\pi & 1 \\
nb^\pi & 0
\end{pmatrix}\) has g-Drazin inverse.

Proof. \((1) \Rightarrow (2)\) Let \(M = \begin{pmatrix}
a & 1 \\
0 & 0
\end{pmatrix}\) and \(p = \begin{pmatrix}
b^\pi & 0 \\
0 & b^\pi
\end{pmatrix}\). Since \(b^\pi ab^d = 0\), we see that
\[
pM(1-p) = 0, M(1-p) = \begin{pmatrix}
bb^\pi ab^d & bb^d \\
bb^d & 0
\end{pmatrix}.
\]

Obviously, \([M(1-p)]^d = \begin{pmatrix}
0 & b^d \\
bb^d & -ab^d
\end{pmatrix}\). In view of [16, Lemma 2.2], \(pM = \begin{pmatrix}
b^\pi a & b^\pi \\
b^\pi b & 0
\end{pmatrix}\) has g-Drazin inverse.
Let $N = \begin{pmatrix} b_0 & 1 \\ b_{-1} & 0 \end{pmatrix}$. Then we have

$$N = \begin{pmatrix} 1 & 0 \\ 0 & b_0 \end{pmatrix} \begin{pmatrix} b_0 & 1 \\ b_{-1} & 0 \end{pmatrix},$$

$$\begin{pmatrix} b_0 & b_0 \\ b_{-1} & 0 \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ b_{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b_0 \end{pmatrix}. $$

By using Cline’s formula (see [6, Theorem 2.1]), $N$ has g-Drazin inverse.

(2) ⇒ (1) Let $e = \begin{pmatrix} b^{d} & 0 \\ 0 & 1 \end{pmatrix}$. Then $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where

$$\alpha = \begin{pmatrix} b^{d}a b^{d} & b^{d} \\ b^{d} & 0 \end{pmatrix}, \beta = \begin{pmatrix} b^{d}a b^{d} & 0 \\ b^{d} & 0 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 0 & b^{n} \\ 0 & 0 \end{pmatrix}, \delta = \begin{pmatrix} b^{n}a & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\alpha^{\#} = \begin{pmatrix} 0 & b^{d} \\ b^{d} & -ab^{d} \end{pmatrix}, \alpha^{\#} = \begin{pmatrix} b^{n} & 0 \\ 0 & b^{n} \end{pmatrix},$$

$$\beta + \gamma + \delta = \begin{pmatrix} b^{d}a b^{n} & b^{n}a \\ b^{n} & 0 \end{pmatrix}.$$

and

$$(\beta + \gamma + \delta)\alpha = \begin{pmatrix} b^{d}a b^{n} & b^{n}a \\ b^{n} & 0 \end{pmatrix} \begin{pmatrix} b^{d}a b^{d} & b^{d} \\ b^{d} & 0 \end{pmatrix} = 0.$$

Moreover, we have

$$\beta + \gamma + \delta = \begin{pmatrix} b^{d}a + b^{n}a & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} b^{n} & 0 \\ 0 & b^{n} \end{pmatrix};$$

$$\begin{pmatrix} b^{n} & b^{n} \\ b^{n} & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^{n} \\ b^{n} & 0 \end{pmatrix} \begin{pmatrix} b^{d}a + b^{n}a & 1 \\ b & 0 \end{pmatrix};$$

$$\begin{pmatrix} b^{n} & b^{n} \\ b^{n} & 0 \end{pmatrix} = \begin{pmatrix} b^{n} & 0 \\ 0 & b^{n} \end{pmatrix} \begin{pmatrix} b^{n} & 0 \\ 0 & b^{n} \end{pmatrix};$$

$$\begin{pmatrix} b^{n} & 1 \\ b^{n} & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^{n} \\ b^{n} & 0 \end{pmatrix} \begin{pmatrix} b^{n} & 1 \\ b^{n} & 0 \end{pmatrix}.$$ 

