Filomat 37:14 (2023), 4639–4647
https://doi.org/10.2298/FIL2314639C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The g-Drazin invertibility in a Banach algebra

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Abstract. We present necessary and sufficient conditions under which the anti-triangular matrix $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ over a Banach algebra has g-Drazin inverse. New additive results for g-Drazin inverse are obtained. Then we apply our results to 2×2 operator matrices and generalize many known results, e.g., [5, Theorem 2.2], [13, Theorem 2.1] and [14, Theorem 4.1].

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with an identity. An element a in $\mathcal{A}$ has g-Drazin inverse provided that there exists some $b \in \mathcal{A}$ such that

$$b = bab, ab = ba, a - a^2b \in \mathcal{A}^{qnil}.$$

Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + ax \in \mathcal{A}^{-1} \text{ whenever } ax = xa\}$. That is, $x \in \mathcal{A}^{qnil} \Leftrightarrow \lim\limits_{n\to\infty} \parallel x^n \parallel^{\frac{1}{n}} = 0$. We say that $a \in \mathcal{A}$ has Drazin inverse a^D if $\mathcal{A}^{qnil}$ is replaced by the set $\mathcal{A}^{nil}$ of all nilpotents in $\mathcal{A}$. As is well known, $a \in \mathcal{A}$ has Drazin (resp. g-Drazin) inverse if and only if there exists an idempotent $e \in \mathcal{A}$ such that $ae = ea, a - e$ is invertible and $ae \in \mathcal{A}^{nil}(resp.\mathcal{A}^{qnil})$. The Drazin and g-Drazin inverses play important roles in matrix and operator theory. They also were extensively studied in ring theory under strongly π-regularity and quasipolarity (see [3–5, 9, 12, 15, 16, 18]).

The solutions to singular systems of differential equations are determined by the Drazin (g-Drazin) inverses of certain anti-triangular block complex matrices (see [1]). This inspires to investigate the Drazin (g-Drazin) invertibility for the anti-triangular matrix $M = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \in M_2(\mathcal{A})$ with $a, b \in \mathcal{A}^d$. In [8], Patrício and Hartwig considered the case $ab = ba$ for the Drazin inverse of M. For bounded linear operators on Banach spaces, a new expression of M^D was given under the same condition (see [12, Theorem 3.8]). Also Bu et al. gave the alternative representation of M^D for subblock complex matrices. In [16, Theorem 2.3], Zhang and Mosić presented the g-Drazin inverse of M under the condition $bab^\pi = 0$. In [15, Theorem 2.6] the g-Drazin inverse M^d under the conditions $b^dab^\pi = 0, b^\pi ba = 0$ has been investigated. For the anti-triangular operator matrix M over a complex Hilbert space, Yu and Deng characterized its Drazin inverse under wider

2020 *Mathematics Subject Classification*. 15A09, 47C05, 16U99
Keywords. g-Drazin inverse; anti-triangular matrix; operator matrix; Banach algebra
Received: 04 August 2022; Revised: 14 September 2022; Accepted: 08 October 2022
Communicated by Dijana Mosić
Research supported by the Natural Science Foundation of Zhejiang Province, China (No. LY21A010018).
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conditions $b^\pi ab^D = 0, b^\pi ab = b^\pi ba$ and $b^\pi ab^D = 0, b^\pi ab(b^\pi a)^\pi = 0, (b^\pi a)^D b^\pi ab = 0$ (see [14, Theorem 4.1]). These conditions were also considered in [17, Theorem 2.12].

In Section 2, we present necessary and sufficient conditions under which the anti-triangular matrix $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ over a Banach algebra has g-Drazin inverse. [14, Theorem 4.1] and [17, Theorem 2.12] are thereby extended to a more general setting.