Since $N$ has g-Drazin inverse, by using Cline’s formula, $\beta + \gamma + \delta$ has g-Drazin inverse. Clearly, we have

$$\begin{pmatrix} b^{n} & b^{n} \\ b^{n} & 0 \end{pmatrix}^{d} = \begin{pmatrix} b^{n} & 1 \\ b^{n} & 0 \end{pmatrix} \begin{pmatrix} b^{n} & 0 \\ 0 & b^{n} \end{pmatrix}^{d} = N^{d} \begin{pmatrix} 1 & 0 \\ 0 & b^{n} \end{pmatrix}.$$ 

Therefore

$$(\beta + \gamma + \delta)^{d} = \begin{pmatrix} b^{d}a + b^{n}a & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} b^{n} & 0 \\ 0 & b^{n} \end{pmatrix} \begin{pmatrix} b^{n} & 0 \\ 0 & b^{n} \end{pmatrix}^{d} = \begin{pmatrix} b^{n} & 0 \\ 0 & b^{n} \end{pmatrix} N^{d} \begin{pmatrix} b^{n} & 0 \\ 0 & b^{n} \end{pmatrix}.$$ 

In light of Lemma 2.1, $M$ has g-Drazin inverse. In fact, we get

$$M^{d} = \sum_{i=0}^{\infty} a^{i}a^{\#}[\beta + \gamma + \delta]^{d+1} + \sum_{i=0}^{\infty} [a^{\#}]^{i+1}(\beta + \gamma + \delta)^{i}(\beta + \gamma + \delta)^{i}$$

$$= a^{\#}(\beta + \gamma + \delta)^{d} + \sum_{i=0}^{\infty} [a^{\#}]^{i+1}(\beta + \gamma + \delta)^{i}(\beta + \gamma + \delta)^{i}.$$
Obviously, we have
\[
\begin{align*}
(\begin{array}{cc}
\pi a & 1 \\
\pi b & 0
\end{array}) &= (\begin{array}{cc}
1 & 0 \\
0 & \pi b
\end{array}) (\begin{array}{cc}
\pi a & 1 \\
\pi b & 0
\end{array}), \\
(\begin{array}{cc}
\pi a & \pi a \\
\pi b & 0
\end{array}) &= (\begin{array}{cc}
\pi a & 1 \\
\pi b & 0
\end{array})(\begin{array}{cc}
1 & 0 \\
0 & \pi b
\end{array}), \\
(\begin{array}{cc}
\pi a & \pi b \\
\pi b & 0
\end{array}) &= (\begin{array}{cc}
\pi a & 0 \\
0 & \pi b
\end{array})(\begin{array}{cc}
a & 1 \\
b & 0
\end{array}), \\
(\begin{array}{cc}
\pi a & a \\
\pi b & 0
\end{array}) &= (\begin{array}{cc}
1 & 0 \\
0 & \pi b
\end{array})(\begin{array}{cc}
a & 1 \\
b & 0
\end{array}), \\
(\begin{array}{cc}
\pi a & b \\
\pi b & 0
\end{array}) &= (\begin{array}{cc}
1 & 0 \\
0 & \pi b
\end{array})(\begin{array}{cc}
1 & 0 \\
0 & \pi b
\end{array}).
\end{align*}
\]
Therefore we complete the proof by repeatedly using Cline’s formula (see [6, Theorem 2.1]).

**Corollary 2.3.** Let \( a, b, \pi a \in \mathcal{A} \). If \( \pi a b = 0, \pi ab = \pi ab \), then \( \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \) has \( g \)-Drazin inverse.

**Proof.** Let \( N = \begin{pmatrix} \pi a & 1 \\ \pi b & 0 \end{pmatrix} \). We check that
\[
(ab)(bb) = \pi ab = \pi ba = bab = (bb)(ab).
\]
As the argument in [8], \( N \) has \( g \)-Drazin inverse. This completes the proof by Theorem 2.2.

Yu et al. characterized the Drazin invertibility of an anti-triangular matrix over a complex Hilbert space by using solutions of certain operator equations (see [14, Theorem 4.1]). We now generalize their main results to the \( g \)-Drazin inverse in a Banach algebra by using ring technique as follows.

**Corollary 2.4.** Let \( a, b, \pi a \in \mathcal{A} \). If \( \pi ab = 0, \pi ab = \pi ab \), then \( \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \) has \( g \)-Drazin inverse.

**Proof.** Let \( N = \begin{pmatrix} \pi a & 1 \\ \pi b & 0 \end{pmatrix} \). By hypothesis, we have
\[
(b)(bb) = \pi ab = \pi ba = bab = (bb)(ab).
\]
As the argument in [8], \( N \) has \( g \)-Drazin inverse. We obtain the result by Theorem 2.2.

**Corollary 2.5.** Let \( a, b, \pi a \in \mathcal{A} \). If \( \pi ab = 0, (\pi a)^2 \pi ab = 0, \pi ab (\pi a)^2 = 0 \), then \( \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \) has \( g \)-Drazin inverse.

**Proof.** Let \( N = \begin{pmatrix} \pi a & 1 \\ \pi b & 0 \end{pmatrix} \). Then \( N = P + Q \), where
\[
P = \begin{pmatrix} (\pi a)^2 (\pi a) & (\pi a)(\pi a) \\ 0 & 0 \end{pmatrix}, \
Q = \begin{pmatrix} (\pi a)(\pi a) & (\pi a)(\pi a) \\ (\pi a)(\pi a) & (\pi a)(\pi a) \end{pmatrix}
\]
Clearly, \( P \) has \( g \)-Drazin inverse and \( PQ = 0 \).
Write $\begin{pmatrix} (b^a)(b^a)^n & (b^a)(b^a)^n(b^a) b^n a^n \\ 1 & 0 \end{pmatrix} = K + L$, where

$$K = \begin{pmatrix} (b^a)(b^a)^n & 0 \\ 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & (b^a)^n b^n b^n a^n \\ 1 & 0 \end{pmatrix}.$$ 

Obviously, $(b^a)(b^a)^n$ is quasinilpotent. Hence, $K$ is quasinilpotent.

One easily checks that

$$(b^a)^n(b^a)b^n = [1 - (b^a)(b^a)](b^a)b^n = b^a b^n - b^a b^n a^n(1 - b^a)b^n = b^a b^n - b^a b^n a^n + (b^a)(b^a)b^n d^n b^2 = b^a b.$$ 

Then $(b^a)^n(b^a)b^n$ is quasinilpotent. By using Cline’s formula, $(b^a)^n(b^a)b^n a^n$ is quasinilpotent. Accordingly, $L$ is quasinilpotent.

By hypothesis, we check that

$$[(b^a)^n(b^a)]((b^a)^n(b^a)b^n a^n) = (b^a)^n(b^a)(b^a)b^n a^n = (b^a)^n(b^a)(b^a)b^n a^n = 0.$$ 

By Lemma 2.1, $\begin{pmatrix} (b^n a^n b^n a^n) & (b^n a^n b^n a^n) \\ 1 & 0 \end{pmatrix}$ is quasinilpotent. We verify that

$$\begin{pmatrix} b^a b^a a^n & (b^a)^n \\ (b^a)^n b^n b^n b^n & 0 \end{pmatrix}$$

$$= \begin{pmatrix} b^a b^a a^n & (b^a)^n \\ (b^a)^n b^n b^n b^n & 0 \end{pmatrix}$$

$$= \begin{pmatrix} b^a b^a a^n & (b^a)^n \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} b^a b^a a^n & (b^a)^n \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} b^a b^a a^n & (b^a)^n \\ 1 & 0 \end{pmatrix}$$

By using Cline’s formula, $Q$ has quasinilpotent. In view of Lemma 2.1, $N$ has g-Drazin inverse. According to Theorem 2.2, we complete the proof. □

We are now ready to prove the following:

**Theorem 2.6.** Let $a, b, b^n a^n \in \mathcal{A}$. If $b^n (a b^2) = 0$ and $b^n (a b) = 0$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.