Let $a, b \in \mathcal{A}^d$. Many authors have studied when $a + b \in \mathcal{A}$ has Drazin (g-Drazin) inverse. In [14, Theorem 2.1], Yang and Liu considered the conditions $ab^2 = 0$ and $aba = 0$. In [2, Theorem 2.4], the authors extend to the conditions $ab^2 = 0$ and $b^\pi aba = 0$. In [9, Theorem 3.1], for the setting of complex matrices, Shakoor et al. investigated the Drazin inverse of $a + b$ under the conditions $ab^2 = 0$ and $a^2ba = 0$. These conditions were also considered in [10, Theorem 3.1]. We refer the reader for more related papers, e.g., [11, 13, 16, 18].

In Section 3, we apply our results to establish some new additive results. Let $a, b, ab, (ab)^\pi a \in \mathcal{A}^d$. If $ab^2 = 0, (ab)^\pi a(ab)^d = 0$ and $(ab)^\pi aba = 0$, we prove that $a + b \in \mathcal{A}^d$. This also extends the existing results above.

Let X, Y be Banach spaces and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $(*)$, where $A \in \mathcal{L}(X), B \in \mathcal{L}(X, Y), C \in \mathcal{L}(Y, X)$ and $D \in \mathcal{L}(Y)$. Then M is a bounded linear operator on $X \oplus Y$. Finally, in the last section, we split M into the sum of two block operator matrices. We then establish new results for the g-Drazin inverse of 2×2 block operator matrix M. These also recover some known results, e.g., [5, Theorem 2.2].

Throughout the paper, we use $\mathcal{A}^d$ to denote the set of all g-Drazin invertible elements in $\mathcal{A}$. Let $a \in \mathcal{A}^d$. The spectral idempotent $1 - aa^d$ is denoted by a^π. $\mathcal{L}(X)$ denotes the Banach algebra of all bounded linear operators on the Banach space X. $\mathbb{C}^{n\times n}$ stands for the Banach algebra of all $n \times n$ complex matrices.

2. Anti-triangular matrices over Banach algebra

The aim of this section is to investigate the g-Drazin invertibility of the operator matrix $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ over a Banach algebra $\mathcal{A}$. We begin with

Lemma 2.1. *(see [16, Lemma 1.3]) Let $a, b \in \mathcal{A}^d$. If $ab = 0$, then $a + b \in \mathcal{A}^d$ and*

$$(a + b)^d = \sum_{i=0}^{\infty} b^i b^\pi (a^d)^{i+1} + \sum_{i=0}^{\infty} (b^d)^{i+1} a^i a^\pi.$$

Theorem 2.2. *Let $a, b \in \mathcal{A}^d$ and $b^\pi ab^d = 0$. Then the following are equivalent:*

(1) $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.

(2) $\begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}$ has g-Drazin inverse.

(3) $\begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix}$ has g-Drazin inverse.

Proof. (1) $\Rightarrow$ (2) Let $M = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ and $p = \begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix}$. Since $b^\pi ab^d = 0$, we see that

$$pM(1 - p) = 0, M(1 - p) = \begin{pmatrix} bb^d abb^d & bb^d \\ b^2 b^d & 0 \end{pmatrix}.$$

Obviously, $[M(1 - p)]^\# = \begin{pmatrix} 0 & b^d \\ bb^d & -ab^d \end{pmatrix}$. In view of [16, Lemma 2.2], $pM = \begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix}$ has g-Drazin inverse.

Let $N = \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}$. Then we have

$$\begin{aligned} N &= \begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix}\begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}, \\ \begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix} &= \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix}. \end{aligned}$$

By using Cline's formula (see [6, Theorem 2.1]), N has g-Drazin inverse.

(2) ⇒ (1) Let $e = \begin{pmatrix} bb^d & 0 \\ 0 & 1 \end{pmatrix}$. Then $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_e$, where

$$\alpha = \begin{pmatrix} bb^dabb^d & bb^d \\ b^2b^d & 0 \end{pmatrix}, \beta = \begin{pmatrix} bb^dab^\pi & 0 \\ bb^\pi & 0 \end{pmatrix},$$
$$\gamma = \begin{pmatrix} 0 & b^\pi \\ 0 & 0 \end{pmatrix}, \delta = \begin{pmatrix} b^\pi a & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\alpha^\# = \begin{pmatrix} 0 & b^d \\ bb^d & -ab^d \end{pmatrix}, \alpha^\pi = \begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix},$$
$$\beta + \gamma + \delta = \begin{pmatrix} bb^dab^\pi + b^\pi a & b^\pi \\ bb^\pi & 0 \end{pmatrix},$$

and

$$(\beta + \gamma + \delta)\alpha = \begin{pmatrix} bb^dab^\pi + b^\pi a & b^\pi \\ bb^\pi & 0 \end{pmatrix}\begin{pmatrix} bb^dabb^d & bb^d \\ b^2b^d & 0 \end{pmatrix} = 0.$$

Moreover, we have

$$\begin{aligned} \beta + \gamma + \delta &= \begin{pmatrix} bb^da + b^\pi a & 1 \\ b & 0 \end{pmatrix}\begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix}; \\ \begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix} &= \begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix}\begin{pmatrix} bb^da + b^\pi a & 1 \\ b & 0 \end{pmatrix}; \\ \begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix} &= \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix}, \\ \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix}\begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}. \end{aligned}$$

Since N has g-Drazin inverse, by using Cline's formula, $\beta + \gamma + \delta$ has g-Drazin inverse. Clearly, we have

$$\begin{aligned} \begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix}^d &= \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}(N^d)^2\begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix} \\ &= N^d\begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} &(\beta + \gamma + \delta)^d \\ = &\begin{pmatrix} bb^da + b^\pi a & 1 \\ b & 0 \end{pmatrix}\Big[\begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix}^d\Big]^2\begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix} \\ = &\begin{pmatrix} bb^da + b^\pi a & 1 \\ b & 0 \end{pmatrix}N^d\begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix}N^d\begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix}. \end{aligned}$$

In light of Lemma 2.1, M has g-Drazin inverse. In fact, we get

$$\begin{aligned} M^d &= \sum_{i=0}^{\infty}\alpha^i\alpha^\pi[(\beta+\gamma+\delta)^d]^{i+1} + \sum_{i=0}^{\infty}[\alpha^\#]^{i+1}(\beta+\gamma+\delta)^i(\beta+\gamma+\delta)^\pi \\ &= \alpha^\pi(\beta+\gamma+\delta)^d + \sum_{i=0}^{\infty}[\alpha^\#]^{i+1}(\beta+\gamma+\delta)^i(\beta+\gamma+\delta)^\pi. \end{aligned}$$

(2) ⇔ (3) Obviously, we have

$$\begin{aligned}
\begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix}\begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix},\\
\begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix} &= \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix};\\
\begin{pmatrix} b^\pi a & b^\pi \\ b^\pi b & 0 \end{pmatrix} &= \begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix}\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix},\\
\begin{pmatrix} ab^\pi & b^\pi \\ bb^\pi & 0 \end{pmatrix} &= \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}\begin{pmatrix} b^\pi & 0 \\ 0 & b^\pi \end{pmatrix};\\
\begin{pmatrix} ab^\pi & b^\pi \\ bb^\pi & 0 \end{pmatrix} &= \begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix},\\
\begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & b^\pi \end{pmatrix}\begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix}.
\end{aligned}$$

Therefore we complete the proof by repeatedly using Cline's formula (see [6, Theorem 2.1]). □

Corollary 2.3. *Let $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi ab^d = 0, abb^\pi = b^\pi ba$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Let $N = \begin{pmatrix} ab^\pi & 1 \\ bb^\pi & 0 \end{pmatrix}$. We check that

$$(ab^\pi)(bb^\pi) = abb^\pi = b^\pi ba = b^\pi bab^\pi = (bb^\pi)(ab^\pi).$$

As the argument in [8], N has g-Drazin inverse. This completes the proof by Theorem 2.2. □

Yu et al. characterized the Drazin invertibility of an anti-triangular matrix over a complex Hibert space by using solutions of certain operator equations (see [14, Theorem 4.1]). We now generalize their main results to the g-Darzin inverse in a Banach algebra by using ring technique as follows.