**Proof.** Since $b^a (ab^2) = 0$, we have $b^a ab^2 = b^a ab^2 (b^2) = 0, b^a a(b^a b^2) = 0, b^a a(b^a b^2) b^a a = 0$. 

$$b^a b^2 = b^a b^2 (b^2) = 0, b^a a(b^a b^2) = 0, b^a a(b^a b^2) b^a a = 0.$$
Let $N = \begin{pmatrix} b^n a & 1 \\ b & 0 \\ 0 & 0 \end{pmatrix}$. Then $N^2 = \begin{pmatrix} (b^n a)^2 + b^n b & b^n a \\ b^n bb^n a & b^n b \end{pmatrix}$. Write $N^2 = P + Q$, where

$$P = \begin{pmatrix} (b^n a)^2 & b^n a \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} b^n b \\ b^n bb^n a \\ b^n b \end{pmatrix}.$$ 

Obviously, we have

$$PQ^2 = \begin{pmatrix} (b^n a)^2 b^n b & b^n ab \\ 0 & 0 \end{pmatrix}, \quad PQP = \begin{pmatrix} (b^n a)^2 b^n b & b^n ab \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b^n b \\ b^n bb^n a \\ b^n b \end{pmatrix} = 0.$$ 

By virtue of [2, Theorem 2.4], $N^2$ has g-Drazin inverse. It follows from [7, Corollary 2.2] that $N$ has g-Drazin inverse. In light of Theorem 2.2, the result follows.

**Corollary 2.7.** Let $a, b \in \mathcal{A}$. If $ab^2 = 0$ and $aba = 0$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.

**Proof.** Since $ab^2 = 0$, $ab^2 = a - (ab^2)(b^2) = a \in \mathcal{A}$. By Cline’s formula, $b^n a \in \mathcal{A}$. This completes the proof by Theorem 2.6.

**Corollary 2.8.** Let $a, b, b^n a \in \mathcal{A}$. If $b^n ab = 0$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.

**Proof.** Since $b^n ab = 0$, we see that $b^n ab^2 = 0$ and $b^n (aba) = 0$. So the corollary is true by Theorem 2.6.

3. Additive properties

In this section we establish some elementary additive properties of g-Drazin inverse in a Banach algebra. The following fact will also be used in our subsequent investigations.

**Theorem 3.1.** Let $a, b, ab, (ab)^n a \in \mathcal{A}$. If $ab^2 = 0$, $(ab)^n a(ab)^d = 0$ and $(ab)^n aba = 0$, then $a + b \in \mathcal{A}$.

**Proof.** Obviously, we have $a + b = (1, b) \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$. In view of Cline’s formula, it suffices to prove

$$M = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} (1, b) = \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}$$

has g-Drazin inverse. Write $M = K + L$, where

$$K = \begin{pmatrix} a & ab \\ 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$ 

Let $H = \begin{pmatrix} a & 1 \\ ab & 0 \end{pmatrix}$ and $N = \begin{pmatrix} (ab)^n a & 1 \\ (ab)^n ab & 0 \end{pmatrix}$. One easily checks that

$$(ab)^n a[(ab)^n a]^2 = (ab)^n a(ab)^d(ab)a = 0,$$

$$(ab)^n a[(ab)^n ab][(ab)^n ab]^2 = (ab)^n a(ab)^d(ab)a = 0.$$ 

In light of Corollary 2.7, $N$ has g-Drazin inverse. By hypothesis, $(ab)^n a(ab)^d = 0$. According to Theorem 2.2, $H$ has g-Drazin inverse. Clearly,

$$H = \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix} \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}.$$ 

By using Cline’s formula, $K$ has g-Drazin inverse. Since $ab^2 = 0$, we have $KL = 0$. In light of Lemma 2.1, $M$ has g-Drazin inverse. Therefore $a + b \in \mathcal{A}$. 

Corollary 3.2. Let \( a, b, ab, (ab)^{\pi} \in \mathcal{A}^d \). If \( a^2b = 0, (ab)^{4}b(ab)^{\pi} = 0 \) and \( bab(ab)^{\pi} = 0 \), then \( a + b \in \mathcal{A}^d \).