Corollary 2.4. *Let $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi ab^d = 0, b^\pi ab = b^\pi ba$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Let $N = \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}$. By hypothesis, we have

$$(b^\pi a)(b^\pi b) = b^\pi ab = b^\pi ba = (b^\pi b)(b^\pi a).$$

As the argument in [8], N has g-Drazin inverse. We obtain the result by Theorem 2.2. □

Corollary 2.5. *Let $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi ab^d = 0, (b^\pi a)^d b^\pi ab = 0, b^\pi ab(b^\pi a)^\pi = 0$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Let $N = \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}$. Then $N = P + Q$, where

$$P = \begin{pmatrix} (b^\pi a)^2(b^\pi a)^d & (b^\pi a)(b^\pi a)^d \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} (b^\pi a)(b^\pi a)^\pi & (b^\pi a)^\pi \\ (b^\pi a)^\pi(b^\pi b) & 0 \end{pmatrix}.$$

Clearly, P has g-Drazin inverse and $PQ = 0$.

Write $\begin{pmatrix} (b^\pi a)(b^\pi a)^\pi & (b^\pi a)^\pi(b^\pi b)(b^\pi a)^\pi \\ 1 & 0 \end{pmatrix} = K + L$, where

$$K = \begin{pmatrix} (b^\pi a)(b^\pi a)^\pi & 0 \\ 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & (b^\pi a)^\pi(b^\pi b)(b^\pi a)^\pi \\ 1 & 0 \end{pmatrix}.$$

Obviously, $(b^\pi a)(b^\pi a)^\pi$ is quasinilpotent. Hence, K is quasinilpotent.

One easily checks that

$$\begin{array}{rcl} (b^\pi a)^\pi(b^\pi b) & = & [1 - (b^\pi a)^d(b^\pi a)](b^\pi b) \\ & = & b^\pi b - (b^\pi a)^d b^\pi a(1 - b^d b)b \\ & = & b^\pi b - (b^\pi a)^d b^\pi ab + (b^\pi a)^d(b^\pi ab^d)b^2 \\ & = & b^\pi b. \end{array}$$

Then $(b^\pi a)^\pi(b^\pi b)$ is quasinilpotent. By using Cline's formula, $(b^\pi a)^\pi(b^\pi b)(b^\pi a)^\pi$ is quasinilpotent. Accordingly, L is quasinilpotent.

By hypothesis, we check that

$$\begin{array}{rl} & [(b^\pi a)(b^\pi a)^\pi][(b^\pi a)^\pi(b^\pi b)(b^\pi a)^\pi] \\ = & (b^\pi a)^\pi(b^\pi a)(b^\pi b)(b^\pi a)^\pi] \\ = & (b^\pi a)^\pi(b^\pi ab)(b^\pi a)^\pi \\ = & 0. \end{array}$$

By Lemma 2.1, $\begin{pmatrix} (b^\pi a)(b^\pi a)^\pi & (b^\pi a)^\pi(b^\pi b)(b^\pi a)^\pi \\ 1 & 0 \end{pmatrix}$ is quasinilpotent. We verify that

$$\begin{array}{rl} & \begin{pmatrix} b^\pi a(b^\pi a)^\pi & (b^\pi a)^\pi \\ (b^\pi a)^\pi b^\pi b & 0 \end{pmatrix} \\ = & \begin{pmatrix} (b^\pi a)^\pi & 0 \\ 0 & (b^\pi a)^\pi \end{pmatrix}\begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}, \\ & \begin{pmatrix} (b^\pi a)(b^\pi a)^\pi & (b^\pi a)^\pi \\ (b^\pi b)(b^\pi a)^\pi & 0 \end{pmatrix} \\ = & \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}\begin{pmatrix} (b^\pi a)^\pi & 0 \\ 0 & (b^\pi a)^\pi \end{pmatrix} \\ = & \begin{pmatrix} 1 & 0 \\ 0 & (b^\pi b)(b^\pi a)^\pi \end{pmatrix}\begin{pmatrix} (b^\pi a)(b^\pi a)^\pi & (b^\pi a)^\pi \\ 1 & 0 \end{pmatrix}, \\ & \begin{pmatrix} (b^\pi a)(b^\pi a)^\pi & (b^\pi a)^\pi(b^\pi b)(b^\pi a)^\pi \\ 1 & 0 \end{pmatrix} \\ = & \begin{pmatrix} (b^\pi a)(b^\pi a)^\pi & (b^\pi a)^\pi \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (b^\pi b)(b^\pi a)^\pi \end{pmatrix}. \end{array}$$