Proof. Since \((\mathcal{A}, \cdot)\) is a Banach algebra, \((\mathcal{A}, *)\) is a Banach algebra with the multiplication \( x \circ y = y \cdot x \). Then we complete the proof by applying Theorem 3.1 to the Banach algebra \((\mathcal{A}, *)\). \( \square \)

We are now ready to generalize [9, Theorem 3.1] as follow:

Theorem 3.3. Let \( a, b, ab \in \mathcal{A}^d \). If \( ab^2 = 0 \) and \( (ab)^{\pi}a^2ba = 0 \), then \( a + b \in \mathcal{A}^d \).

Proof. Let \( M = \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix} \). Write \( M = K + L \), where

\[
K = \begin{pmatrix} a & ab \\ 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.
\]

Let \( H = \begin{pmatrix} a & 1 \\ ab & 0 \end{pmatrix} \). By hypothesis, we check that

\[
(ab)^{\pi}a(ab)^{\pi} = 0, (ab)^{\pi}a(ab)a = 0.
\]

According to Theorem 2.6, \( H \) is g-Drazin inverse. As in the proof of Theorem 3.1, by using Cline’s formula, \( K \) has g-Drazin inverse. Since \( ab^2 = 0 \), it follows by Lemma 2.1 that \( M \) has g-Drazin inverse. Observing that

\[
a + b = (1, b) \begin{pmatrix} a \\ 1 \end{pmatrix}, \quad M = \begin{pmatrix} a \\ 1 \end{pmatrix}(1, b),
\]

by using Cline’s formula again, \( a + b \) has g-Drazin inverse. \( \square \)

Corollary 3.4. Let \( a, b, ab \in \mathcal{A}^d \). If \( a^2b = 0 \) and \( bab^2(ab)^{\pi} = 0 \), then \( a + b \in \mathcal{A}^d \).

Proof. Similarly to Corollary 3.2, we obtain the result by Theorem 3.3. \( \square \)

Corollary 3.5. Let \( a, b, ab \in \mathcal{A}^d \). If \( ab^2 = 0 \) and \( a^2ba = 0 \), then \( a + b \in \mathcal{A}^d \).

Proof. This is obvious by Theorem 3.3. \( \square \)

4. Operator matrices over Banach spaces

In this section we apply our results to establish g-Drazin invertibility for the block operator matrix \( M \) as in (\( \ast \)). Throughout this section, we always assume that \( A, D, BC \in \mathcal{L}(X)^d \). We come now to extend [14, Theorem 3.1] as follows.

Theorem 4.1. If \((BC)^{\pi}ABCA = 0, (BC)^{\pi}ABC\pi B = 0, DCA = 0 \) and \( DCB = 0 \), then \( M \) has g-Drazin inverse.

Proof. Write \( M = P + Q \), where

\[
P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.
\]

Clearly, \( Q^2 = 0 \), and so \( PQ^2 = 0 \). Moreover, we have

\[
PQ = \begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix}, \quad (PQ)^{\pi} = \begin{pmatrix} BC^{\pi} & 0 \\ DC^{\pi}[BC]^{\pi} & 0 \end{pmatrix} = \begin{pmatrix} (BC)^{\pi} & 0 \\ 0 & 0 \end{pmatrix}, \quad (PQ)^{\pi} = \begin{pmatrix} BC^{\pi} & 0 \\ -DC[BC]^{\pi} & 1 \end{pmatrix} = \begin{pmatrix} (BC)^{\pi} & 0 \\ 0 & 1 \end{pmatrix}.
\]
We easily check that
\[
(PQ)^d P^2 Q P = \begin{pmatrix}
BC^d & 0 \\
-DC(BC)^d & I
\end{pmatrix}
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}
\begin{pmatrix}
BC & 0 \\
0 & D
\end{pmatrix}
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}
= \begin{pmatrix}
BC & 0 \\
0 & D
\end{pmatrix}
\begin{pmatrix}
BC & 0 \\
0 & D
\end{pmatrix}
= 0.
\]

Therefore we complete the proof by Theorem 3.3. □

**Corollary 4.2.** If \((BC)^d A^{\circ} B C A = 0, (BC)^d A B C B = 0, B D C = 0 \) and \(B D^2 = 0\), then \(M\) has g-Drazin inverse.