By using Cline's formula, Q has quasinilpotent. In view of Lemma 2.1, N has g-Drazin inverse. According to Theorem 2.2, we complete the proof. □

We are now ready to prove the following:

Theorem 2.6. *Let $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi(ab^2) = 0$ and $b^\pi(aba) = 0$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Since $b^\pi(ab^2) = 0$, we have

$$b^\pi ab^d = b^\pi ab^2(b^d)^3 = 0, b^\pi a(b^\pi b)^2 = 0, b^\pi a(b^\pi b)b^\pi a = 0.$$

Let $N = \begin{pmatrix} b^\pi a & 1 \\ b^\pi b & 0 \end{pmatrix}$. Then $N^2 = \begin{pmatrix} (b^\pi a)^2 + b^\pi b & b^\pi a \\ b^\pi b b^\pi a & b^\pi b \end{pmatrix}$. Write $N^2 = P + Q$, where

$$P = \begin{pmatrix} (b^\pi a)^2 & b^\pi a \\ 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} b^\pi b & 0 \\ b^\pi b b^\pi a & b^\pi b \end{pmatrix}.$$

Obviously, we have

$$PQ^2 = \begin{pmatrix} (b^\pi a)^2 b^\pi b & b^\pi ab \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b^\pi b & 0 \\ b^\pi b b^\pi a & b^\pi b \end{pmatrix} = 0,$$
$$PQP = \begin{pmatrix} (b^\pi a)^2 b^\pi b & b^\pi ab \\ 0 & 0 \end{pmatrix}\begin{pmatrix} (b^\pi a)^2 & b^\pi a \\ 0 & 0 \end{pmatrix} = 0.$$

By virtue of [2, Theorem 2.4], N^2 has g-Drazin inverse. It follows from [7, Corollary 2.2] that N has g-Drazin inverse. In light of Theorem 2.2, the result follows. □

Corollary 2.7. *Let $a, b \in \mathcal{A}^d$. If $ab^2 = 0$ and $aba = 0$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Since $ab^2 = 0$, $ab^\pi = a - (ab^2)(b^d)^2 = a \in \mathcal{A}^d$. By Cline's formula, $b^\pi a \in \mathcal{A}^d$. This completes the proof by Theorem 2.6. □

Corollary 2.8. *Let $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi ab = 0$, then $\begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$ has g-Drazin inverse.*

Proof. Since $b^\pi ab = 0$, we see that $b^\pi ab^2 = 0$ and $b^\pi(aba) = 0$. So the corollary is true by Theorem 2.6. □

3. Additive properties

In this section we establish some elementary additive properties of g-Drazin inverse in a Banach algebra. The following fact will also be used in our subsequent investigations.

Theorem 3.1. *Let $a, b, ab, (ab)^\pi a \in \mathcal{A}^d$. If $ab^2 = 0, (ab)^\pi a(ab)^d = 0$ and $(ab)^\pi\, aba = 0$, then $a + b \in \mathcal{A}^d$.*

Proof. Obviously, we have $a + b = (1, b)\begin{pmatrix} a \\ 1 \end{pmatrix}$. In view of Cline's formula, it suffices to prove

$$M = \begin{pmatrix} a \\ 1 \end{pmatrix}(1, b) = \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}$$

has g-Drazin inverse. Write $M = K + L$, where

$$K = \begin{pmatrix} a & ab \\ 1 & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

Let $H = \begin{pmatrix} a & 1 \\ ab & 0 \end{pmatrix}$ and $N = \begin{pmatrix} (ab)^\pi a & 1 \\ (ab)^\pi ab & 0 \end{pmatrix}$. One easily checks that

$$(ab)^\pi a[(ab)^\pi ab]^2 = (ab)^\pi a(ab)^\pi(aba)b = 0,$$
$$(ab)^\pi a[(ab)^\pi ab](ab)^\pi a = (ab)^\pi a(ab)^\pi(aba) = 0.$$

In light of Corollary 2.7, N has g-Drazin inverse. By hypothesis, $(ab)^\pi a(ab)^d = 0$. According to Theorem 2.2, H has g-Drazin inverse. Clearly,

$$H = \begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, K = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & ab \end{pmatrix}.$$

By using Cline's formula, K has g-Drazin inverse. Since $ab^2 = 0$, we have $KL = 0$. In light of Lemma 2.1, M has g-Drazin inverse. Therefore $a + b \in \mathcal{A}^d$. □

Corollary 3.2. *Let* $a, b, ab, b(ab)^{\pi} \in \mathcal{A}^d$. *If* $a^2b = 0, (ab)^d b(ab)^{\pi} = 0$ *and* $bab(ab)^{\pi} = 0$, *then* $a + b \in \mathcal{A}^d$

Proof. Since $(\mathcal{A}, \cdot)$ is a Banach algebra, $(\mathcal{A}, *)$ is a Banach algebra with the multiplication $x * y = y \cdot x$. Then we complete the proof by applying Theorem 3.1 to the Banach algebra $(\mathcal{A}, *)$. □

We are now ready to generalize [9, Theorem 3.1] as follow:

Theorem 3.3. *Let* $a, b, ab \in \mathcal{A}^d$. *If* $ab^2 = 0$ *and* $(ab)^{\pi} a^2 ba = 0$, *then* $a + b \in \mathcal{A}^d$.

Proof. Let $M = \begin{pmatrix} a & ab \\ 1 & b \end{pmatrix}$. Write $M = K + L$, where

$$K = \begin{pmatrix} a & ab \\ 1 & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}.$$

Let $H = \begin{pmatrix} a & 1 \\ ab & 0 \end{pmatrix}$. By hypothesis, we check that

$$(ab)^{\pi} a(ab)^2 = 0, (ab)^{\pi} a(ab)a = 0.$$

According to Theorem 2.6, H is g-Drazin inverse. As in the proof of Theorem 3.1, by using Cline's formula, K has g-Drazin inverse. Since $ab^2 = 0$, it follows by Lemma 2.1 that M has g-Drazin inverse. Observing that

$$\begin{aligned} a + b &= (1, b)\begin{pmatrix} a \\ 1 \end{pmatrix}, \\ M &= \begin{pmatrix} a \\ 1 \end{pmatrix}(1, b), \end{aligned}$$

by using Cline's formula again, $a + b$ has g-Drazin inverse. □

Corollary 3.4. *Let* $a, b, ab \in \mathcal{A}^d$. *If* $a^2b = 0$ *and* $bab^2(ab)^{\pi} = 0$, *then* $a + b \in \mathcal{A}^d$.

Proof. Similarly to Corollary 3.2, we obtain the result by Theorem 3.3. □

Corollary 3.5. *Let* $a, b, ab \in \mathcal{A}^d$. *If* $ab^2 = 0$ *and* $a^2ba = 0$, *then* $a + b \in \mathcal{A}^d$.

Proof. This is obvious by Theorem 3.3. □

4. Operator matrices over Banach spaces

In this section we apply our results to establish g-Drazin invertibility for the block operator matrix M as in (∗). Throughout this section, we always assume that $A, D, BC \in \mathcal{L}(X)^d$. We come now to extend [14, Theorem 3.1] as follows.