**Proof.** Write \(M = P + Q\), where
\[
P = \begin{pmatrix}
A & B \\
C & 0
\end{pmatrix}, Q = \begin{pmatrix}
0 & 0 \\
0 & D
\end{pmatrix}.
\]

In light of Theorem 4.1, \(P\) has g-Drazin inverse. Since \(PQ^2 = 0\) and \(PQP = 0\), we complete the proof by [2, Theorem 2.4]. □

**Theorem 4.3.** If \((BC)^d A(BC)^d = 0, (BC)^d BCA = 0, (BC)^d BCB = 0, DCA = 0 \) and \(DCB = 0\), then \(M\) has g-Drazin inverse.

**Proof.** Write \(M = P + Q\), where
\[
P = \begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}, Q = \begin{pmatrix}
0 & 0 \\
C & 0
\end{pmatrix}.
\]

Then \(Q^2 = 0\) and \(PQ = \begin{pmatrix}
BC & 0 \\
DC & 0
\end{pmatrix}\); hence,
\[
(PQ)^d = \begin{pmatrix}
(BC)^d & 0 \\
0 & 0
\end{pmatrix}, (PQ)^{\circ} = \begin{pmatrix}
(BC)^{\circ} & 0 \\
0 & I
\end{pmatrix}.
\]

By hypothesis, we verify that
\[
(PQ)^d P(QP)^d = \begin{pmatrix}
BC^d & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}
\begin{pmatrix}
BC & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
(BC)^d & 0 \\
0 & 0
\end{pmatrix}
= \begin{pmatrix}
BC^d & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
BC & 0 \\
0 & D
\end{pmatrix}
\begin{pmatrix}
A & B \\
0 & D
\end{pmatrix}
= 0.
\]

This completes the proof by Theorem 3.1. □

**Corollary 4.4.** If \((BC)^d A(BC)^d = 0, (BC)^d BCA = 0, (BC)^d BCB = 0, B D C = 0 \) and \(B D^2 = 0\), then \(M\) has g-Drazin inverse.

**Proof.** Write \(M = P + Q\), where
\[
P = \begin{pmatrix}
A & B \\
C & 0
\end{pmatrix}, Q = \begin{pmatrix}
0 & 0 \\
0 & D
\end{pmatrix}.
\]

In light of Theorem 4.3, \(P\) is g-Drazin inverse. As in the proof of Corollary 4.2, \(M\) has g-Drazin inverse. □
Theorem 4.5. If \((CB)^n ABC = 0, A(BC)^n ABC = 0, ABD = 0\) and \(CBD = 0\), then \(M\) has g-Drazin inverse.

Proof. Write \(M = P + Q\), where

\[
P = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, 
Q = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}.
\]

Then

\[
PQ^2 = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}^2 = \begin{pmatrix} 0 & ABD \\ 0 & CBD \end{pmatrix},
\]

\[
(PQ)^n P^2 QP = \begin{pmatrix} 0 & A \end{pmatrix} \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix} \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} = \begin{pmatrix} I & -AB/(CB)^n \\ 0 & (CB)^n \end{pmatrix} \begin{pmatrix} A^2 BC & 0 \\ CABC & 0 \end{pmatrix}.
\]

Clearly, \(P\) and \(Q\) have g-Drazin inverses. Moreover, \(PQ^2 = 0\) and \((PQ)^n P^2 QP = 0\), and therefore we complete the proof by Theorem 3.3. \(\square\)

We now generalize [5, Theorem 2.2] as follow.

Corollary 4.6. If \((CB)^n ABC = 0, A(BC)^n ABC = 0, BDC = 0\) and \(BD^2 = 0\), then \(M\) has g-Drazin inverse.

Proof. As in the proof of Corollary 4.2, we are through by Theorem 4.5. \(\square\)

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