Theorem 4.1. *If* $(BC)^{\pi} ABCA = 0, (BC)^{\pi} ABCB = 0, DCA = 0$ *and* $DCB = 0$, *then* M *has g-Drazin inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

Clearly, $Q^2 = 0$, and so $PQ^2 = 0$. Moreover, we have

$$\begin{aligned} PQ &= \begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix}, \\ (PQ)^d &= \begin{pmatrix} (BC)^d & 0 \\ DC[(BC)^d]^2 & 0 \end{pmatrix} = \begin{pmatrix} (BC)^d & 0 \\ 0 & 0 \end{pmatrix}, \\ (PQ)^{\pi} &= \begin{pmatrix} (BC)^{\pi} & 0 \\ -DC(BC)^d & I \end{pmatrix} = \begin{pmatrix} (BC)^{\pi} & 0 \\ 0 & I \end{pmatrix}. \end{aligned}$$

We easily check that

$$\begin{array}{cl} & (PQ)^{\pi}P^2QP \\ = & \begin{pmatrix} (BC)^{\pi} & 0 \\ -DC(BC)^d & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \\ = & \begin{pmatrix} (BC)^{\pi}A & (BC)^{\pi}B \\ 0 & D \end{pmatrix}\begin{pmatrix} BCA & BCB \\ 0 & 0 \end{pmatrix} \\ = & 0. \end{array}$$

Therefore we complete the proof by Theorem 3.3. □

Corollary 4.2. *If* $(BC)^{\pi}ABCA = 0, (BC)^{\pi}ABCB = 0, BDC = 0$ *and* $BD^2 = 0$, *then* M *has g-Drazin inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

In light of Theorem 4.1, P has g-Drazin inverse. Since $PQ^2 = 0$ and $PQP = 0$, we complete the proof by [2, Theorem 2.4]. □

Theorem 4.3. *If* $(BC)^{\pi}A(BC)^d = 0, (BC)^{\pi}BCA = 0, (BC)^{\pi}BCB = 0, DCA = 0$ *and* $DCB = 0$, *then* M *has g-Drazin inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

Then $Q^2 = 0$ and $PQ = \begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix}$; hence,

$$(PQ)^d = \begin{pmatrix} (BC)^d & 0 \\ 0 & 0 \end{pmatrix}, (PQ)^{\pi} = \begin{pmatrix} (BC)^{\pi} & 0 \\ 0 & I \end{pmatrix}.$$

By hypothesis, we verify that

$$\begin{array}{cl} & (PQ)^{\pi}P(PQ)^d \\ = & \begin{pmatrix} (BC)^{\pi} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} (BC)^d & 0 \\ 0 & 0 \end{pmatrix} \\ = & \begin{pmatrix} (BC)^{\pi} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} A(BC)^d & 0 \\ 0 & 0 \end{pmatrix} \\ = & 0, \\ & (PQ)^{\pi}PQP \\ = & \begin{pmatrix} (BC)^{\pi} & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \\ = & \begin{pmatrix} (BC)^{\pi}BCA & (BC)^{\pi}BCB \\ DC(BC)^{\pi}A & DC(BC)^{\pi}B \end{pmatrix} \\ = & 0. \end{array}$$

This completes the proof by Theorem 3.1. □

Corollary 4.4. *If* $(BC)^{\pi}A(BC)^d = 0, (BC)^{\pi}BCA = 0, (BC)^{\pi}BCB = 0, BDC = 0$ *and* $BD^2 = 0$, *then* M *has g-Drazin inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}.$$

In light of Theorem 4.3, P is g-Drazin inverse. As in the proof of Corollary 4.2, M has g-Drazin inverse. □

Theorem 4.5. *If* $(CB)^{\pi}CABC = 0, A(BC)^{\pi}ABC = 0, ABD = 0$ *and* $CBD = 0$, *then* M *has g-Drazin inverse.*

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}.$$

Then

$$\begin{aligned} PQ^2 &= \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}^2 \\ &= \begin{pmatrix} 0 & ABD \\ 0 & CBD \end{pmatrix}, \\ (PQ)^{\pi}P^2QP &= \begin{pmatrix} 0 & AB \\ 0 & CB \end{pmatrix}^{\pi}\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}^2\begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}\begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \\ &= \begin{pmatrix} I & -AB(CB)^d \\ 0 & (CB)^{\pi} \end{pmatrix}\begin{pmatrix} A^2BC & 0 \\ CABC & 0 \end{pmatrix}. \end{aligned}$$

Clearly, P and Q have g-Drazin inverses. Moreover, $PQ^2 = 0$ and $(PQ)^{\pi}P^2QP = 0$, and therefore we complete the proof by Theorem 3.3. □

We now generalize [5, Theorem 2.2] as follow.

Corollary 4.6. *If* $(CB)^{\pi}CABC = 0, A(BC)^{\pi}ABC = 0, BDC = 0$ *and* $BD^2 = 0$, *then* M *has g-Drazin inverse.*

Proof. As in the proof of Corollary 4.2, we are through by Theorem 4.5. □

Acknowledgement

The authors are thankful the referees for the valuable suggestions which led to improvement of the paper.

References

[1] C. Bu; K. Zhang and J. Zhao, Representation of the Drazin inverse on solution of a class singular differential equations, *Linear Multilinear Algebra*, **59**(2011), 863–877.
[2] H. Chen and M. Sheibani, G-Drazin inverses for operator matrices, *Operators and Matrices*, **14**(2020), 23–31.
[3] J. Cui and J. Chen, When is a 2×2 matrix ring over a commutative local ring quasipolar? *Comm. Algebra*, **39**(2011), 3212–3221.
[4] J. Cui and J. Chen, Quasipolar triangular matrix rings over local rings, *Comm. Algebra*, **40**(2012), 784–794.
[5] E. Dopazo and M.F. Martínez-Serrano, Further results on the representation of the Drazin inverse of a 2×2 block matrix, *Linear Algebra Appl.*, **432**(2010), 1896–1904.
[6] Y. Liao; J. Chen and J. Cui, Cline's formula for generalized Drazin inverse, *Bull. Malays. Math. Sci. Soc.*, **37**(2014), 37-42.
[7] D. Mosić, A note on Cline's formula for the generalized Drazin inverse, *Linear Multilinear Algebra*, **63**(2014), 1106–1110.
[8] P. Patricío and R.E. Hartwig, The $(2, 2, 0)$ Drazin inverse problem, *Linear Algebra Appl.*, **437**(2012), 2755–2772.
[9] A. Shakoor; I. Ali; S. Wali and A. Rehman, Some formulas on the Drazin inverse for the sum of two matrices and block matrices, *Bull. Iran. Math. Soc.*, 2021, https://doi.org/10.1007/s41980-020-00521-3.
[10] L. Sun; B. Zheng; S. Bai and C. Bu, Formulas for the Drazin inverse of matrices over skew fields, *Filomat*, **30**(2016), 3377–3388.
[11] L. Xia and B. Deng, The Drazin inverse of the sum of two matrices and its applications, *Filomat*, **31** (2017), 5151-5158.
[12] Q. Xu; C. Song and L. Zhang, Solvability of certain quadratic operator equations and representations of Drazin inverse, *Linear Algebra Appl.*, **439** (2013), 291–309.
[13] H. Yang and X. Liu, The Drazin inverse of the sum of two matrices and its applications, *J. Comput. Applied Math.*, **235**(2011), 1412–1417.
[14] A. Yu; X. Wang and C. Deng, On the Drazin inverse of an anti-triangular block matrix, . *Linear Algebra Appl.*, **489**(2016), 274–287.
[15] D. Zhang, Representations for generalized Drazin inverse of operator matrices over a Banach space, *Turk. J. Math.*, **40**(2016), 428–437.
[16] D. Zhang and D. Mosić, Explicit formulae for the generalized Drazin inverse of block matrices over a Banach algebra, *Filomat*, **32**(2018), 5907–5917.
[17] H. Zou; J. Chen and D. Mosić, The Drazin invertibility of an anti-triangular matrix over a ring, *Studia Scient. Math. Hungar.*, **54**(2017), 489–508.
[18] H. Zou; D. Mosić and J. Chen, Generalized Drazin invertibility of the product and sum of two elements in a Banach algebra and its applications, *Turk. J. Math.*, **41**(2017), 548–563.